Defining and Plotting a Parametric Curve in 3D $\alpha(t) = (x(t), y(t), z(t))$

Clear["Global`*"] (* clears all previous assignments so we can reuse them *)

(* Define vector functions in 3D of one variable t, i.e. a curve in 3D *)
(* to typeset the alpha, beta, and sigma using esc alpha esc, ditto beta, ditto sigma *)
$\alpha[t_] = \{\text{Cos}[t], \text{Sin}[t], t\}$
$\beta[t_] = \{t, 2*t, 3*t\}$
$\sigma[t_] = \{\text{Cos}[t], \text{Sin}[t], 0\}$

(* Plot the curves $\alpha$, $\beta$, $\sigma$ with no plotting options defined *)
(* Give the plot an assigned name and use this name in the GraphicsGrid command to plot *)
$\alpha\text{plot1} = \text{ParametricPlot3D}[\alpha[t], \{t, 0, 2*Pi\}];$
$\beta\text{plot1} = \text{ParametricPlot3D}[\beta[t], \{t, 0, 2*Pi\}];$
$\sigma\text{plot1} = \text{ParametricPlot3D}[\sigma[t], \{t, 0, 2*Pi\}];$

(* Plot the curve with some plotting options defined *)
$\alpha\text{plot2} = \text{ParametricPlot3D}[\alpha[t], \{t, 0, 2*Pi\},$
\hspace{1em} PlotStyle $\rightarrow \text{Directive[Red, Thickness[0.015]]}$,
\hspace{1em} AxesLabel $\rightarrow \{x, y, z\}$,
\hspace{1em} LabelStyle $\rightarrow \text{Directive[Black, 15]}];$

$\beta\text{plot2} = \text{ParametricPlot3D}[\beta[t], \{t, 0, 2*Pi\},$
\hspace{1em} PlotStyle $\rightarrow \text{Directive[Red, Thickness[0.015], Dashed]}$,  
\hspace{1em} AxesLabel $\rightarrow \{x, y, z\}$,
\hspace{1em} LabelStyle $\rightarrow \text{Directive[Black, 15]}];$

$\sigma\text{plot2} = \text{ParametricPlot3D}[\sigma[t], \{t, 0, 2*Pi\},$
\hspace{1em} PlotStyle $\rightarrow \text{Directive[Red, Thickness[0.015], DotDashed]}$, 
\hspace{1em} AxesLabel $\rightarrow \{x, y, z\}$,
\hspace{1em} LabelStyle $\rightarrow \text{Directive[Black, 15]}];$

(* Display the two plots side by side *)
GraphicsGrid[}
Display all the plots on one set of axes:

```math
\{\text{aplot1, bplot1, splot1}, \{\text{aplot2, bplot2, splot2}\}\}
```

Now plot the combined plots side by side:

```math
\text{GraphicsGrid}[\{\{\text{plots1, plots2}\}\}]
```
Defining and Plotting Explicit Functions $z=f(x,y)$
Clear["Global`*"] (* clears all previous assingments so we can reuse them *)

(* Define some explicit functions z=f(x,y), z=g(x,y), z=h(x,y), i.e. surfaces in 3D *)
f[x_, y_] = x^2 + y^2
g[x_, y_] = Cos[x] * Sin[y]
h[x_, y_] = 2*x - 3*y + 2

(* Plot surfaces with no plot options *)
R = 5;
plotf1 = Plot3D[f[x, y], {x, -R, R}, {y, -R, R}];
plotg1 = Plot3D[g[x, y], {x, -R, R}, {y, -R, R}];
ploth1 = Plot3D[h[x, y], {x, -R, R}, {y, -R, R}];

(* Plot surfaces with some basic plot options *)
R = 5;
plotf2 = Plot3D[f[x, y], {x, -R, R}, {y, -R, R}, PlotStyle -> Directive[Green]];
plotg2 = Plot3D[g[x, y], {x, -R, R}, {y, -R, R}, PlotStyle -> Directive[Red], Mesh -> False ];
ploth2 = Plot3D[h[x, y], {x, -R, R}, {y, -R, R}, PlotStyle -> Directive[Blue, Opacity[0.6]], Mesh -> False ];

(* Plot surfaces with some more plot options *)
Plot3D[f[x, y], {x, -R, R}, {y, -R, R},
    ImageSize -> {400, 400},
    ColorFunction -> "SunsetColors", Background -> Black,
    Boxed -> False, Mesh -> 20,
    PlotStyle -> Directive[Opacity[0.8]] ]

(* Plot surfaces side by side *)
GraphicsGrid[
    {{plotf1, plotg1, ploth1}, {plotf2, plotg2, ploth2}}]

(* Group plots on one set of axes then display pairs side by side *)
plot1 = Show[ {plotf1, plotg1, ploth1} ];
plot2 = Show[ {plotf2, plotg2, ploth2} ];
GraphicsGrid[ {{plot1, plot2}} ]
\[ x^2 + y^2 \]

\[ \cos(x) \sin(y) \]

\[ 2 + 2x - 3y \]
Defining and Plotting Implicit Functions $f(x,y,z)=c$ where $c$ is some constant

- Recall Implicit Functions $f(x,y)=c$ and their relation to $z=f(x,y)$

```mathematica
Clear["Global`*"] (* clears all previous assignments so we can reuse them *)

(* define an expression in the variables x and y *)
f[x_, y_] = x * Cos[y];

(* ContourPlot is how you plot an implicit function $f(x,y)=c$ *)
plot1 = ContourPlot[{f[x, y] == -1, f[x, y] == 1/2},
{x, -2, 2}, {y, -2, 2}, AxesLabel -> {x, y},
ContourStyle -> Directive[Black, Thickness[0.01]]];

(* In terms of the surface $z=f(x,y)$,
how do we geometrically interpret $f(x,y)=c$? *)
plot2 = Plot3D[{f[x, y], -1, 1/2}, {x, -2, 2}, {y, -2, 2},
PlotStyle -> {Directive[Green, Specularity[White, 20]],
Directive[Black, Opacity[0.6]],
Directive[Black, Opacity[0.4]]},
Lighting -> "Neutral", Mesh -> False,
ImageSize -> {500, 500}, AxesLabel -> {x, y, z}];

(* Plot the ContourPlot and the Surface with z planes side by side *)
GraphicsGrid[{{{plot1, plot2}}}]`

(* Just for fun lets make an animation for various values of the "z-slice" *)
P1 = Manipulate[ContourPlot[f[x, y] == c, {x, -3, 3},
{y, -3, 3}, AxesLabel -> {x, y}, ContourStyle ->
Directive[Black, Thickness[0.01]]], {c, -2, 3, 1}];
P2 = Manipulate[Plot3D[{f[x, y], c}, {x, -3, 3},
{y, -3, 3}, PlotStyle -> {Directive[Green],
Directive[Black, Opacity[0.6]]}, Mesh -> False,
ImageSize -> {500, 500}, AxesLabel -> {x, y, z},
ImageSize -> {200, 200}], {c, -2, 3, 1}];

GraphicsGrid[{{P1, P2}}, ImageSize -> {1200, 1200}];
```
Implicit Functions $f(x,y,z)=c$ and their relation to $w=f(x,y,z)$ (ToDo When Get Faster Machine--ContourPlot3D)

(* Define two expression each depending on three variables *)
F[x_, y_, z_] = x^2 + y^2 + z^2
G[x_, y_, z_] = Cos[x] * Sin[y] * z

(* ContourPlot3D is how to plot $F(x,y,z)=c$ *)
(* ContourPlot3D[{F[x,y,z]==1,F[x,y,z]==4},
 {x,-2,2},{y,-2,2},{z,-2,2},ContourStyle→
 {Directive[Blue], Directive[Green,Opacity[0.5]] } ] *)

$x^2 + y^2 + z^2$

$z \cos[x] \sin[y]$
Curve in 2D $\alpha(t)=(x(t),y(t))$, Curve $\beta(t)=f(\alpha(t))$ on Surface $z=f(x,y)$ in 3D
Clear["Global`*"] (* clears all 
previous assingments so we can reuse them *)

(* Define a function/surface z=
  f(x,y) on which we will "project" a curve *)
f[x_, y_] = x^2 + y^2*Sin[x*y]

(* Define a 2D curve α(t) in the (x,y)-plane *)
x[t_] = Cos[t]
y[t_] = Sin[t]
α2[t_] = {x[t], y[t]}

(* Trick: Think of the 2D curve α(t)
  as a 3D curve in the (x,y,z=0)-plane *)
α3[t_] = {x[t], y[t], 0}

(* Define a 3D curve β(t)=
  f(α(t)) which is constrained to the surface z=f(x,y) *)
αf[t_] = {x[t], y[t], f[x[t], y[t]]}

(* Plots of the above *)
α2plot = ParametricPlot[α2[t], {t, 0, 2*Pi},
   PlotStyle -> Directive[Black, Thickness[0.01]],
   AxesLabel -> {x, y}, Frame -> True];
α3plot = ParametricPlot3D[α3[t], {t, 0, 2*Pi},
   PlotStyle -> Directive[Black, Thickness[0.01]],
   AxesLabel -> {x, y, z}, Boxed -> True];
αfplot = ParametricPlot3D[αf[t], {t, 0, 2*Pi},
   PlotStyle -> Directive[Black, Thickness[0.015]],
   AxesLabel -> {x, y, z}, Boxed -> True];
fplot = Plot3D[f[x, y], {x, -1, 1}, {y, -1, 1}, PlotStyle ->
   Directive[Red, Opacity[0.8], Specularity[White, 20]],
   Mesh -> False, AxesLabel -> {x, y, z},
   Boxed -> True, ImageSize -> {500, 500}];

(* Bring the 2D and 3D plots together *)
plot1 = Show[{αfplot, fplot, α3plot}];
GraphicsGrid[{{α2plot, α3plot, plot1}}]
Bring All The Above Together: Curve of Intersection Problems

(In Class Problem) Find the curve of intersection of the two surfaces \( f(x, y) = x + 2y - 2 \) and \( g(x, y) = 2x + y + 3 \).

(Hint: Do the algebra work by hand)
(Extra: Come up with plots to check your answer)
(An Answer):

Remember that \( f(x,y) = x + 2y - 2 \) \( g(x,y) = 2x + y + 3 \) are explicit functions of \( x \) and \( y \) which we interpret to mean \( z = x + 2y - 2 \) and \( z = 2x + y + 3 \). To be an intersection curve the \( x \), \( y \) and \( z \)'s must all be equal. Our equations scream for us to focus on the \( z \)'s first (since they are already "solved" for \( z \)).

```mathematica
Clear["Global`*"
(* clears all previous assingments so we can reuse them *)

(* We want to find when the \( z \) values equal, so set them equal OR find when their difference is 0 *)
Solve[x + 2*y - 2 - (2*x + y + 3) == 0, x] (* option 1,
Solve for \( x \) in terms of \( y \), i.e. find \( x(y) \) *)
Solve[x + 2*y - 2 - (2*x + y + 3) == 0, y] (* option 2,
Solve for \( y \) in terms of \( x \), that is find \( y(x) \) *)

(* At this point we can write at least two solutions *)
\( \alpha[y_] = \{-5 + y, y, 2*(-5 + y) + y + 3\}; \)
(* \( y \) is playing the role of the parameter \( t \) *)
\( \beta[x_] = \{x, 5 + x, x + 2*(5 + x) - 2\}; \)
(* \( x \) is playing the role of the parameter \( t \) *)

PossibleAnswer1 = Simplify[\( \alpha[t] \)]
PossibleAnswer2 = Simplify[\( \beta[t] \)]

(* plot these curves *)
SolutionCurve1 = ParametricPlot3D[\( \alpha[y] \), \{y, 0, 5\},
PlotStyle -> Directive[Green, Thickness[0.015]]];
SolutionCurve2 = ParametricPlot3D[\( \beta[x] \), \{x, -1, 0\},
PlotStyle -> Directive[Green, Thickness[0.015]]];

(* plot the two original surfaces *)
SurfacesPlot =
Plot3D[\{x + 2*y - 2, 2*x + y + 3\}, \{x, -5, 5\}, \{y, -5, 5\},
PlotStyle -> \{Directive[Red], Directive[Blue, Opacity[0.8]]\}];

GraphicsGrid[\{\{Show[SolutionCurve1, SurfacesPlot],
Show[SolutionCurve2, SurfacesPlot]\}\} ]
```

\([\{x \to -5 + y\}]\)
PossibleAnswer1 = \{-5 + t, t, -7 + 3t\}

PossibleAnswer2 = \{t, 5 + t, 8 + 3t\}

(HomeWork 1) Find the curve of intersection of the two surfaces \(x + 3y + 4z - 2 = 0\) and \((x - 1) + (2y + 2) - (z - 2) = 0\)

(Hint : Do the algebra work by hand)
(Extra : Come up with plots to check your answer)

(An Answer) : So this problem is like the In Class Problem but both surfaces are Implicit.

Don’t panic when you see an implicit form of a function. As a start, you can always try to turn it into an explicit form by solving for the “easiest” variable.
Clear["Global`*"]
(* clears all previous assingments so we can reuse them *)

(* So the surfaces are implicit f(x,y,z)=c form, so remedy this "problem" by solving for the explicit form x=f(y,z). *)
foo = Expand[(x - 1) + (2*y + 2) - (z - 2) == 0]
foo1 = Solve[foo, x]
foo2 = Solve[x + 3*y + 4*z - 2 == 0, x]
a[y_, z_] = -3 - 2*y + z;
b[y_, z_] = 2 - 3*y - 4*z;

(* So now we have two explicit functions for x as a function of y and z. Now repeat steps from Homework 1 *)
SurfacesPlot = ContourPlot3D[{x + 3*y + 4*z - 2 == 0,  
    (x - 1) + (2*y + 2) - (z - 2) == 0, {x, -10, 10}, {y, -10, 10}, {z, -10, 10},  
    ContourStyle -> {Directive[Red], Directive[Blue, Opacity[0.8]]},  
    AxesLabel -> {x, y, z}};

(* Solve for y as a function of z *)
Solve[a[y, z] - b[y, z] == 0, y]

(* Define a curve with parameter z and show its plot along with the two surfaces *)
Solution[z_] = {-3 - 2*(5 - 5*z) + z, 5 - 5*z, z}
SolutionPlot = ParametricPlot3D[Solution[z], {z, 0, 2},  
    PlotStyle -> Directive[Green, Thickness[0.015]], AxesLabel -> {x, y, z}];

foo3 = GraphicsGrid[
    {Show[SolutionPlot, SurfacesPlot ]}, ImageSize -> {500, 500}]
3 + x + 2*y - z == 0
{{x -> -3 - 2*y + z}}
{{x -> 2 - 3*y - 4*z}}
{{y -> -5 (-1 + z)}}
{3 - 2 (5 - 5*z) + z, 5 - 5*z, z}
3 + x + 2*y - z == 0

{{x -> -3 - 2*y + z}}

{{x -> 2 - 3*y - 4*z}}

{{y -> -5 (-1 + z)}}

{3 - 2 (5 - 5*z) + z, 5 - 5*z, z}
\[3 + x + 2y - z = 0\]

\[
\{(x \rightarrow -3 - 2y + z)\}
\]

\[
\{(x \rightarrow 2 - 3y - 4z)\}
\]

\[
\{(y \rightarrow -5 (-1 + z))\}
\]

\[
\{-3 - 2 (5 - 5z) + z, 5 - 5z, z\}
\]
(HomeWork 2) Find the curve of intersection of the two surfaces \( f(x,y) = -x + 2y \) and 
\( x + (y+2) + 2(z+3) = 0 \).

(Hint: Do the algebra work by hand)
(Extra: Come up with plots to check your answer)

(An Answer):

Am I trying to trick you by giving you one implicit and one explicit function? No. Just follow
the same steps as in previous examples.

```math
SurfacesPlot = Plot3D[
   {-x + 2*y, -1/2*(x + y) - 4}, {x, -5, 5}, {y, -5, 5},
   PlotStyle -> {Directive[Red],
              Directive[Blue, Opacity[0.8]]}];

CurvePlot = ParametricPlot3D[
   {(5*y + 8), y, -(5*y + 8) + 2*y}, {y, -5, 5},
   PlotStyle -> Directive[Green, Thickness[0.015]]];

Show[SurfacesPlot, CurvePlot]
```
(HomeWork, A Little More Challenging) Find the curve of intersection of the two surfaces \( x^2y - z = 5 \) and \( xy - z = 4 \)

(Hint: Do the algebra work by hand)
(Extra: Come up with plots to check your answer)

(An Answer)

```math
SurfacesPlot = ContourPlot3D[
  \{x^2*y - z == 5, x*y - z == 4\}, \{x, -5, 5\}, \{y, -5, 5\}, \{z, -5, 5\},
  ContourStyle -> {Directive[Red],
  Directive[Blue, Opacity[0.8]]}];

\[a[x_] = \{x, 1/(x^2 - x), 1/(x - 1) - 4\}\]

(* Notice from the formula for \(a(x)\) the problem values \(x=0\) and \(x=1\) so take note when plotting *)

CurvePlot1 = ParametricPlot3D[\[a[x]\], \{x, -5, -0.1\},
  PlotStyle -> Directive[Green, Thickness[0.015]]];

CurvePlot2 = ParametricPlot3D[\[a[x]\], \{x, 0.1, 0.9\},
  PlotStyle -> Directive[Green, Thickness[0.015]]];

CurvePlot3 = ParametricPlot3D[\[a[x]\], \{x, 1.01, 5\},
  PlotStyle -> Directive[Green, Thickness[0.015]]];

Show[SurfacesPlot, CurvePlot1, CurvePlot2, CurvePlot3]
```

\(\{x, \frac{1}{-x + x^2}, -4 + \frac{1}{-1 + x}\}\)