Flashcard 2, 3, 8, 9, 11, 12, 13, 20 ; Exam 4j, 6a

2) $\sum_{n=1}^{\infty} \frac{(-7)^n}{8n^6+9^n}$

Series : $\sum (-1)^k a_k$, $a_k \geq 0$

absolute conv : does $\sum |(-1)^k a_k|$ converge?

Thm: abs convergence $\Rightarrow$ convergence (of original series)

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 7^n}{9^n+8n^6}$ abs. convergence? $\boxed{\text{look at } \sum \frac{7^n}{9^n+8n^6}}$ $\approx \sum \frac{7^n}{9^n} = \sum \left(\frac{7}{9}\right)^n$

as $n \to \infty$, behaves our series like this

geometric, with ratio $7/9 < 1$, hence this converges.

– comp. test or – lim comp. test.

Now use limit comparison Test with $\sum \left(\frac{7}{9}\right)^n$

(alternatively: use comparison test here)

If we use Alt. Series Test, it would tell us just about $\sum \frac{(-1)^n \cdot 7^n}{9^n+8n^6}$, but wouldn't tell us about $\sum \frac{7^n}{9^n+8n^6}$

$$\lim_{n\to\infty} \frac{\frac{7^n}{9^n+8n^6}}{\frac{7^n}{9^n}} = \lim_{n\to\infty} \frac{\cancel{7^n}}{9^n+8n^6} \cdot \frac{9^n}{\cancel{7^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{8n^6}{9^n}} = 1$$

Goal is to understand conv/div of $\sum \frac{7^n}{9^n+8n^6}$, which is not alternating.

$\Rightarrow$ since $\sum \left(\frac{7}{9}\right)^n$ converges, $\sum \frac{7^n}{9^n+8n^6}$ converges as well

Since series absolutely converges, we conclude that $\sum \frac{(-7)^n}{9^n+8n^6}$ converges as well

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OTOH, if Q just asked about $\sum \frac{(-7)^n}{9^n+8n^6}$ converging or diverging, (not abs. conv.) then alternating series test would be enough

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If it had been the case that $\sum a_k$ diverged, then we'd next look at $\sum (-1)^k a_k$ to determine conv/div. (resp.)

$\rightsquigarrow$ conditional conv. (resp.)

$\rightsquigarrow$ divergence.

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3) $\sum \frac{(-1)^n}{4n^{1/4}+7}$

if we try to look at abs. conv, we take

$\sum \frac{1}{4n^{1/4}+7} \approx \sum \frac{1}{4n^{1/4}} = \frac{1}{4} \sum \frac{1}{n^{1/4}}$ diverges. (p-series with $p = 1/4 \leq 1$)

Then use Limit Comparison Test to conclude that $\sum \frac{1}{4n^{1/4}+7}$ diverges as well.

So we don't have absolute convergence.

Do we have conditional convergence?

$\sum \frac{(-1)^n}{4n^{1/4}+7}$, use Alternating Series Test: check terms $\frac{1}{4n^{1/4}+7}$ are non-increasing ✓

for each n, $4(n+1)^{1/4}+7 > 4n^{1/4}+7$ so reciprocals are non-inc.

$\lim_{n\to\infty} \frac{1}{4n^{1/4}+7} = 0$ ✓

$\Rightarrow$ series converges.

Yes, have conditional convergence.

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p-series test : $\sum \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

different conclusions

Alternating Series Test : $\sum \underline{(-1)^k} \cdot c_k$ $\rightsquigarrow$ check $c_{k+1} \leq c_k$, $\lim_{k\to\infty} c_k = 0$

$\Rightarrow$ conclude series converges

Example:

Alternating harmonic series : $\sum (-1)^n \frac{1}{n}$ converges by A.S.T.

OTOH

Harmonic Series: $\sum \frac{1}{n}$ diverges by p-series test.

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8) $\sum (-1)^n \left(\sqrt{n^2+1} - n\right) \cdot \underline{\left(\sqrt{n^2+1} + n\right)}$

$$= \sum \frac{(-1)^n \left(n^2+1-n^2\right)}{\sqrt{n^2+1}+n} = \sum \frac{(-1)^n}{\sqrt{n^2+1}+n}$$

want to understand abs. conv.: $\sum \frac{1}{\sqrt{n^2+1}+n} \overset{\text{carry out Limit Comparison Test}}{\approx} \sum \frac{1}{\sqrt{n^2}+n} = \sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ diverges.

does not converge absolutely

Check conditional conv: $\sum \frac{(-1)^n}{\sqrt{n^2+1}+n} \approx \sum \frac{(-1)^n}{2n}$ converges

So we have conditional convergence.

apply Alt. Series Test. terms non-inc. $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0$

9) $\sum_{n=1}^{\infty} \cos(n\pi)\left(\frac{n!}{7^n}\right)$

$\cos(\pi) = -1$
$\cos(2\pi) = 1$
$\cos(3\pi) = -1$
$\vdots$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n!}{7^n}$

Look at $\lim_{n\to\infty} \frac{(-1)^n n!}{7^n} \neq 0$

So by Divergence Test, series diverges

11) $\sum_{n=1}^{\infty} \frac{3+2\sin n}{7n^{5/4}+5\cos n}$ compare to $\sum \frac{1}{n^{5/4}}$, which converges by p-series test.

Limit Comparison Test. $\Rightarrow$ converges.

20) $\sum_{n=1}^{\infty} \sin\left(\frac{2n^2+4}{n^4+2}\right)$ $\left(\approx \sum \sin\left(\frac{1}{n^2}\right) \approx \sum \frac{1}{n^2}\right)$ ($\sin x \approx x$ when x is near 0)

use Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{(2n^2+4)\cdot\frac{1}{n^4}}{(n^4+2)\cdot\frac{1}{n^4}}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin\left(\frac{2/n^2+4/n^4}{1+2/n^4}\right)}{\frac{1}{n^2}} =$$

do L'Hôpital's Rule & directly here, compute the limit

since $\sum \frac{1}{n^2}$ converges by p-series test,

$\sum \sin\left(\frac{2n^2+4}{n^4+2}\right)$ converges as well.

#4, on practice exam:

$\sum_{n=3}^{\infty} \frac{n!}{n^n}$

Apply Ratio Test:

$$\lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\cancel{(n+1)}\, n^n}{\cancel{(n+1)}\cdot(n+1)^n}\right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \boxed{\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = e^{-1}} < 1$$

So converges.

Given $\sum a_k$, knowing $a_k \to 0$ does not necessarily tell you that $\sum a_k$ converges !!

$\sum \frac{1}{k}$ vs $\sum \frac{(-1)^k}{k}$

How do we compute $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-1}$? $= L$

$$\ln\left(\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n-1}\right) \quad (= \ln L)$$

$$= \lim \ln\left(\left(1-\frac{1}{n}\right)^{n-1}\right) = \lim_{n\to\infty} (n-1)\cdot\ln\left(1-\frac{1}{n}\right)$$

$n \to \infty$

$$= \lim_{n \to \infty} \frac{\ln\left(1-\frac{1}{n}\right) = \ln(1-n^{-1})}{\frac{1}{n-1} = (n-1)^{-1}}$$

apply L'Hopital's Rule

$$= \lim_{n \to \infty} \frac{\frac{-1 \cdot -n^{-2}}{1-n^{-1}}}{-1(n-1)^{-2}}$$

$$= \lim_{n \to \infty} \frac{\frac{\frac{1}{n^2}}{1-\frac{1}{n}}}{(-1)\cdot\frac{1}{(n-1)^2}} = \lim_{n \to \infty} \frac{\frac{1}{n^2}}{1-\frac{1}{n}} \cdot (n-1)^2 \cdot (-1)$$

$$= \lim_{n \to \infty} \frac{\frac{(n-1)^2}{n^2}(-1)}{1-\frac{1}{n}} = -1$$

$$\Rightarrow -1 = \ln L \Rightarrow e^{-1} = L.$$