REFERENCES

[BK] R. Bryant and L. Kovacs, *Lie representations and groups of prime power order*, Jour. London Math. Soc. **17** (1978), 415–421.

[Ca] A. Caranti, *Automorphism groups of p-groups of class 2 and exponent p^2: a classification on 4 generators*, Annali di Mathematica **134** (1983), 93–146.

[C] Charles W. Curtis, *Modular representations of finite groups with split (B,N)-pairs*, Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin, 1970, pp. 57–95.

[CR] Charles W. Curtis and Irving Reiner, *Methods of Representation Theory, Vol. II*, John Wiley & Sons, New York, 1987.

[D] Jill Dietz, *Stable splittings of classifying spaces of metacyclic p-groups*, Ph.D. dissertation Northwestern University, 1991.

[HL] H. Heineken and H. Liebeck, *The occurence of finite groups in the automorphism group of nilpotent groups of class 2*, Arch. Math. **25** (1974), 8–16.

[M] John Martino, *Stable splittings and the Sylow 2-subgroups of $SL_3(\mathbf{F}_q)$, q odd*, Ph.D. dissertation Northwestern University, 1988.

[MP1] John Martino and Stewart Priddy, *The complete stable splitting for the classifying space of a finite group*, Topology **31** (1992), 143–156.

[MP2] ———, *Stable homotopy classification of $BG_p^\wedge$*, to appear.

[MiP] Stephen Mitchell and Stewart Priddy, *Stable splittings derived from the Steinberg module*, Topology **22** (1983), 285–298.

[N] Goro Nishida, *Stable homotopy type of classifying spaces of finite groups*, Algebraic and topological theories (Kinosaki, 1984), Kinokuniya, Tokro, 1986, pp. 391–404.

[P] Stewart Priddy, *On characterizing summands in the classifying space of a group, II*, Homotopy theory and related topics (Kinosaki, 1988), Springer, Berlin, 1990, pp. 175–183.

[R] Forrest Richen, *Modular representations of split BN pairs*, Trans. AMS **140** (1969), 435–460.

[S] Robert Steinberg, *Prime power representations of finite linear groups II*, Can. J. Math. **9** (1957), 347–351.

[W] Ursula Webb, *The occurence of groups as automorphisms of nilpotent p-groups*, Arch. Math. **37** (1981), 481–498.

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Equivariant vector bundles on the Lubin-Tate moduli space

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Introduction

Lubin and Tate showed that the functor of deformations of a formal group F of dimension 1 and height n over $\mathbb{Z}/p\mathbb{Z}$ is representable by the formal scheme $X = \operatorname{Spf} \mathbb{Z}_p[\![u_1, \cdots, u_{n-1}]\!]$ over $\mathbb{Z}_p$. They also described an action of the étale group scheme $G = \operatorname{Aut}(F)$ over $\mathbb{Z}_p$ on the moduli space X. Here we will study this action by a consideration of certain G-equivariant vector bundles on X. In particular, we show that there is an étale, surjective map:

$$X \otimes \mathbb{Q}_p \longrightarrow \mathbb{P}^{n-1} \otimes \mathbb{Q}_p$$

of rigid analytic spaces over $\mathbb{Q}_p$ which converts the action of G on $X \otimes \mathbb{Q}_p$ into a linear action on projective space.

Following Drinfeld, we will work in the more general setting of formal A-modules, where A is a complete discrete valuation ring with finite residue field. (The case considered by Lubin and Tate is when $A = \mathbb{Z}_p$.) Part I is a summary of the basic results of this theory, which are due to Lazard, Honda, Lubin, Cartier, Drinfeld, Hazewinkel and many others. In Part II we consider extensions and deformations of formal A-modules. The main results here are due to Lubin-Tate and Drinfeld; we have expressed them in the language of rigidified extensions and the universal additive extension, following Grothendieck and Mazur-Messing.

In Part III we introduce the moduli space $X = \operatorname{Spf} A[\![u_1, \cdots, u_{n-1}]\!]$ of deformations of a formal A-module F of dimension 1 and height n. This is a formal scheme over A, with an action of the étale group scheme $G = \operatorname{Aut}(F)$. We use the theory of the universal additive extension E of F to construct some natural

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G-equivariant vector bundles on X, and from these bundles construct the tangent bundle of X, the bundle of exterior i-forms, and the canonical line bundle of X.

In Part IV, we restrict equivariant bundles to the general fibre $X \otimes K$, which is a rigid analytic space over the quotient field K of A, isomorphic to the open unit polydisc of dimension $n-1$. We show that the G-bundle $\mathcal{L}ie(E)$ of rank n is flat over $X \otimes K$. Taking the image of its horizontal sections in the quotient line bundle $\mathcal{L}ie(F)$, we construct an étale, surjective map from $X \otimes K$ to $\mathbb{P}^{n-1} \otimes K$. As a corollary, we obtain the differentiability of the action of G on X.

After this paper was written, it was pointed out to the authors that in 1978 Lafaille [**G.L79**] proved, what in this paper are Corollary 23.21 and Corollary 23.26. Our argument differs very little from his. We have nevertheless decided to include the proof in order to keep this paper as complete as possible.

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Part I. Formal A-modules

1. Homomorphisms

Let A be a complete, discrete valuation ring with uniformizing parameter π and finite residue field $k = A/\pi A$. Let q be the cardinality of k and K the quotient field of A.

Let R be a commutative A-algebra, with structure map $i : A \to R$. By definition, a formal A-module F of dimension n over R is a commutative formal group of dimension n over R, together with a ring homomorphism $\theta : A \to \mathrm{End}_R(F)$ such that the endomorphism $\theta(a) = a_F$ of F acts by the scalar $i(a)$ on the tangent space $\mathrm{Lie}(F)$, for all a in A.

The formal group structure on F is given by a formal comultiplication on the algebra $R[\![X_1, X_2, \dots, X_n]\!] = R[\![X]\!]$. Once parameters have been chosen, the comultiplication is described by n power series in $2n$ variables:

(1.1)
$$F(X,Y) = (F_1(X_1, \cdots, X_n, Y_1, \cdots, Y_n), \cdots, F_n(X_1, \cdots, X_n, Y_1, \cdots, Y_n))$$

which satisfy the usual associative, commutative, and identity laws:

$$(1.2)\qquad \begin{cases} F(X,0) = X, F(0,Y) = Y \\ F(X, F(Y,Z)) = F(F(X,Y),Z) \\ F(X,Y) = F(Y,X). \end{cases}$$

The endomorphism a_F is given by n series in n variables

$$(1.3)\qquad a_F(X) = (a_1(X_1, \cdots, X_n), \cdots, a_n(X_1, \cdots, X_n)),$$

which satisfy

$$(1.4)\qquad F(a_F X, a_F Y) = a_F(F(X,Y)).$$

The condition on the action of a_F on $\mathrm{Lie}(F)$ is simply that

$$(1.5)\qquad a_i(X) \equiv i(a) X_i \quad (\bmod \deg 2)$$

for $i = 1, 2, \cdots, n$.

We will usually not distinguish between the formal A-module F over R and the collection of series $\{F(X,Y), a_F(X)\}$, which depend on the choice of parameters X of the algebra $R[\![X]\!]$. The latter are often referred to as "formal group laws" or "formal A-module laws" in the literature. A different choice of parameters X' for this algebra would result in a law $\{F'(X',Y'), a_{F'}(X')\}$isomorphic to the original one over R. We write $X +_F Y$ for $F(X,Y)$.

If F is a formal A-module over R and $\alpha : R \to S$ is a homomorphism of A-algebras, then the series $\{\alpha F(X,Y), \alpha a_F(X)\}$ define a formal A-module over S. We will write this module simply as αF.

A homomorphism $f : F \to F'$ of formal A-modules over R is a homomorphism of formal groups (given by n' power series $X' = f(X)$ in n variables) which satisfies

$$(1.6)\qquad f \circ a_F = a_{F'} \circ f \quad \text{in} \quad R[\![X']\!].$$

The set $\mathrm{Hom}_R(F, F')$ of all homomorphisms over R is an A-module, with addition and A-multiplication defined using the operations of F'. We will write this A-module simply as $\mathrm{Hom}(F, F')$ if the base ring R is fixed. If $F = F'$ we write $\mathrm{End}(F)$ for $\mathrm{Hom}(F, F)$; this is an A-algebra under composition.

An example of a formal A-module, which is central to the theory, is the module $\mathbb{G}_a$ of dimension one. This is defined by the series

$$(1.7)\qquad \begin{cases} \mathbb{G}_a(X,Y) = X + Y \\ a_{\mathbb{G}_a}(X) = i(a) \cdot X \end{cases}$$

We have an injection of A-algebras:

$$(1.8)\qquad \begin{aligned} R &\hookrightarrow \mathrm{End}_R(\mathbb{G}_a) \\ \alpha &\longmapsto f_\alpha(X) = \alpha X. \end{aligned}$$

2. Invariant differentials

Let F be a formal A-module over R. The R-module $\omega(F)$ of "invariant differentials" on F is defined to be the submodule of the differentials $\omega(X) =$

$f(X)dX = \sum_{i=1}^{n} f_i(X)dX_i$ of $R[\![X]\!]$ over R which satisfy:

$$\begin{cases} \omega(F(X,Y)) = \omega(X) + \omega(Y) \\ \omega(a_F(X)) = i(a)\omega(X) \qquad a \in A. \end{cases} \tag{2.1}$$

PROPOSITION 2.2. *The R-module $\omega(R)$ is free of rank $n = \dim(F)$, with a basis $\omega_i(X)$ of differentials which satisfy*

$$\omega_i(X) \equiv dX_i \qquad \text{mod deg } 2.$$

Every invariant differential on F is closed.

PROOF. The $n \times n$ matrix of power series over R:

$$\left(\frac{\partial}{\partial X_j} F_i(0, Y)\right) = (B_{ij}(Y))$$

is congruent, mod degree 1, to the identity. Hence it is invertible, with inverse

$$(A_{ij}(Y)) \equiv I \qquad (\text{mod deg } 1).$$

It is proved in [**Hon70**, Prop. 1.1] that the differentials

$$\omega_i(X) = \sum_{j=1}^{n} A_{ij}(X)dX_j \equiv dX_i \qquad (\text{mod deg } 2)$$

from an R-basis of the space of differentials on X which satisfy $\omega(F(X,Y)) = \omega(X) + \omega(Y)$. In [**Hon70**, Prop. 1.3] it is shown that each ω_i is closed.

To complete the proof, we must show that every element in this free R-module satisfies $\omega(a_F(X)) = i(a)\omega(X)$. If $\omega = \sum_i f_i(X)dX_i$ is translation invariant, then by the above $\omega = \sum_i f_i(0)\omega_i$. The differential

$$\omega(a_F(X)) = \sum_i f_i(a_F(X))da_i(X)$$

is also translation invariant [**Hon70**, Prop. 1.2], and has linear term

$$\omega(a_F(X)) \equiv i(a) \sum f_i(0)dX_i \qquad (\text{mod deg } 2)$$

by our hypothesis on the series $a_F(X)$. Hence it is equal to $i(a) \cdot \omega(X)$. □

If $f : F \longrightarrow F'$ is a homomorphism of formal A-modules over R, there is an induced map of free R-modules

$$\begin{aligned} f^* &: \omega(F') \longrightarrow \omega(F) \\ \sum g_i(X')dX_i' &\longmapsto \sum g_i(f(X))df_i(X). \end{aligned} \tag{2.3}$$

The functor dual to $\omega(F)$ is the R-module $\mathrm{Lie}(F)$ of invariant derivations

$$D(X) = \sum_{i=1}^{n} f_i(X) \frac{\partial}{\partial X_i}$$

of $R[\![X]\!]$ over R. This R-module is also free of rank n, with dual basis over R

$$D_j(X) = \sum_i B_{ij}(X) \frac{\partial}{\partial X_i},$$

where $B_{ij}(Y) = \frac{\partial}{\partial X_j} F_i(0, Y) \equiv \delta_{ij}$ (mod deg 1). The formula

$$\langle \omega = \sum g_i dX_i, \quad D = \sum h_i \frac{\partial}{\partial X_i} \rangle = \sum_i g_i(0) \cdot h_i(0)$$

defines a non-degenerate pairing of free R-modules:

$$(\ ,\ ) : \omega(F) \times \mathrm{Lie}(F) \longrightarrow R. \tag{2.4}$$

On the formal A-module $\mathbb{G}_a$ we have $\omega(F) = RdX$ and $\mathrm{Lie}(F) = R\frac{\partial}{\partial X}$. The pairing $\langle adX, b\partial/\partial X \rangle$ is equal to ab.

If $f : F \to F'$ is a homomorphism of formal A-modules over R, there is an induced map of R-modules

$$f_* : \mathrm{Lie}(F) \longrightarrow \mathrm{Lie}(F'). \tag{2.5}$$

This can be defined as the adjoint of f^* using the pairing (2.4):

$$\langle f^*\omega', D \rangle_F = \langle \omega', f_* D \rangle_{F'}.$$

In particular, the endomorphism a_F acts on the R-modules $\omega(F)$ and $\mathrm{Lie}(F)$ by multiplication by $i(a)$.

3. Logarithms

Let F be a formal A-module over R and $f : F \longrightarrow \mathbb{G}_a$ a homomorphism of formal A-modules over R. Then

$$\omega = f^*(dX') = df(X) = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(X) \cdot dX_i$$

is an invariant differential on F. This gives a homomorphism of R-modules

$$\begin{aligned} d : \mathrm{Hom}(F, \mathbb{G}_a) &\longrightarrow \omega(F) \\ f &\longmapsto df(X). \end{aligned} \tag{3.1}$$

PROPOSITION 3.2. *1) If R is a flat (= torsion-free) A-algebra, then the map of* (1.8) *induces an isomorphism $R \xrightarrow{\sim} \mathrm{End}(\mathbb{G}_a)$ and the map d of* (3.1) *is an injection.*

2) If R is a K-algebra, then the map d of (3.1) *is an isomorphism, so $\mathrm{Hom}(F, \mathbb{G}_a)$ is a free R-module of rank $n = \dim(F)$.*

PROOF. 1) Since A is a discrete valuation ring, R is flat if and only if $i(a)$ is not a zero divisor in R for all $a \neq 0$ in A. Let $f(X) = \sum_{k\geq 1} a_k X^k$ be an endomorphism of $G = \mathbb{G}_a$, so $f \circ \pi_G = \pi_G \circ f$. But $\pi_G(X) = i(\pi) \cdot X$, so

$$a_k(i(\pi))^k = i(\pi) \cdot a_k \qquad \text{for all } k \geq 1.$$

Hence $a_k \cdot i(\pi^k - \pi) = 0$. For $k \geq 2$ the element $\pi^k - \pi$ is a uniformizer in A, so by our hypothesis of flatness, $i(\pi^k - \pi)$ is not a zero divisor in R. Thus $a_k = 0$ for $k \geq 2$ and $f(X) = a_1 X$. This shows the map of (1.8) is an isomorphism.

To show that $d : \operatorname{Hom}(F, \mathbb{G}_a) \longrightarrow \omega(F)$ is an injection, we observe that $df(X) = 0$ implies that $f \equiv 0 \pmod{\deg 2}$. The identity $f \circ \pi_F = \pi_{\mathbb{G}_a} \circ f = i(\pi) \cdot f$ then shows that $f = 0$. Indeed, if $a_k X^k$ is the leading term of $f(X)$ we find that $a_k \cdot i(\pi^k - \pi) = 0$.

For 2), assume R is a K-algebra. In particular, R is a flat A-algebra so the map d of (3.1) is an injection. To show that d is surjective, we break into two cases. When $\operatorname{char}(K) = 0$, the invariant differentials ω on F are all formally exact: $\omega = df$. This follows from the fact that they are closed; we choose the primitive uniquely by insisting that $f(0) = 0$. Then $f : F \longrightarrow \mathbb{G}_a$ is a homomorphism of formal A-modules by (2.1).

If $\operatorname{char}(K) = p$ then $p_F(X) = 0$ and F is isomorphic, as a formal group, to n copies of $\mathbb{G}_a$. We must show there is a unique homomorphism of formal A-modules $f : F \to \mathbb{G}_a$ with $df(X) = \beta_1 dX_1 + \cdots + \beta_r dX_r$ in $\omega(F)$. The homomorphisms from $\mathbb{G}_a^n$ to $\mathbb{G}_a$ all have the form

$$f(X) = \sum_{i=1}^{n} f_i(X_i), \qquad \text{with} \qquad f_i(X_i) = \sum_{k\geq 0} a_i(k) X_i^{p^k}$$

and we claim there is a unique series of this form which satisfies

$$(3.3) \qquad \begin{cases} f \circ \pi_F = i(\pi) \cdot f \\ a_i(0) = \beta_i, \qquad i = 1, 2, \cdots, n. \end{cases}$$

This follows from the fact that when $k \geq 1$, $i(\pi^{p^k} - \pi)$ is a unit in R, so we may solve for the coefficients of f successively. To show f is a homomorphism of formal A-modules, we must check that $f \circ \zeta_F = i(\zeta) \cdot f$ for all $\zeta \in k$ (as $A = k[\![\pi]\!]$). But $i(\zeta)^{-1} \cdot f \circ \zeta_F = g$ is another series which satisfies (3.3), as $\zeta_F \circ \pi_F = \pi_F \circ \zeta_F$. Hence $g = f$ as claimed. □

From Proposition 3.2 we may conclude the following. Let F be a formal A-module of dimension 1 over R and let ω be a basis of $\omega(F)$ over R. If R is A-flat, so injects into the K-algebra $R \otimes K$, there is a unique isomorphism

$$(3.4) \qquad f : F \xrightarrow{\sim} \mathbb{G}_a \qquad \text{over } R \otimes K$$

with $df = \omega$. We call f a logarithm for F, and its inverse $e = f^{-1} : \mathbb{G}_a \xrightarrow{\sim} F$ an exponential. We then have

$$(3.5) \qquad \begin{cases} F(X,Y) &= e(f(X) + f(Y)) \\ a_F(X) &= e(af(X)), \qquad a \in A. \end{cases}$$

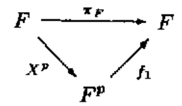
4. The height

In this section, F is a formal A-module of dimension 1 over R. If R is a field, then either $i(\pi) = 0$ in R or $i(\pi)$ is a unit in R. The latter case, when R is a K-algebra, was considered in the previous section. We now consider the case when $i(\pi) = 0$.

LEMMA 4.1. *Assume that R is a field and that $i(\pi) = 0$ in R. Then either $\pi_F = 0$ in $End(F)$ or there is an integer $n \geq 1$ such that*

$$\pi_F(X) = f(X^{q^n}) \qquad \textit{with} \qquad f'(0) \neq 0.$$

PROOF. Let ω be a basis for $\omega(F)$ over R. Since $\pi_F^*(\omega) = i(\pi) \cdot \omega = 0$, we have $\pi_F(X) = f_1(X^p)$. Let F^p be the conjugate formal A-module over the field R, which has characteristic p. Then we have a commutative diagram of morphisms



If $f_1'(0) \neq 0$, the same argument using differentials shows that $f_1 = f_2(X^p)$. If $\pi_F \neq 0$, we eventually have

$$\pi_F(X) = f(X^{p^h}) \qquad \text{for some } h \geq 1, \text{with } f'(0) \neq 0.$$

Write $q = p^f$. We must show that $h \equiv 0(\mod f)$, so that $\pi_F(X) = f(X^{q^n})$. Since $\pi_F \circ \zeta_F = \zeta_F \circ \pi_F$ for all $(q-1)^{\text{st}}$ roots of unity ζ in A^*, we have $i(\zeta) = i(\zeta^{p^h})$. Hence $(q-1)$ divides $(p^h - 1)$, which implies that f divides h. □

If the second case occurs in Lemma 4.1, we say the formal A-module F has *height* n over R. More generally, if R is a complete, local, Noetherian ring with maximal ideal P containing $i(\pi)$, we say F has height n over R if its reduction has height n over the field R/P. We call such rings "local A-algebras" for short.

PROPOSITION 4.2. *Assume R is a local A-algebra, and F is a formal A-module of dimension 1 and height n over R. If G is a formal A-module of dimension 1 over R, then reduction of homomorphisms induces an injection of A-modules*

$$\operatorname{Hom}_R(F, G) \hookrightarrow \operatorname{Hom}_{R/P}(F, G).$$

PROOF. Assume $f : F \longrightarrow G$ satisfies $f(X) \equiv 0 \mod P^k)$. We will show that $f(X) \equiv 0 \mod P^{k+1}$. Since $R = \varprojlim R/P^k$, this will establish the claim

Since f is a homomorphism of A-modules, $f \circ \pi_F = \pi_G \circ f$. Since $\pi_G(X) = i(\pi) \cdot X + \cdots$ and $i(\pi)$ is in P, we have $\pi_G \circ f(X) \equiv 0 \mod P^{k+1}$. Hence $f \circ \pi_F(X) = 0 \mod P^{k+1}$. But $f \circ \pi_F(X) \equiv f(\alpha X^{q^n} + \cdots) \mod P^{k+1}$, where the substitution is made in the R/P-module P^k/P^{k+1} and $\alpha \neq 0$. Hence $f \equiv 0 \mod P^{k+1}$. □

COROLLARY 4.3. *If R is a local A-algebra and F has height n over R, then* $\mathrm{Hom}(F, \mathbb{G}_a) = 0$.

PROOF. By the above proposition, it is enough to prove that

$$\mathrm{Hom}_{R/P}(F, \mathbb{G}_a) = 0.$$

Since any homomorphism $f : F \longrightarrow \mathbb{G}_a$ over R/P must satisfy

$$f \circ \pi_F(X) = f(\alpha X^{q^n} + \cdots) = i(\pi) f(X) = 0$$

we have $f = 0$. □

5. A-typical modules

When $A = k[[\pi]]$ has characteristic p, we may construct formal A-modules of dimension 1 over the ring R as follows. Let

$$(5.1)\qquad \begin{cases} F(X,Y) &= X + Y \\ \zeta_F(X) &= i(\zeta) \cdot X, \qquad \zeta \in k \\ \pi_F(X) &= i(\pi) \cdot X + \sum_{k \geq 1} a_k X^{q^k}, \qquad a_k \in R. \end{cases}$$

These series uniquely determine a formal A-module F over R, and one can show that any formal A-module of dimension one over R is isomorphic to one of this form. If R is a flat A-algebra, the logarithm $f : F \to \mathbb{G}_a$ over $R \otimes K$ with $df = dX$ has the form

$$(5.2)\qquad f(X) = X + \sum_{k \geq 1} b_k X^{q^k}, \qquad b_k \in R \otimes K.$$

The coefficients b_k of this series are completely determined by the identity $f \circ \pi_F = i(\pi) \cdot f$. If R is a local A-algebra, then F has height n over R provided $a_1, a_2, \ldots, a_{n-1}$ lie in P, the maximal ideal of R, but $a_n \not\equiv 0(\mod P)$.

When A has characteristic zero, it is more difficult to write down the series $F(X,Y)$ and $a_F(X)$ defining the general formal A-module of dimension 1 over R. However, it is possible to normalize the choice of coordinate on the algebra $R[[X]]$ so that the logarithm (when R is A-flat) for the co-multiplication has the form (5.2). These formal A-modules are called "A-typical", and are convenient for many computations. A detailed description of their properties is given in Hazewinkel's book **[Haz78]**; we review some of the theory in this section.

Let $A[\underline{v}]$ be the flat A-algebra $A[v_1, v_2, v_3, \ldots]$ of polynomials in an infinite number of variables v_i. Let $f(X) = f[\underline{v}](X)$ be the unique series with coefficients $A[\underline{v}] \otimes K = K[v_1, v_2, v_3, \ldots]$ which satisfies

$$(5.3)\qquad f(X) = X + \sum_{i \geq 1} \frac{v_i}{\pi} f^{q^i}(X^{q^i})$$

where $f^{q^i}(X)$ is the series obtained from $f(X)$ by replacing each variable v_j by v_j^q. This is Hazewinkel's "functional equation" **[Haz78**, 21.5]. The expansion of $f(X)$ has the form

$$(5.4)\qquad f(X) = \sum_{k \geq 0} b_k X^{q^k} = X + \frac{v_1}{\pi} X^q + \left(\frac{v_2}{\pi} + \frac{v_1^{q+1}}{\pi^2} \right) X^{q^2} + \cdots.$$

The coefficients b_k in the expansion of $f(X)$ may be calculated using the following recursion, which is a consequence of the definition of f:

$$(5.5)\qquad \begin{cases} b_0 = 1 \\ \pi b_k = b_0 v_k + b_1 v_{k-1}^q + b_2 v_{k-2}^{q^2} + \cdots + b_{k-1} v_1^{q^{k-1}}. \end{cases}$$

From this it follows, by induction on k, that

$$(5.6)\qquad \pi^k \cdot b_k \in A[\underline{v}].$$

Moreover, we have the congruence

$$f(X) \equiv X + \frac{v_k}{\pi} X^{q^k} \qquad \mod (v_1, \cdots, v_{k-1}), \quad \deg q^k + 1$$

PROPOSITION 5.7 (**[Haz78**, 21.5]). *The series*

$$F(X,Y) = f^{-1}(f(X) + f(Y))$$
$$a_F(X) = f^{-1}(af(X)) \qquad a \in A$$

have coefficients in $A[\underline{v}]$, and define a formal A-module $F[\underline{v}]$ of dimension 1 over $A[\underline{v}]$ with logarithm $f[\underline{v}]$. We have

$$\pi_{F[\underline{v}]}(X) \equiv v_k X^{q^k} \qquad \mod (\pi, v_1, \cdots, v_{k-1}), \quad \deg q^k + 1.$$

If F is any formal A-module of dimension one over R, we say F is "A-typical" if it is the specialization of $F[\underline{v}]$ with respect to a homomorphism of A-algebras $A[\underline{v}] \longrightarrow R$. Such homomorphisms are given simply by specifying the images of each variable v_i in R. Hazewinkel shows **[Haz78**, 21.5.6] that any formal A-module of dimension 1 over R is isomorphic to an A-typical one, so working with A-typical formal A-modules entails no loss of generality. We will often do so.

The modules F described by (5.1) are all A-typical, as the series defining the universal A-typical module $F[\underline{v}]$ (when A has characteristic p) have this form.

If R is A-flat, the module F is A-typical if and only if its logarithm over $R \otimes K$ has the form of (5.2). Finally, A-typical modules have:

$$\zeta_F(X) = i(\zeta) \cdot X \tag{5.8}$$

for all $(q-1)^{\text{st}}$ roots of unity ζ in A^*.

Part II. Extensions and deformations

6. Symmetric 2-cohomology and extensions

Let R be an A-algebra, and let F and F' be formal A-modules of dimensions n and n' over R.

A 1-cochain on F with values in F' is a set of n' series

$$f(X) = (f_1(X), \dots, f_{n'}(X))$$

in n variables $X = (X_1, \cdots, X_n)$ with no constant terms. The set of all 1-cochains forms an A-module, with addition $(f+g)(X) = f(X) +_{F'} g(X)$ and A-multiplication $af(X) = a_{F'} f(X)$ coming from F'. The coboundary $\delta f = \{\, \Delta f(X,Y), \delta_a f(X) a \in A \,\}$ is defined by

$$\begin{cases} \Delta f(X,Y) = f(Y) -_{F'} f(X +_F Y) +_{F'} f(X) \\ \delta_a f(X) = a_{F'} f(X) -_{F'} f(a_F X) \end{cases} \tag{6.1}$$

The kernel of δ is a sub A-module of the 1-cochains, which is the 1-cohomology $H^1(F,F')$. Since $\Delta f(X,Y) = 0$ if and only if $f(X +_F Y) = f(X) +_{F'} f(Y)$, and $\delta_a f(X) = 0$ if and only if $f(a_F X) = a_{F'} f(X)$, we have an identification

$$H^1(F,F') = \operatorname{Hom}(F,F'). \tag{6.2}$$

A 2-cochain on F with values in F' is a set $\{\, \Delta(X,Y), \delta_a(X) a \in A \,\}$ where $\Delta(X,Y)$ is a set of n' series in $2n$ variables and, for $a \in A$, $\delta_a(X)$ is a set of n' series in n variables. These series have no constant terms, and form an A-module via the operations on F'. We say the 2-cochain $\{\, \Delta, \delta_a \,\}$ is a symmetric 2-cocycle if the following identities hold [**Dri74**, §4]:

(6.3)

$$\begin{cases} \Delta(X,Y) = \Delta(Y,X) \\ \Delta(Y,Z) +_{F'} \Delta(X, Y +_F Z) = \Delta(X +_F Y, Z) +_{F'} \Delta(X,Y) \\ \delta_a(X) +_{F'} \delta_a(Y) +_{F'} \Delta(a_F X, a_F Y) = a_{F'} \Delta(X,Y) +_{F'} \delta_a(X +_F Y) \\ \delta_a(X) +_{F'} \delta_b(X) +_{F'} \Delta(a_F X, b_F X) = \delta_{a+b}(X) \\ a_{F'} \delta_b(X) +_{F'} \delta_a(b_F X) = \delta_{ab}(X) \end{cases}$$

The symmetric 2-cocycles form a sub A-module of the 2-cochains, which contains the A-module of coboundaries – those cocycles of the form

$$\delta f = \{\, \Delta f(X,Y), \delta_a f(X) \,\},$$

where f is a 1-cochain (6.1). The quotient A-module is the symmetric 2-cohomology $H^2(F,F')_s$ [LT2, §2; D1, §4].

We now show how classes in $H^2(F,F')_s$ correspond to extensions of F by F' in the category of formal A-modules over R, up to the usual equivalence relation. If E is a formal A-module over R, we say the sequence of homomorphisms

$$0 \longrightarrow F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \longrightarrow 0 \tag{6.4}$$

is exact if the associated sequence of free R-modules

$$0 \to \operatorname{Lie}(F') \xrightarrow{\alpha_*} \operatorname{Lie}(E) \xrightarrow{\beta_*} \operatorname{Lie}(F) \to 0$$

is exact. An extension of F by F' is, by definition, an exact sequence of formal A-modules as in (6.4); we say two extensions are equivalent if there is an isomorphism $i : E \longrightarrow E'$ of formal A-modules over R which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{F'}} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1_F} & & \\ 0 & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

commute. Let $\operatorname{Ext}(F,F')$ denote the set of equivalence classes.

Proposition 6.5. *Let $\{\, \Delta, \delta_a \,\}$ be a symmetric 2-cocycle on F with values in F'. Then the formal A-module E with coordinate ring $R[\![X',X]\!]$ and operations*

$$E((X',X),(Y',Y)) = (F'(X',Y') +_{F'} \Delta(X,Y), F(X,Y))$$
$$a_E(X',X) = (a_{F'} X' + \delta_a(X), a_F X)$$

is an extension of F by F'. The homomorphism α is defined by $\alpha(X') = (X',0)$ and the homomorphism β is defined by $\beta(X',X) = X$.

The equivalence class of E in $\operatorname{Ext}(F,F')$ depends only on the cohomology class of $\{\, \Delta, \delta_a \,\}$ in $H^2(F,F')_s$.

Proof. The identities (6.3) satisfied by $\{\, \Delta, \delta_a \,\}$ show that the multiplication on E is commutative and associative, that a_E is an endomorphism of the formal group underlying E, and finally that the map taking a to a_E is a ring homomorphism from A to the endomorphism ring of the formal group. Thus E is a formal A-module, which is easily seen to be an extension of F by F'.

If $\{\, \Delta, \delta_a \,\}' = \{\, \Delta, \delta_a \,\} + \delta f$ is cohomologous to $\{\, \Delta, \delta_a \,\}$, the map $i(X',X) = (X' +_{F'} f(X), X)$ gives an isomorphism from E to E' which renders the extensions equivalent.

Conversely, every extension of F by F' can be put in the form of Proposition 6.5. For the formal implicit function theorem shows that we may choose a section $s : F \to E$, consisting of $n + n'$ series in n variables X with no constant term, such that $\beta \circ s(X) = X$. This gives coordinates (X', X) on E for which $\alpha(X') = (X', 0), \beta(X', X) = X$, and $s(X) = (0, X)$. The series

$$\begin{aligned} s(X +_F Y) -_E s(X) -_E s(Y) &= (\Delta(X, Y), 0) \\ s(a_F X) -_E a_E s(X) &= (\delta_a(X), 0) \end{aligned} \tag{6.6}$$

then define a 2-cocycle $\{\Delta, \delta_a\}$ on F with values in F'. The class of this cocycle in $H^2(F, F')_s$ is independent of the choice of section s, and the extension E is equivalent to the one defined in Proposition 6.5. Thus we have established a bijection of sets

$$H^2(F, F')_s = \operatorname{Ext}(F, F'). \tag{6.7}$$

The isomorphism (6.7) gives $\operatorname{Ext}(F, F')$ the structure of an A-module; it is a bifunctor, like $\operatorname{Hom}(F, F')$, which is covariant in F' and contravariant in F. If E is an extension defined by the data $\{\Delta(X, Y), \delta_a(X)\}$ and

$$\begin{cases} g = g(U): & G \longrightarrow F \\ g' = g'(X'): & F' \longrightarrow G' \end{cases} \tag{6.8}$$

are homomorphisms of formal A-modules, then $g'Eg$ is, by definition, the extension of G by G' defined by the data $\{g'\Delta(gU, gV), g'\delta_a(gU)\}$. □

Remark. One can also define the A-module $\operatorname{Ext}^2(F, F') = H^3(F, F')_s$ using symmetric 3-cocycles on F with values in F' modulo coboundaries of symmetric 2-cochains. This A-module is always *trivial* (cf. [**Mac50**, Prop. 4], [**Hea59**], [**Laz55**, Prop. 1]), a fact which has important consequences for deformation theory.

7. First order deformations

We now specialize to the case when F is a formal A-module of dimension 1 and $F' = \mathbb{G}_a$. Then $\operatorname{Ext}(F, \mathbb{G}_a) = H^2(F, \mathbb{G}_a)_s$ is an R-module, as R is a subring of $\operatorname{End}(\mathbb{G}_a)$ by (1.8). Following Lubin-Tate and Drinfeld, we give an interpretation of the R-module

$$\operatorname{Ext}(F, \mathbb{G}_a) \otimes_R \operatorname{Lie}(F) = \operatorname{Hom}_R(\omega(F), \operatorname{Ext}(F, \mathbb{G}_a)) \tag{7.1}$$

using deformation theory.

A formal A-module G over the ring $R[\epsilon]/(\epsilon^2)$ is a deformation of F provided $G \equiv F$ and $a_G \equiv a_F$ (modulo ϵ). We say two deformations G and G' are $\star$-*isomorphic* if there is an isomorphism $\varphi : G \to G'$ over $R[\epsilon]/(\epsilon^2)$ such that $\varphi \equiv X$ mod ϵ. This gives an equivalence relation on the set of all deformations of F, and the equivalence classes form an R-module (using addition and R-multiplication of the linear term in the expansion $G = F + \epsilon B$, $a_G = a_F + \epsilon b$).

PROPOSITION 7.2. *The R-module of $\star$-isomorphism classes of deformations G of F to the ring $R[\epsilon]/(\epsilon^2)$ is isomorphic to the R-module $\operatorname{Ext}(F, \mathbb{G}_a) \otimes_R \operatorname{Lie}(F)$.*

If $\{\Delta, \delta_a\}$ is a 2-cocycle on F with values in $\mathbb{G}_a$ and $D = h(X)\partial/\partial X$ is an invariant derivation of F, then the series

$$\begin{cases} G(X, Y) &= F(X, Y) + \epsilon\Delta(X, Y)h(F(X, Y)) \\ a_G(X) &= a_F(X) + \epsilon\delta_a(X)h(X) \end{cases} \tag{7.3}$$

define a deformation of F over $R[\epsilon]/(\epsilon^2)$. The $\star$-isomorphism class of G depends only on the image of $\{\Delta, \delta_a\} \otimes D$ in $H^2(F, \mathbb{G}_a)_s \otimes_R \operatorname{Lie}(F) = \operatorname{Ext}(F, \mathbb{G}_a) \otimes_R \operatorname{Lie}(F)$, and every deformation G of F has the form (7.3).

PROOF. This may be checked directly, using the fact that $h(X)$ is a constant multiple of the series

$$\frac{\partial}{\partial X} F(X, Y)\bigg|_{(0,X)} = F_1(0, X).$$

□

We may restate Proposition 7.2 in a more invariant manner by introducing the additive formal A-module $\mathbb{G}_a \otimes M$ of dimension n over R, where M is a free R-module of rank n. This is defined to be the additive formal A-module with $\operatorname{Lie}(\mathbb{G}_a \otimes M) \simeq M$ (canonically). We then have canonical isomorphisms of R-modules

$$\begin{cases} \operatorname{Hom}(F, \mathbb{G}_a \otimes M) = \operatorname{Hom}(F, \mathbb{G}_a) \otimes_R M \\ \operatorname{Ext}(F, \mathbb{G}_a \otimes M) = \operatorname{Ext}(F, \mathbb{G}_a) \otimes_R M \end{cases} \tag{7.4}$$

COROLLARY 7.5. *Let F be a formal A-module of dimension 1 over R. There is a natural map of R-modules*

$$d : \operatorname{Hom}(F, \mathbb{G}_a \otimes \operatorname{Lie}(F)) \longrightarrow R,$$

which is injective when R is A-flat and an isomorphism when R is a K-algebra. Moreover, the R-module

$$\operatorname{Ext}(F, \mathbb{G}_a \otimes \operatorname{Lie}(F))$$

is isomorphic to the set of $\star$-isomorphism classes of deformations of F to the ring $R[\epsilon]/(\epsilon^2)$.

PROOF. The statements on Hom are simply restatements of Proposition 3.2, using the isomorphism $\langle\ ,\ \rangle : \operatorname{Lie}(F) \otimes \omega(F) \xrightarrow{\sim} R$. The statement on Ext is a restatement of Proposition 7.2. □

8. Rigidified extensions

We define a rigidified extension of F by $\mathbb{G}_a$ as an extension of formal A-modules over R:

$$0 \longrightarrow \mathbb{G}_a \xrightarrow{\alpha} E \xrightarrow{\beta} F \longrightarrow 0 \tag{8.1}$$

together with a splitting of the sequence of Lie algebras (cf. [MM, §2])

$$0 \to \mathrm{Lie}(\mathbb{G}_a) \xrightarrow{\alpha_*} \mathrm{Lie}(E) \underset{s}{\overset{\beta_*}{\rightleftarrows}} \mathrm{Lie}(F) \tag{8.2}$$

in the category of free R-modules. Equivalently, a rigidified extension is an extension together with an invariant differential ω_E on E such that

$$\alpha^*(\omega_E) = dX' \quad \text{in} \quad \omega(\mathbb{G}_a). \tag{8.3}$$

The R-module $\mathrm{RigExt}(F, \mathbb{G}_a)$ of rigidified extension classes fits into a 4 term exact sequence [**Kat79**, §5.2]:

$$\mathrm{Hom}(F, \mathbb{G}_a) \xrightarrow{d} \omega(F) \longrightarrow \mathrm{RigExt}(F, \mathbb{G}_a) \longrightarrow \mathrm{Ext}(F, \mathbb{G}_a) \longrightarrow 0. \tag{8.4}$$

Indeed, two rigidifications of a fixed extension differ by an element in

$$\mathrm{Hom}(\mathrm{Lie}(F), \mathrm{Lie}(\mathbb{G}_a)) = \omega(F),$$

and the splittings of a trivial extension form a principal homogeneous space for $\mathrm{Hom}(F, \mathbb{G}_a)$.

When R is A-flat, so injects into the K-algebra $R \otimes K$, we can give an explicit description of $\mathrm{RigExt}(F, \mathbb{G}_a)$ following Honda, Fontaine [**Fon77**], and Katz [**Kat79**, §5.1]. If $g(X)$ is a power series with no constant term and coefficients in $R \otimes K$, we say the 2-cocycle $\delta g = \{\Delta g(X,Y), \delta_a g(X)\}$ is integral if the series

$$\begin{aligned} \Delta g(X,Y) &= g(Y) - g(X +_F Y) + g(X) \\ \delta_a g(X) &= i(a) g(X) - g(a_F X) \qquad a \in A \end{aligned}$$

have coefficients in R.

PROPOSITION 8.5. *Assume R is A-flat. Then there is an isomorphism of R-modules*

$$\mathrm{RigExt}(F, \mathbb{G}_a) = \frac{\{\, g(X) \in R \otimes K[\![X]\!] \mid g(0) = 0, \delta g \text{ and } dg \text{ are integral} \,\}}{\{\, g(X) \in R[\![X]\!] : g(0) = 0 \,\}}.$$

PROOF. Let (E, ω_E) be a class in $\mathrm{RigExt}(F, \mathbb{G}_a)$. Write E in the form of Proposition 6.5; then

$$\omega_E = dX' + \nu(X). \tag{8.6}$$

By Proposition 3.2, there is a unique homomorphism

$$f_E : E \longrightarrow \mathbb{G}_a \tag{8.7}$$

over $R \otimes K$ such that $df_E = \omega_E$. The map f_E is given by a series

$$f_E(X', X) = X' + g(X) \tag{8.8}$$

where $g(X) \in R \otimes K[\![X]\!]$ satisfies $g(0) = 0$ and $dg(X) = \nu(X)$. Since f_E is a homomorphism, we have

$$\delta g = \{\, \Delta, \delta_a \,\} \tag{8.9}$$

where $\{\, \Delta, \delta_a \,\}$ is the 2-cocycle over R describing the extension E. Hence both dg and δg are integral.

The definition of $g(X)$ required a formal splitting of the extension E over R, to write it in the form of Proposition 6.5. If we change the formal section, and describe E by the cocycle $\{\, \Delta', \delta'_a \,\} = \{\, \Delta, \delta_a \,\} + \delta h$, we find that $g' = g + h$. Since the series h has coefficients in R, the series g is well-defined in the quotient by integral series.

Conversely given a series g with δg and dg integral, we define an extension E of F by $\mathbb{G}_a$ over R using the symmetric 2-cocycle $\delta g = \{\, \Delta, \delta_a \,\}$. We define a differential ω_E on E by the formula

$$\omega_E = dX' + dg(X). \tag{8.10}$$

The class of (E, ω_E) in $\mathrm{RigExt}(F, \mathbb{G}_a)$ depends only on g, up to the addition of an integral series. This gives the bijection of Proposition 8.5. □

We can give a deformation-theoretic interpretation of the R-module:

$$\mathrm{RigExt}(F, \mathbb{G}_a) \otimes_R \mathrm{Lie}(F) = \mathrm{RigExt}(F, \mathbb{G}_a \otimes \mathrm{Lie}(F))$$

similar to the second assertion of Corollary 7.5. Let ω_F be a basis for the free R-module $\omega(F)$ of invariant differentials on F. Then ω_F gives an isomorphism of R-modules:

$$\mathrm{RigExt}(F, \mathbb{G}_a) \otimes_R \mathrm{Lie}(F) \xrightarrow{\sim} \mathrm{RigExt}(F, \mathbb{G}_a).$$

We claim that the elements in this R-module are in one to one correspondence with $\star$-isomorphism classes of deformations (G, ω_G) of the pair (F, ω_F) to the ring $R[\epsilon]/(\epsilon^2)$.

Indeed, assume (E, ω_E) defines a rigidified extension of F by $\mathbb{G}_a$. Define the deformation G of F over $R[\epsilon]/(\epsilon^2)$, using the cocycle $\{\, \Delta, \delta_a \,\}$ arising from a formal splitting of E and the invariant derivation $D = h(X)\partial/\partial X$ dual to ω_F, as in Proposition 7.2. Then the differential

$$\omega_G(X) = \omega_F(X) + \epsilon\nu(X) \tag{8.11}$$

is invariant on G, where $\nu(X)$ is given by (8.6). If R is A-flat, then $f(X)+\epsilon g(X)$ is a logarithm for the group G over $R\otimes K$, where f is the logarithm associated to ω on F and $g(X)$ is given by (8.8).

PROPOSITION 8.12. *Assume F is an A-typical group of dimension 1 over the flat A-algebra R. Then*

$$\mathrm{RigExt}(F,\mathbb{G}_a)\approx \frac{\{g(X)=\sum_{k\geq 0} m_k X^{q^k} : i(\pi)^k\cdot m_k\in R, dg \text{ and } \delta g \text{integral}\}}{\{g(X)=\sum_{k\geq 0} m_k X^{q^k} : m_k\in R\}}.$$

PROOF. Every $\star$-isomorphism class of deformation of F to $R[\epsilon]/(\epsilon^2)$ is represented by an A-typical group G. The logarithm $f(X)+\epsilon g(X)$ associated to ω_G is then a series of the form (5.2), so

$$g(X)=\sum_{k\geq 0} m_k X^{q^k} \qquad \text{with} \qquad m_k\in R\otimes K$$

with dg and δg integral. We must show that this implies that $i(\pi)^k\cdot m_k$ lies in R, which we write as $\pi^k\cdot m_k$ for simplicity.

Since dg is integral, m_0 lies in R. Assume that $k\geq 1$ and that we have shown that $\pi^j m_j$ lies in R for all $j<k$. Consider the coefficient of X^{q^k} in the integral series

$$\delta_\pi g(X)=i(\pi)g(X)-g(\pi_F X).$$

This is equal to

$$\pi m_k - m_k\pi^{q^k} + \text{ terms in } \pi^{1-k}R.$$

Since $k\geq 1$, this shows that πm_k lies in $\pi^{1-k}R$, so $\pi^k m_k$ lies in R. $\square$

We call a power series $g(X)=\sum m_k X^{q^k}$, with dg and δg integral, a "quasi-logarithms" for the A-typical group F. In the next section we will calculate the quasi-logarithms on the universal A-typical group $F[\underline{v}]$ over the ring

$$A[\underline{v}]=A[v_1,v_2,\dots].$$

9. Universal quasi-logarithms

We let $F[\underline{v}]$ be the universal A-typical group over $A[\underline{v}]$ with logarithm

$$f(X)=\sum_{b\geq 0} b_k(\underline{v})X^{q^k}$$

over $K[\underline{v}]$ defined by (5.3). For $i=1,2,3,\cdots$ we let D_i be the derivation

(9.1) $$D_i b(\underline{v})=\frac{\partial}{\partial v_i}b(\underline{v}) \quad \text{of} \quad K[\underline{v}].$$

PROPOSITION 9.2. *For $i=1,2,3,\cdots$ the series*

$$g_i(X)=\sum_{k\geq 0} D_i b_k(\underline{v})X^{q^k}\equiv \frac{X^{q^i}}{\pi}+\cdots \qquad \text{mod } deg\ q^i+1$$

is a quasi-logarithm for the group $F[\underline{v}]$. We have:

$$\left.\begin{aligned}\delta g_i(X,Y)&\equiv \frac{1}{\pi}(X^{q^i}+Y^{q^i}-(X+Y)^{q^i})\\ \delta_a g_i(X)&\equiv\frac{1}{\pi}(a-a^{q^i})X^{q^i}\end{aligned}\right\}\qquad \text{mod } deg\ q^i+1$$

PROOF. Let $G_i[\underline{v}]=\alpha_i(F[\underline{v}])$ be the deformation of $F[\underline{v}]$ to $A[\underline{v}][\epsilon]/(\epsilon^2=0)$ given by the homomorphism

$$\begin{aligned}\alpha_i : A[\underline{v}]&\longrightarrow A[\underline{v}][\epsilon]/(\epsilon^2)\\ v_j&\longmapsto v_j \qquad\qquad j\neq i\\ v_i&\longmapsto v_i+\epsilon.\end{aligned}$$

Then the logarithm for $G_i[\underline{v}]$ has the form $f+\epsilon g_i$, where $g_i(X)$ is defined by Proposition 9.2. Hence g_i is a quasi-logarithm for $F[\underline{v}]$, and the calculation of δg_i is immediate. $\square$

COROLLARY 9.3. *For any $\alpha_0,\alpha_1,\alpha_2,\dots$ in $A[\underline{v}]$, the series*

$$g(X)=\alpha_0 f+\alpha_1 g_1+\alpha_2 g_2+\cdots$$

is a quasi-logarithm for the group $F[\underline{v}]$.

We now consider specializations of the universal quasi-logarithms g_i to formal A-modules F over local A-algebras R. Recall that a "local A-algebra" is complete, local, and Noetherian with maximal ideal P containing $i(\pi)$. We assume F is A-typical of dimension 1 and height n over R; then $F=\alpha(F[\underline{v}])$ for a unique homomorphism of A-algebras

$$\alpha : A[\underline{v}]\longrightarrow R.$$

By Proposition 5.7 we have:

$$\pi_{F[\underline{v}]}(X)\equiv v_k X^{q^k} \qquad \text{mod } (\pi,v_1,\cdots,v_{k-1}),\quad deg(q^k+1)$$

Since F has height n over R we therefore have

(9.4) $$\begin{cases}\alpha(v_i)\in P & i=1,2,\dots,n-1\\ \alpha(v_n)\in R^*\end{cases}$$

When R is A-flat, so injects into $R\otimes K$, we may define the specialization of the quasi-logarithms $f,g_1,\dots,g_{n-1}$ on $F[\underline{v}]$ via the map α to obtain classes

(9.5) $$f_0=\alpha(f), f_1=\alpha(g_1),\dots,f_{n-1}=\alpha(g_{n-1}) \quad \in \mathcal{R}\mathrm{ig}\mathcal{E}\mathrm{xt}(F,\mathbb{G}_a).$$

For general local A-algebra R, we cannot specialize the quasi-logarithms g_i, but we may define the specialization of their coboundaries δg_i, which are symmetric 2-cocycles on $F[\underline{v}]$ over $A[\underline{v}]$. For $i = 1, 2, \ldots, n-1$ define the classes

$$\delta f_i = \alpha(\delta g_i) = \{ \Delta^i(X,Y), \delta^i_a(X) \} \quad \text{in } \operatorname{Ext}(F, \mathbb{G}_a). \tag{9.6}$$

By Proposition 9.2 we have the congruences

$$\delta^i_\pi(X) \equiv (1 - \pi^{q^i-1})X^{q^i} \quad \bmod \deg q^i + 1 \tag{9.7}$$

PROPOSITION 9.8. *Let F be a formal A-module of dimension one and height n over the local A-algebra R. Then the R-module $Ext(F, \mathbb{G}_a)$ is free of rank $(n-1)$ with basis $\{ \delta f_1, \delta f_2, \ldots, \delta f_{n-1} \}$ and the R-module* RigExt$(F, \mathbb{G}_a)$ *is free of rank n. When R is A-flat, the quasi-logarithms $\{ f_0, f_1, f_2, \ldots, f_{n-1} \}$ give a basis of* RigExt$(F, \mathbb{G}_a)$.

PROOF. The congruence (9.7) and the argument in [**LT66**, Prop. 2.6] combine to show that the elements $\{ \delta f_1, \ldots, \delta f_{n-1} \}$ give a basis for $\operatorname{Ext}(F, \mathbb{G}_a) = H^2(F, \mathbb{G}_a)_s$. We show here that the classes δf_i are independent over R, and leave the proof that they span to the reader.

Assume that $\sum \alpha_i(\delta f_i) = 0$ in $H^2(F, \mathbb{G}_a)_s$. In particular, there is a series $h(X)$ in $R[\![X]\!]$ such that

$$\sum_{i=1}^{n-1} \alpha_i \delta^i_\pi(X) = \delta_\pi h(X) = h(\pi_F X) - i(\pi) h(X). \tag{9.9}$$

Since $i(\pi)$ is in P and

$$\pi_F X \equiv g(X^{q^n}) \qquad (\bmod P) \tag{9.10}$$

the coefficients of X^{q^i} on the right hand side of (9.9) are in P for $i = 1, 2, \ldots, n-1$. But by (9.7) we have

$$\delta^i_\pi(X) \equiv X^{q^i} + \cdots \qquad (\bmod P). \tag{9.11}$$

Hence $\alpha_1 \equiv 0 \bmod P$, which implies $\alpha_2 \equiv 0 \bmod P$, etc. Thus $\alpha_i \equiv 0 \bmod P$ for all i, and the left hand side of (9.9) is a series with all coefficients in P. Since $g'(0) \neq 0$ in (9.10), this implies that $h(X)$ is in $P[\![X]\!]$.

Now assume, by induction, we have shown that $\alpha_i \equiv 0 \bmod P^{k-1}$ for all i and that $h(X)$ is in $P^{k-1}[\![X]\!]$. Then $i(\pi)h(X)$ is in $P^k[\![X]\!]$ and the coefficients of X^{q^i} on the right hand side of (9.9) lie in P^k, for $i \leq n-1$. By (9.11) this shows that $\alpha_1 \equiv 0 \bmod P^k$, hence $\alpha_2 \equiv 0 \bmod P^k$, etc. Similarly $\alpha_i \equiv 0 \bmod P^k$ for all i and the left hand side of (9.9) is a series with all coefficients in P^k. Using (9.10) and the fact that $g'(0) \neq 0$, we see that $h(X) \in P^k[\![X]\!]$. This induction shows that $\alpha_i \equiv 0 \bmod P^k$ for all $k \geq 1$. Since $\bigcap_{k \geq 1} P^k = 0$ in R, $\alpha_i = 0$ for $i = 1, 2, \cdots, n-1$. □

Since $\operatorname{Hom}(F, \mathbb{G}_a) = 0$ by Corollary 4.3, we have an exact sequence

$$0 \longrightarrow \omega(F) \longrightarrow \operatorname{RigExt}(F, \mathbb{G}_a) \longrightarrow \operatorname{Ext}(F, \mathbb{G}_a) \longrightarrow 0 \tag{9.12}$$

following (8.4). Since $\omega(F)$ is free of rank 1 and $\operatorname{Ext}(F, \mathbb{G}_a)$ is free of rank $n-1$, $\operatorname{RigExt}(F, \mathbb{G}_a)$ is a free R-module of rank n. The elements $\{ f_0, f_1, \ldots, f_{n-1} \}$ give a basis when R is A-flat, as f_0 spans the image of $\omega(F)$ and the images $\{ \delta f_1, \ldots, \delta f_{n-1} \}$ give a basis of $\operatorname{Ext}(F, \mathbb{G}_a)$.

COROLLARY 9.13. *If $R \to R'$ is a homomorphism of local A-algebras, the induced maps of free R'-modules*

$$\operatorname{Ext}_R(F, \mathbb{G}_a) \otimes R' \to \operatorname{Ext}_{R'}(F, \mathbb{G}_a)$$
$$\operatorname{RigExt}_R(F, \mathbb{G}_a) \otimes R' \to \operatorname{RigExt}_{R'}(F, \mathbb{G}_a)$$

are isomorphisms.

10. A-divided powers

Let R be a flat A-algebra. We say an ideal $I \subseteq R$ has "A-divided powers" provided

$$I^{q^j} \subseteq \pi^j R \qquad \text{for all } j \geq 1. \tag{10.1}$$

(We write $\pi^j R$ for $i(\pi)^j R$ throughout this section.)

As examples, the ideal $I = \pi R$ has A-divided powers. Another case of importance is when R is the ring of integers in a finite field extension L of K. Let P be the maximal ideal of R and e the ramification index of L over K, so $(P)^e = \pi R$. The ideal P has A-divided powers provided

$$\frac{q^j}{e} \geq j \qquad \text{for all } j \geq 1. \tag{10.2}$$

This occurs precisely when $e \leq q$. We note, for future reference, that the inequality of (10.2) will certainly hold for all $j \geq J(e,q)$, where $J(e,q)$ is an integer depending only on e and q.

PROPOSITION 10.3. *Let F and F' be two A-typical formal A-modules of dimension 1 over R, and let $I \subseteq R$ be an ideal with A-divided powers. If φ and ψ are elements of* $\operatorname{Hom}(F, F')$ *with $\varphi \equiv \psi \bmod I$, then*

$$\varphi^* = \psi^* : \operatorname{RigExt}(F', \mathbb{G}_a) \longrightarrow \operatorname{RigExt}(F, \mathbb{G}_a).$$

PROOF. Let $\alpha \in \operatorname{RigExt}(F', \mathbb{G}_a)$, and represent α by a quasi-logarithm

$$g(X) = \sum m_k X^{q^k} \qquad \text{with} \quad \pi^k m_k \in R.$$

Then $\varphi^* g(X)$ is represented by the quasi-logarithm $g(\varphi X)$ and $\psi^* g(X)$ is represented by $g(\psi X)$. Hence we must show that

$$\varphi \equiv \psi \bmod I \Longrightarrow g(\varphi) \equiv g(\psi) \bmod R.$$

Write $\varphi = \psi + \Delta$ where $\Delta(X) \in I[\![X]\!]$. Then

$$g(\varphi) - g(\psi) = \sum m_k((\psi + \Delta)^{q^k} - \psi^{q^k}) = \sum m_k \left(\sum_{i=1}^{q^k} \binom{q^k}{i} \Delta^i \psi^{q^k - i} \right).$$

But for $i \geq 1$, $\binom{q^k}{i}\Delta^i$ has coefficients in $\pi^k R$. (This uses the fact that q is in πR). Since $m_k \cdot \pi^k$ is in R, this establishes the claim. □

11. The universal additive extension

Let R be a local A-algebra and let F be a formal A-module of dimension 1 and height n over R. Combining Corollary 4.3 and Proposition 9.8, we have seen that

$$(11.1) \qquad \begin{cases} \mathrm{Hom}(F, \mathbb{G}_a) = 0 \\ \mathrm{Ext}(F, \mathbb{G}_a) \quad \text{is a free } R\text{-module of rank } n-1. \end{cases}$$

These facts combine to prove the existence of a universal additive extension E of F over R.

Let $M = \mathrm{Hom}_R(\mathrm{Ext}(F, \mathbb{G}_a), R)$, which is a free R-module of rank $n-1$, and let $F' = \mathbb{G}_a \otimes M$ be the associated additive formal A-module of dimension $n-1$. Then

$$\mathrm{Ext}(F, F') = \mathrm{Ext}(F, \mathbb{G}_a) \otimes M = \mathrm{End}_R(\mathrm{Ext}(F, \mathbb{G}_a)).$$

Let

$$(11.2) \qquad 0 \longrightarrow F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \longrightarrow 0$$

be an extension corresponding to the identity map in $\mathrm{End}_R(\mathrm{Ext}(F, \mathbb{G}_a))$. If $0 \to F' \to E' \to F \to 0$ is another extension in this class, there is a *unique* isomorphism $i : E \to E'$ over R which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

commute. Indeed, the difference of two isomorphisms would induce a homomorphism from F to F', and $\mathrm{Hom}(F, F') = \mathrm{Hom}(F, \mathbb{G}_a) \otimes M = 0$. Hence the sequence (11.2) is well-defined up to a unique isomorphism over R, and we call E the universal additive extension of F.

PROPOSITION 11.3. *If $0 \to G' \to E' \to F \to 0$ is any extension of F by an additive A-module G', there are unique homomorphisms over R:*

$$i : E \to E', \quad g' : F' \to G'$$

such that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & G' & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

commutes.

PROOF. Write $G' = \mathbb{G}_a \otimes M'$. Then

$$\mathrm{Hom}(F', G') = \mathrm{Ext}(F, \mathbb{G}_a) \otimes M' = \mathrm{Ext}(F, G').$$

This gives the homomorphism g'. The unicity of i follows from

$$\mathrm{Hom}(F, G') = \mathrm{Hom}(F, \mathbb{G}_a) \otimes M' = 0.$$

□

Proposition 11.3 states that any additive extension of F arises uniquely by push-out from the universal additive extension. In particular, any endomorphism of F lifts uniquely to an endomorphism of E. We also have a canonical isomorphism

$$(11.4) \qquad \omega(E) = \mathrm{RigExt}(F, \mathbb{G}_a).$$

Indeed, let α be a class in $\mathrm{RigExt}(F, \mathbb{G}_a)$, viewed as an extension E' of F by $\mathbb{G}_a$ and a differential $\omega_{E'}$ which pulls back to dX'. The universality of E gives a homomorphism $i : E \to E'$, and $i^*(\omega_{E'})$ is the associated class in $\omega(E)$. The restriction of i to F' is the map $g' : F' \to \mathbb{G}_a$ which is the image of α in $\mathrm{Ext}(F, \mathbb{G}_a) = \mathrm{Hom}(F', \mathbb{G}_a)$. Hence

$$(11.5) \qquad \omega(F') = \mathrm{Ext}(F, \mathbb{G}_a)$$

and the exact sequence (9.12) is given by the cotangent sequence

$$(11.6) \qquad 0 \longrightarrow \omega(F) \xrightarrow{\beta^*} \omega(E) \xrightarrow{\alpha^*} \omega(F') \longrightarrow 0.$$

Its dual, in the category of R-modules, is the exact sequence:

$$(11.7) \qquad 0 \longrightarrow \mathrm{Lie}(F') \xrightarrow{\alpha_*} \mathrm{Lie}(E) \xrightarrow{\beta_*} \mathrm{Lie}(F) \longrightarrow 0$$

of Lie algebras, where the terms are free R-modules of rank $n-1, n$, and 1, respectively.

Part III. Equivariant bundles on the moduli space

12. The universal deformation

Henceforth we fix an integer $n \geq 1$. Let

$$A[\![\underline{u}]\!] = A[\![u_1, u_2, \cdots, u_{n-1}]\!] \tag{12.1}$$

be the local A-algebra of power series in the variables u_i. Then $A[\![\underline{u}]\!]$ is a regular local ring of dimension n, with maximal ideal $(\pi, u_1, u_2, \cdots, u_{n-1})$ and residue field

$$A[\![\underline{u}]\!]/(\pi, u_1, \cdots, u_{n-1}) = A/\pi A = k \tag{12.2}$$

finite of order q.

Let $F = F[\underline{u}]$ be the A-typical formal A-module of height n over $A[\![\underline{u}]\!]$ which is obtained from the universal A-typical module $F[\underline{v}]$ defined in §5 by the specialization

$$\begin{array}{lccl} & F = \alpha F[\underline{v}] & & \\ \alpha : A[\underline{v}] & \longrightarrow & A[\![\underline{u}]\!] & \\ v_i & \longmapsto & u_i & i = 1, 2, \cdots, n-1 \\ v_n & \longmapsto & 1 & \\ v_i & \longmapsto & 0 & i \geq n+1. \end{array} \tag{12.3}$$

It follows from Proposition 5.7 that we have the congruence

$$\pi_F(X) \equiv X^{q^n} \qquad \bmod (\pi, u_1, \cdots, u_{n-1}), \quad \deg q^n + 1. \tag{12.4}$$

Hence the reduction $F \otimes k$ of F modulo the maximal ideal of $A[\![\underline{u}]\!]$ has height n. In fact, we have [**Haz78**, 3.2.4]:

$$\pi_{F\otimes k}(X) = X^{q^n} \quad \text{in } k[\![X]\!]. \tag{12.5}$$

Since the A-module $F \otimes k$ is defined over the finite field with q elements, the series

$$\varphi(X) = X^q \tag{12.6}$$

gives an endomorphism of $F \otimes k$, which satisfies

$$\varphi^n = \pi \qquad \text{in} \qquad \operatorname{End}(F \otimes k). \tag{12.7}$$

It is not difficult to show that $\operatorname{End}(F \otimes k)$ is the commutative A-algebra $A[\varphi]$ of integers in the totally ramified extension $K(\varphi) = K(\sqrt[n]{\pi})$ of K.

Now let R be any local A-algebra – recall that this means that R is complete, local, and Noetherian, with maximal ideal P containing $i(\pi)$. The R-module

$$\operatorname{Hom}(A[\![\underline{u}]\!], R) \tag{12.8}$$

of all continuous homomorphisms of topological A-algebras is isomorphic to $(P)^{n-1}$. Indeed, such a homomorphism $\beta : A[\![\underline{u}]\!] \to R$ is completely determined

by the images $\beta(u_i)$, which must lie in P. We say that the formal A-module F' over R is a deformation of $F \otimes k$ provided

$$F' \equiv F \otimes k \mod P. \tag{12.9}$$

Since R/P is a field containing k, the field of definition for $F \otimes k$, this congruence is meaningful; in particular, F' has height n over R. The key deformation-theoretic result over local A-algebras is the following theorem of Lubin-Tate [**LT66**, §3, for $A = \mathbb{Z}_p$] and Drinfeld [**Dri74**, §4].

PROPOSITION 12.10. *Let F' be a deformation of $F \otimes k$ over the local A-algebra R. Then there is a unique element*

$$\beta \in \operatorname{Hom}(A[\![\underline{u}]\!], R) = P^{n-1}$$

such that the specialization $\beta F = F[\beta \underline{u}]$ is $\star$-isomorphic to F' over R. Moreover, the $\star$-isomorphism

$$h : \beta F \xrightarrow{\sim} F' \qquad \text{over } R$$

is unique.

PROOF. This is exactly as in Lubin-Tate; the key cohomological calculations are as follows. First:

$$\operatorname{Ext}^2(F', \mathbb{G}_a) = 0. \tag{12.11}$$

This insures that deformations from R/I to R exist when $I^2 = 0$, and that the versal deformation space is smooth. Next, since F' has height n

$$\operatorname{Ext}^1(F', \mathbb{G}_a) \quad \text{is free of rank } n-1 \text{ over } R. \tag{12.12}$$

This gives the dimension $(= n-1)$ of the tangent space to the versal deformation space. Finally, since F' has finite height

$$\operatorname{Hom}(F', \mathbb{G}_a) = 0. \tag{12.13}$$

This shows the generic deformation has no non-trivial automorphisms, and that the moduli space exists. □

Proposition 12.10 states that the functor of $\star$-isomorphism classes of deformations of $F \otimes k$ to local A-algebras is representable by the formal scheme

$$X = \operatorname{Spf} A[\![\underline{u}]\!]. \tag{12.14}$$

Indeed, $X(R) = P^{n-1}$. It also shows that $F = F[\underline{u}]$ is a universal deformation of $F \otimes k$ over X. The general fibre

$$X \otimes K \tag{12.15}$$

is a rigid analytic space over K, isomorphic to the $(n-1)$-dimensional polydisc. We will study this analytic space, and the K-algebra $K\{\{\underline{u}\}\}$ of rigid analytic functions on it, more thoroughly in Part IV. Here we simply note that its points over an algebraic closure $\bar{K}$ of K are given by

$$(X \otimes K)(\bar{K}) = X(\bar{A}) = (\bar{m})^{n-1}, \tag{12.16}$$

where $\bar{A}$ is the integral closure of A in $\bar{K}$ and $\bar{m}$ is the maximal ideal

$$\{\, \alpha \in \bar{A} : \operatorname{ord}(\alpha) > 0 \,\}$$

If $\beta \in (\bar{m})^{n-1}$ we let F_β be the corresponding specialization of $F = F[\underline{u}]$ over the flat A-algebra $R = A[\beta]$.

13. The canonical lifting

In this section, we study the specialization $F_{\underline{0}}$ of the universal deformation $F = F[\underline{u}]$. This is a formal A-module over the local A-algebra $R = A$, which is obtained by the homomorphism $\beta : A[\![\underline{u}]\!] \longrightarrow A$ with $\beta(u_i) = 0$ for all i. Hence $F_{\underline{0}}$ is an A-typical formal A-module of height n, which is obtained from $F[\underline{v}]$ via the specialization

$$\begin{aligned} \alpha : A[\underline{v}] &\longrightarrow A \\ v_i &\longmapsto 0 \qquad i \neq n \\ v_n &\longmapsto 1. \end{aligned} \tag{13.1}$$

We call $F_{\underline{0}}$ the "canonical lifting" of $F \otimes k$.

Since $F_{\underline{0}} \equiv F \otimes k \mod \pi A$ we have

$$\pi_{F_{\underline{0}}}(X) \equiv X^{q^n} \qquad (\bmod\ \pi A) \tag{13.2}$$

by (12.5). As for the logarithm of $F_{\underline{0}}$, we have the following formula.

LEMMA 13.3. *The logarithm f_0 of $F_{\underline{0}}$ is given by the series*

$$\begin{aligned} f_0(X) &= X + \frac{X^{q^n}}{\pi} + \frac{X^{q^{2n}}}{\pi^2} + \cdots \\ &= \sum_{k \geq 0} \frac{X^{q^{kn}}}{\pi^k}. \end{aligned}$$

PROOF. This follows from a combination of (13.1) with the functional equation (5.3) for the logarithm of $F[\underline{v}]$. Together they show that $f_0(X)$ satisfies the functional equation

$$f_0(X) = X + \frac{1}{\pi} f_0(X^{q^n})$$

which gives the expansion recursively. □

Let ζ_n be a primitive $(q^n - 1)$ root of unity in $\bar{K}$. Then the ring $A_n = A[\zeta_n]$ is the unramified extension of degree n over A, with residue field $A_n/\pi A_n = k_n =$

$k(\zeta_n)$ the cyclic extension of degree n of k. The automorphism group of A_n over A is cyclic of order n, generated by the Frobenius automorphism σ defined by

$$\sigma(\zeta_n) = \zeta_n^q. \tag{13.4}$$

This automorphism satisfies the congruence

$$\sigma(\alpha) \equiv \alpha^q \pmod{\pi A_n} \tag{13.5}$$

for any element $\alpha \in A_n$.

PROPOSITION 13.6. *The series $\zeta_n(X) = \zeta_n \cdot X$ defines an automorphism of the formal A-module $F_{\underline{0}}$ over A_n. We have*

$$\operatorname{End}_{A_n}(F_{\underline{0}}) = A_n. \tag{13.7}$$

PROOF. By Lemma 13.3, we have $f_0(\zeta_n \cdot X) = \zeta_n \cdot f_0(X)$. Since

$$\begin{aligned} F_{\underline{0}}(X,Y) &= f_0^{-1}(f_0(X) + f_0(Y)) \\ a_{F_{\underline{0}}}(X) &= f_0^{-1}(a \cdot f_0(X)), \end{aligned}$$

we find that

$$\begin{aligned} F_{\underline{0}}(\zeta_n X, \zeta_n Y) &= \zeta_n \cdot F_{\underline{0}}(X,Y) \\ a_{F_{\underline{0}}}(\zeta_n X) &= \zeta_n \cdot a_{F_{\underline{0}}}(X). \end{aligned}$$

This shows $\zeta_n \cdot X$ is an automorphism of $F_{\underline{0}}$, and that the ring $A_n = A[\zeta_n]$ acts as endomorphisms of $F_{\underline{0}}$ over the base $R = A_n$.

Since $F_{\underline{0}}$ has height n, the inclusion $A_n \hookrightarrow \operatorname{End}_{A_n}(F_{\underline{0}})$ shows that $F_{\underline{0}}$ is a group of Lubin-Tate type **[LT65]**. (This also follows from the congruence (13.2)). We may then conclude that A_n is the absolute endomorphism ring of $F_{\underline{0}}$. □

We now give a description of the free A-module $\operatorname{RigExt}(F_{\underline{0}}, \mathbb{G}_a)$ using quasi-logarithms.

PROPOSITION 13.8. *The series*

$$\begin{cases} f_0(X) = \sum_{k \geq 0} \frac{X^{q^{kn}}}{\pi^k} \\ f_i(X) = \frac{1}{\pi} f_0(X^{q^i}) \qquad i = 1, 2, \cdots, n-1 \end{cases}$$

are quasi-logarithms for the formal A-module $F_{\underline{0}}$, and give a basis for the free A-module $\operatorname{RigExt}(F_{\underline{0}}, \mathbb{G}_a)$. These basis elements are eigenvectors for the action of $A_n = \operatorname{End}_{A_n}(F_{\underline{0}})$. We have

$$\alpha^*(f_i) = \sigma^i(\alpha) \cdot f_i \tag{13.9}$$

for all $\alpha \in A_n$ and $i = 0, 1, \cdots, n-1$.

PROOF. The series f_i in Proposition 13.8 are exactly the specializations (9.5) $f_i = \alpha(g_i)$ of the universal quasi-logarithms on the A-module $F[\underline{v}]$ via the homomorphism α of 13.1. Hence they give a basis by Proposition 9.8.

Since any $\alpha \in A$ acts by multiplication by α on $\mathrm{RigExt}(F_{\underline{0}}, \mathbb{G}_a) = \omega(E_{\underline{0}})$, it suffices to show that

$$\zeta_n^*(f_i) = \zeta_n^{q^i} \cdot f_i,$$

which is a special case of (13.9). This follows from the fact that for any quasi-logarithm $g(X)$ on f_0 we have

$$\zeta_n^* g(X) = g(\zeta_n X).$$

□

Finally, we may use the canonical lifting $F_{\underline{0}}$ of $F \otimes k$ to determine the ring $\mathrm{End}_{k_n}(F \otimes k)$, which is also the absolute endomorphism ring of the reduced group $F \otimes k$. By Proposition 4.2 we have an injection:

$$A_n = \mathrm{End}_{A_n}(F_{\underline{0}}) \hookrightarrow \mathrm{End}_{k_n}(F \otimes k).$$

The series $\varphi(X) = X^q$ also defines an endomorphism of $F \otimes k$, by (12.6), which satisfies $\varphi^n = \pi$.

PROPOSITION 13.10. *We have*

$$\mathrm{End}_{k_n}(F \otimes k) = A_n \oplus A_n\varphi \oplus A_n\varphi^2 \oplus \cdots \oplus A_n\varphi^{n-1}$$

where $\varphi\alpha = \alpha^\sigma\varphi$ for all $\alpha \in A_n$. This ring is the absolute endomorphism ring of $F \otimes k$ over $\bar{k}$, and is isomorphic to the maximal order in the division algebra of invariant $\frac{1}{n}$ with center K.

PROOF. This is standard, see [**Haz78**, §23]. □

14. The group action

We now describe the action of the group scheme $G = \mathrm{Aut}(F \otimes k)$ on the formal A-scheme $X = \mathrm{Spf}\, A[\![\underline{u}]\!]$ of deformations of $F \otimes k$ to local A-algebras.

The group G is étale over k, with points

$$(14.1) \qquad \begin{cases} G(k) = A[\varphi]^* = A^* \oplus A\varphi \oplus \cdots \oplus A\varphi^{n-1} \\ G(k_n) = G(\bar{k}) = A_n[\varphi]^* = A_n^* \oplus A_n\varphi \oplus \cdots \oplus A_n\varphi^{n-1}. \end{cases}$$

It is the twist of the constant group scheme $A_n[\varphi]^*$ over k by the one-cocycle of $\mathrm{Gal}(k_n/k)$ taking the generator σ to the automorphism "conjugation by φ". We view G as an étale group scheme over A, which becomes constant over the unramified extension A_n. The torus $T = \mathrm{Aut}(F_0)$ is a subgroup scheme of G, with points

$$(14.2) \qquad \begin{cases} T(A) = A^* \\ T(A_n) = A_n^* \end{cases}$$

and the center $Z = \mathrm{Aut}(F)$ of G is the constant group scheme A^*.

Let b be an element of $G(A_n) = \mathrm{Aut}_{k_n}(F \otimes k)$, given by the invertible power series $b(X)$ with coefficients in $k_n[\![X]\!]$. Lift the series $b^{-1}(X)$ arbitrarily to an invertible series $h(X)$ with coefficients in A_n, and define the formal A-module F' over $A_n[\![\underline{u}]\!]$ by

$$(14.3) \qquad \begin{cases} F'(hX, hY) &= h(F(X,Y)) \\ a_{F'}(hX) &= h(a_F X), \end{cases}$$

where $F = F[\underline{u}]$ is the universal deformation over $A[\![\underline{u}]\!]$. By definition of F', the series h defines an isomorphism of formal A-modules

$$(14.4) \qquad h : F \xrightarrow{\sim} F' \quad \text{over} \quad A_n[\![\underline{u}]\!].$$

Since h reduces modulo $(\pi, u_1, \ldots, u_{n-1})$ to b^{-1}, which is an automorphism of $F \otimes k$, the group F' is also a deformation of $F \otimes k$. Hence, by Proposition 12.10, there is a unique continuous homomorphism of A-algebras

$$(14.5) \qquad \beta : A[\![\underline{u}]\!] \longrightarrow A_n[\![\underline{u}]\!]$$

such that βF is $\star$-isomorphic to F'. Moreover, the $\star$-isomorphism

$$(14.6) \qquad j : \beta F \xrightarrow{\sim} F' \quad \text{over} \quad A_n[\![\underline{u}]\!].$$

is unique.

PROPOSITION 14.7. *The homomorphism β of* (14.5) *depends only on*

$$b \in \mathrm{Aut}(F \otimes k)$$

(and not on the lifting $h(X)$ of $b^{-1}(X)$ to $A_n[\![X]\!]$). It extends to a continuous automorphism of A_n-algebras

$$\beta = \beta(b) : A_n[\![\underline{u}]\!] \xrightarrow{\sim} A_n[\![\underline{u}]\!].$$

The composite map

$$(14.8) \qquad k = k(b) : \beta F \xrightarrow{j} F' \xrightarrow{h^{-1}} F$$

depends only on b, and is the unique isomorphism from βF to F over $A_n[\![\underline{u}]\!]$ which reduces to the automorphism b of $F \otimes k$ mod $\pi, \underline{u}$.

PROOF. The map β depends only on the $\star$-isomorphism class of the deformation F', which is clearly independent of the choice of lifting $h(X)$. It clearly extends to a homomorphism of A_n-algebras; we will soon show (Prop. 14.9) it is an automorphism by proving that the composite $\beta(b) \circ \beta(b^{-1})$ is the identity map.

The isomorphism k clearly reduces to b mod $\pi, \underline{u}$. By Proposition 4.2 it is the unique map from βF to F with this property, so depends only on b. □

PROPOSITION 14.9. *The map $b \to \beta(b)$ defines a representation*

$$\beta : G(A_n) \longrightarrow \mathrm{Aut}_{A_n}(A_n[\![\underline{u}]\!]),$$

so the group $G(A_n)$ acts on (the left of) the A_n-algebra $A_n[\![\underline{u}]\!]$.

PROOF. Let b_1, b_2 be elements of $G(A_n) = \mathrm{Aut}_{k_n}(F \otimes k)$, and write β_i for $\beta(b_i)$ and k_i for $k(b_i)$. We will show that

$$\begin{cases} \beta(b_1 b_2) = \beta(b_1) \circ \beta(b_2) = \beta_1 \circ \beta_2 \\ k(b_1 b_2) = k_1 \circ \beta_1(k_2). \end{cases} \tag{14.10}$$

Indeed, the composite $k_1 \circ \beta_1(k_2)$ is an isomorphism

$$\beta_1\beta_2 F \xrightarrow{\beta_1 k_2} \beta_1 F \xrightarrow{k_1} F$$

which reduces to $b_1 b_2$ in $\mathrm{Aut}(F \otimes k)$. Hence $\beta_1\beta_2 F$ and $\beta(b_1 b_2)F$ are $\star$-isomorphic. By uniqueness, this proves (14.10). □

The left action of $G(A_n)$ on the A_n-algebra $A_n[\![\underline{u}]\!]$ gives, by transport of structure, a right action of the étale group scheme G on the formal scheme $X = \mathrm{Spf}\, A[\![\underline{u}]\!]$ over A. If x is a point of $X(R) = \mathrm{Hom}(A[\![\underline{u}]\!], R)$ and φ is a function in $A[\![\underline{u}]\!]$ we write

$$\langle x, \varphi \rangle = x \circ \varphi \quad \text{in } R. \tag{14.11}$$

(From the dual point of view, $\langle x, \varphi \rangle$ is just the value of the function $\varphi(\underline{u})$ at the point x.) The element $g \in G$ acts on X by the formula:

$$\langle xg, \varphi \rangle = \langle x, g\varphi \rangle. \tag{14.12}$$

This gives a morphism

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto xg \end{aligned}$$

in the category of schemes over A, which satisfies all the diagrams for a group action.

The following result describes the orbits of G on X.

PROPOSITION 14.13. *Let $\underline{x} = (x_1, \cdots, x_{n-1})$ be a point in $X(R) = P^{n-1}$, corresponding to a deformation $F_{\underline{x}}$ of $F \otimes k$ over the local A-algebra R, up to $\star$-isomorphism. The subgroup scheme $G_{\underline{x}}$ of G fixing $\underline{x}$ is equal to the image of the injection*

$$\mathrm{Aut}(F_{\underline{x}}) \hookrightarrow \mathrm{Aut}(F \otimes k) = G$$

under reduction mod P. The G-orbit of $\underline{x}$ consists of those deformations $F_{\underline{y}}$ which are isomorphic to $\underline{x}$ over R.

PROOF. Indeed for $b \in G(R)$, the point $\underline{x}^b$ in $X(R) = P^{n-1}$ corresponds to the deformation $F_{\underline{x}^b}$ over R which is $\star$-isomorphic to

$$F'_{\underline{x}} = h(F_{\underline{x}}(h^{-1}X, h^{-1}Y)).$$

This is always isomorphic to $F_{\underline{x}}$ over R (via h), and is $\star$-isomorphic to $F_{\underline{x}}$ iff b lifts to an automorphism of $F_{\underline{x}}$ over R. □

In particular, Proposition 14.13 shows that the center Z of the group scheme G acts trivially on X, as every lifting $F_{\underline{x}}$ has endomorphisms by A and hence automorphisms by A^*. It also shows that the torus T is precisely the subgroup scheme of G which stabilizes the point $\underline{x} = \underline{0} = (0, 0, \cdots, 0)$ of $X(A) = (\pi A)^{n-1}$.

The action of an element b in $G(A_n) = A_n[\varphi]^*$ on $A_n[\![\underline{u}]\!]$ is completely determined by the images

$$\beta(b) \circ u_i = \sum a_i(b)_J \underline{u}^J, \tag{14.14}$$

as $\beta(b)$ is A_n-linear and continuous. The constant coefficient $a_i(b)_0$ lies in πA_n, and the linear coefficients $a_i(b)_j$ give an invertible Jacobian matrix:

$$\det(a_i(b)_j) \in A_n^*.$$

The element $\sigma \in \mathrm{Aut}(A_n/A)$ acts by conjugation of coefficients

$$\sigma\left(\sum a_J \underline{u}^J\right) = \sum \sigma(a_J) \cdot \underline{u}^J; \tag{14.15}$$

this action is A-linear and normalizes the action of $G(A_n)$. We have the formula

$$\sigma\beta(b)\sigma^{-1} = \beta(\varphi b \varphi^{-1}) \tag{14.16}$$

in $\mathrm{Aut}_A(A_n[\![\underline{u}]\!])$. In particular, the compact K-analytic group

$$\underline{G} = G(A_n) \rtimes \langle \sigma \rangle \tag{14.17}$$

acts on the A-algebra $A_n[\![\underline{u}]\!]$, where σ acts on $G(A_n) = A_n[\varphi]^*$ by the outer automorphism "conjugation by φ".

PROPOSITION 14.18. *The invariants $A_n[\![\underline{u}]\!]^{\underline{G}}$ of $\underline{G}$ on the ring $A_n[\![\underline{u}]\!]$ consist of the constant power series in $\underline{u}$ with coefficients in A.*

PROOF. It suffices to show that $A_n[\![\underline{u}]\!]^{G(A_n)} = A_n$, as $A_n^{\langle\sigma\rangle} = A$ by Galois theory.

To do this, we observe that $G(A_n)$ has a single orbit on the set $X(A_n) = (\pi A_n)^{n-1}$. Indeed, any lifting $F_{\underline{x}}$ of $F \otimes k$ to $R = A_n$ has π-series satisfying

$$\begin{cases} \pi_{F_{\underline{x}}}(X) \equiv \pi X & (\bmod \deg 2) \\ \pi_{F_{\underline{x}}}(X) \equiv X^{q^n} & (\bmod \pi A_n). \end{cases}$$

Hence $F_{\underline{x}}$ is a formal A-module of Lubin-Tate type **[LT65]** with $\mathrm{End}_{A_n}(F_{\underline{x}}) = A_n$. Since all Lubin-Tate modules are isomorphic over A_n, Proposition 14.13 shows that $G(A_n)$ has a single orbit on $X(A_n)$, and in fact, that

$$X(A_n) \simeq G(A_n)/T(A_n) \simeq A_n[\varphi]^*/A_n^* \tag{14.19}$$

as $T(A_n)$ is the stabilizer of $F_{\underline{0}}$.

Consequently, if the series $f(\underline{u}) = \sum a_J \underline{u}^J$ in $A_n[\![\underline{u}]\!]$ is fixed by $G(A_n)$, we have

$$f(\underline{x}) = f(\underline{0}) = a_0$$

for all $\underline{x} \in X(A_n) = (\pi A_n)^{n-1}$. Hence the series

$$f(\underline{u}) - f(\underline{0}) = g(\underline{u})$$

vanishes on $(\pi A_n)^{n-1}$, which implies that $g(\underline{u}) = 0$ and $f(\underline{u}) = a_0$. Indeed, if $g(\underline{u}) \neq 0$, its zero locus is nowhere dense in

$$Y(1) = \left\{ \underline{x} \in (\bar{K})^{n-1} : |x_i| \leq 1/q \quad \text{for all } i \right\},$$

so cannot contain $(\pi A_n)^{n-1}$ **[BGR84**, 5.14]. □

15. Equivariant vector bundles: general theory

Let $\mathcal{O}_X$ denote the sheaf of functions on the formal scheme X over A, so

$$H^0(X, \mathcal{O}_X) = A[\![\underline{u}]\!]. \tag{15.1}$$

An equivariant vector bundle $\mathcal{M}$ on X is by definition a sheaf of $\mathcal{O}_X$-modules, which is locally free of finite rank, together with a right action of G

$$\mathcal{M} \times G \longrightarrow \mathcal{M} \tag{15.2}$$

which is compatible with the right action of G on X.

Since X is affine, the equivariant bundle $\mathcal{M}$ is completely determined by its space of sections

$$M = H^0(X \otimes A_n, \mathcal{M} \otimes A_n), \tag{15.3}$$

which is a free $A_n[\![\underline{u}]\!]$ module of rank $= \mathrm{rank}_{\mathcal{O}_X}(\mathcal{M})$. The group $G(A_n) = A_n[\varphi]^*$ acts A_n-linearly on the left of M, and this action satisfies

$$b(r \cdot m) = \beta(r) \cdot bm \tag{15.4}$$

for $r \in A_n[\![\underline{u}]\!] = R$, where $\beta = \beta(b)$ is defined by (14.5). Since $\mathcal{M}$ is defined over A, the cyclic group $\langle\sigma\rangle = \mathrm{Aut}(A_n/A)$ acts semi-linearly on M, and this normalizes the action of $G(A_n)$. Hence the group $\underline{G} = G(A_n) \rtimes \langle\sigma\rangle$ acts A-linearly on M and this action satisfies

$$g(rm) = g(r) \cdot g(m) \qquad r \in R, m \in M g \in \underline{G} \tag{15.5}$$

Conversely, an A-linear representation of $\underline{G}$ on a free R-module M of rank m determines an equivariant vector bundle $\mathcal{M}$ of rank m on X, provided the action satisfies (15.5) and $M^{(\sigma)}$ is a free $A[\![\underline{u}]\!]$-module of rank m with $M^{(\sigma)} \otimes_{A[\![\underline{u}]\!]} R = M$.

The simplest example of an equivariant bundle $\mathcal{M}$ on X is the trivial bundle: $\mathcal{M} = \mathcal{O}_X$, $M = R$ of rank 1. If $\mathcal{M}$ and $\mathcal{N}$ are two equivariant bundles on X which afford representations M and N of $\underline{G}$, then $\mathcal{M} \oplus \mathcal{N}, \mathcal{M} \otimes \mathcal{N}$, and $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ are equivariant bundles on X which afford the representations

$$\begin{cases} M \oplus N & g(m,n) = (gm, gn) \\ M \otimes_R N & g(m \otimes n) = gm \otimes gn \\ \mathrm{Hom}_R(M,N) & gf(m) = g(f(g^{-1}m)) \end{cases} \tag{15.6}$$

respectively. The equivariant vector bundle $\mathcal{M}^{\check{}} = \mathrm{Hom}(\mathcal{M}, \mathcal{O}_X)$ is called the dual of $\mathcal{M}$, and $M^{\check{}} = \mathrm{Hom}_R(M, R)$ the dual representation.

A homomorphism $f : \mathcal{M} \to \mathcal{N}$ of equivariant bundles on X is a homomorphism of $\mathcal{O}_X$-modules which commutes with the right actions of G. This gives rise to a homomorphism $f : M \to N$ of free R-modules which satisfies

$$f(gm) = g(f(m)) \qquad m \in M, g \in \underline{G}. \tag{15.7}$$

We say f is an isomorphism iff it is an isomorphism in the category of R-modules, and that the sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

of equivariant bundles is exact if the associated sequence of representations of $\underline{G}$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact in the category of R-modules. We have canonical isomorphisms of representations

$$\begin{cases} M & \xrightarrow{\sim} & (M^{\check{}})^{\check{}} \\ M^{\check{}} \otimes N & \xrightarrow{\sim} & \mathrm{Hom}(M,N). \end{cases} \tag{15.8}$$

Let $\mathrm{Hom}_G(\mathcal{N}, \mathcal{M})$ be the A-module of all G-homomorphisms from $\mathcal{N}$ to $\mathcal{M}$. Then

$$\mathrm{Hom}_G(\mathcal{N}, \mathcal{M}) = (M \otimes_R N^{\check{}})^{\underline{G}}. \tag{15.9}$$

In particular:

$$\mathrm{Hom}_G(\mathcal{O}_X, \mathcal{M}) = M^{\underline{G}}. \tag{15.10}$$

PROPOSITION 15.11. *We have* $\mathrm{rank}_A(M^{\underline{G}}) \le \mathrm{rank}_R(M)$. *In particular, the* A*-module*

$$\mathrm{Hom}_G(\mathcal{N}, \mathcal{M})$$

is free of rank $\le \mathrm{rank}(\mathcal{N}) \cdot \mathrm{rank}(\mathcal{M})$.

PROOF. Let $\{m_1, \cdots, m_t\}$ be a maximal set of R-independent elements in $M^{\underline{G}}$; clearly $t \le \mathrm{rank}_R(M)$. If $m \in M^{\underline{G}}$ we have $m = \sum_{i=1}^{T} r_i m_i$ with $g(r_i) = r_i$ for all i, and $g \in \underline{G}$. But by Proposition 14.18, $R^{\underline{G}} = A$ and hence the elements m_i give an A-basis for $M^{\underline{G}}$. □

We may also define the A-modules

$$\mathrm{Ext}^i_G(\mathcal{N}, \mathcal{M}) \tag{15.12}$$

as derived functors of Hom_G in the category of $\mathcal{O}_X$-modules with an action of G. In particular, we define the cohomology modules of an equivariant bundle $\mathcal{M}$ by

$$H^i_G(X, \mathcal{M}) = \mathrm{Ext}^i_G(\mathcal{O}_X, \mathcal{M}). \tag{15.13}$$

These are subtle invariants of the representation M of $\underline{G}$.

We say the equivariant bundle $\mathcal{M}$ has central character $\chi : A^* \longrightarrow A^*$ if, for all $m \in M$ and $a \in A^* = Z(A_n)$ we have

$$a(m) = \chi(a) \cdot m. \tag{15.14}$$

For example, the central character of $\mathcal{M} = \mathcal{O}_X$ is equal to the trivial character $\chi = 1$, as Z acts trivially on X.

Let $\mathbb{N} : A_n[\varphi]^* \to A^*$ be the reduced norm homomorphism; the restriction of $\mathbb{N}$ to the center A^* is equal to the character $\chi(a) = a^n$. If k is an integer, let $\mathcal{O}_X(k)$ be the equivariant bundle of rank 1 on X whose representation $R(k)$ is given by the following twisted action of $\underline{G}$ on R:

$$g_k(r) = (\mathbb{N}b)^k \cdot g(r) \qquad g = b \times \sigma^i. \tag{15.15}$$

Then $\mathcal{O}_X(k)$ has central character $\chi(a) = a^{nk}$. More generally, if $\mathcal{M}$ is an equivariant bundle on X we define the twisted bundle

$$\mathcal{M}(k) = \mathcal{M} \otimes \mathcal{O}_X(k). \tag{15.16}$$

Then $\mathrm{rank}\,\mathcal{M}(k) = \mathrm{rank}\,\mathcal{M}$ and $\mathcal{M}(k)(\ell) = \mathcal{M}(k + \ell)$. If $\mathcal{M}$ has a central character, so does $\mathcal{M}(k)$ and

$$\chi_{\mathcal{M}(k)}(a) = \chi_{\mathcal{M}}(a) \cdot a^{nk}. \tag{15.17}$$

Instead of considering all equivariant bundles $\mathcal{M}$ on X, we can restrict to those with trivial central character χ. These may be viewed as PG-equivariant bundles on X, where PG is the projective group.

$$PG = G/Z. \tag{15.18}$$

We may define the A-modules

$$\begin{cases} \mathrm{Ext}^i_{PG}(\mathcal{N}, \mathcal{M}) \\ H^i_{PG}(X, \mathcal{M}) = \mathrm{Ext}^i_{PG}(\mathcal{O}_X, \mathcal{M}) \end{cases} \tag{15.19}$$

as derived functors of Hom_{PG} in the category of $\mathcal{O}_X$-modules with an action of PG.

16. Equivariant bundles: some exact sequences

We now use the universal deformation F over $A[\![\underline{u}]\!]$ to define some natural G-equivariant bundles on X. Consider for example, the free $R = A_n[\![\underline{u}]\!]$ module

$$M = \mathrm{Lie}(F) \underset{A[\![\underline{u}]\!]}{\otimes} A_n[\![\underline{u}]\!] \tag{16.1}$$

of rank 1. This has a natural action of $\langle\sigma\rangle = \mathrm{Aut}(A_n/A)$; we now describe an A_n-linear action of the group $G(A_n) = \mathrm{Aut}_{k_n}(F \otimes k)$ on M.

Recall that for each $b \in G(A_n)$, there is a unique isomorphism

$$k(b) : \beta F \longrightarrow F$$

of formal A-modules over R which reduces to $b \mod (\pi, \underline{u})$. Here $\beta = \beta(b)$ is the automorphism of $A_n[\![\underline{u}]\!]$ studied in (14.5)-(14.7). Consider the composition

$$M \xrightarrow{\beta_*} \mathrm{Lie}(\beta F) \xrightarrow{k(b)_*} M$$

where the first map is given by the base change $\beta : R \to R$. We define

$$b(m) = k(b)_*(\beta_* m) \qquad m \in M. \tag{16.2}$$

Since $k(b)_*$ is R-linear and $\beta_*(rm) = \beta(r) \cdot \beta_*(m)$, this action of $G(A_n)$ satisfies (15.5). Hence there is an equivariant line bundle (= vector bundle of rank 1) $\mathcal{L}\mathrm{ie}(F)$ on X with

$$M = H^0(X \otimes A_n, \mathcal{L}\mathrm{ie}(F) \otimes A_n). \tag{16.3}$$

The central character of $\mathcal{L}\mathrm{ie}(F)$ is given by the formula

$$\chi_{\mathcal{L}\mathrm{ie}(F)}(a) = a. \tag{16.4}$$

Indeed, for $b = a$ in A^* we have $\beta = 1$ and $k(b) = a_F$.

Let E be the universal additive extension of F over $A[\![\underline{u}]\!]$ and let F' be the additive A-module $F' = \mathbb{G}_a \otimes \mathrm{Hom}_R(\mathrm{Ext}(F, \mathbb{G}_a), R)$. We have an exact sequence of free $A[\![\underline{u}]\!]$-modules:

$$0 \longrightarrow \mathrm{Lie}(F') \longrightarrow \mathrm{Lie}(E) \longrightarrow \mathrm{Lie}(F) \longrightarrow 0. \tag{16.5}$$

Similar to the above, once we extend scalars to $R = A_n[\![\underline{u}]\!]$, there is an action of $G(A_n)$ on the free R-modules in this exact sequence. We summarize this result as follows.

PROPOSITION 16.6. *There is an exact sequence of equivariant vector bundles on X:*

$$0 \longrightarrow \mathcal{L}ie(F') \longrightarrow \mathcal{L}ie(E) \longrightarrow \mathcal{L}ie(F) \longrightarrow 0$$

of ranks $n-1, n$, and 1, respectively, which gives the sequence (16.5) of free $A[\![\underline{u}]\!]$-modules on taking global sections. The central character of these bundles is given by: $\chi_{\mathcal{L}ie}(a) = a$. □

Taking the dual of the exact sequence in Proposition 16.6 gives an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \varpi(F) & \longrightarrow & \varpi(E) & \longrightarrow & \varpi(F') & \longrightarrow & 0 \\ & & \| & & \| & & \\ & & \mathcal{R}\mathrm{ig}\mathcal{E}\mathrm{xt}(F, \mathbb{G}_a) & & \mathcal{E}\mathrm{xt}(F, \mathbb{G}_a) & & \end{array}$$

of equivariant bundles on X of ranks $1, n$, and $n-1$, respectively, and central character $\chi_\varpi(a) = a^{-1}$. We may define the action of $G(A_n)$ on the sections over A_n, using the formula:

$$b(m) = (k(b)^{-1})^*(\beta_* m). \tag{16.7}$$

Indeed, the functors $\omega(F)$, $\mathrm{RigExt}(F, \mathbb{G}_a)$, and $\mathrm{Ext}(F, \mathbb{G}_a)$ are contravariant and commute with base change.

If $\mathcal{M}$ is any equivariant vector bundle on X, the fibre $\mathcal{M}_{\underline{0}}$ over $\underline{x} = \underline{0}$ is a free A-module, and the torus $T(A_n) = A_n^*$ which stabilizes $\underline{x} = \underline{0}$ acts on the A_n-module $M_{\underline{0}} = \mathcal{M}_{\underline{0}} \otimes A_n$. We can determine this action on the vector bundles $\mathcal{L}$ie and ϖ, using Proposition 13.8.

PROPOSITION 16.8. *The torus $T(A_n)$ acts on $\mathrm{Lie}(F)_0$ via the identity character $\epsilon(\alpha) = \alpha$ of A_n^*, and acts on $\mathrm{Lie}(E)_0$ via the direct sum of the n distinct characters $\epsilon_i(\alpha) = \sigma^i(\alpha) \quad i = 0, 1, 2, \ldots, n-1$.*

17. The tangent bundle and the canonical line bundle

Let $\mathcal{T}_X$ denote the tangent bundle of the (smooth) formal scheme

$$X = \mathrm{Spf}\, A[\![u_1, \ldots, u_{n-1}]\!]$$

over A. Then $\mathcal{T}_X$ is a vector bundle of rank $n-1$ on X; the sections

$$T = H^0(X \otimes A_n, \mathcal{T}_X \otimes A_n)$$

are in one to one correspondence with the deformations of the universal group F from $R = A_n[\![\underline{u}]\!]$ to $R[\epsilon]/(\epsilon^2)$, up to $\star$-isomorphism. We now define an action of $G(A_n)$ on T, which gives $\mathcal{T}_X$ the structure of an equivariant bundle on X with trivial central character.

For $t \in T$, let

$$\begin{cases} F_t(X,Y) = F(X,Y) + \epsilon G(X,Y) \\ a_{F_t}(X) = a_F(X) + \epsilon g_a(X) \end{cases}$$

be the corresponding deformation. If $b \in G(A_n)$, the deformation $b(t)$ is given by first conjugating the above series by $\beta = \beta(b)$ to obtain a deformation βF, then using the isomorphism $k = k(b) : \beta F \xrightarrow{\sim} F$ to obtain a new deformation of F:

$$\begin{cases} F_{b(t)}(X,Y) = k(\beta F(k^{-1}X, k^{-1}Y) + \epsilon \beta G(k^{-1}X, k^{-1}Y)) \\ a_{F_{b(t)}}(X) = k(\beta a_F(k^{-1}X) + \epsilon \beta g_a(k^{-1}X)). \end{cases} \tag{17.1}$$

PROPOSITION 17.2. *The isomorphism of free R-modules in Proposition 7.2:*

$$\theta : \mathrm{Ext}(F, \mathbb{G}_a) \otimes \mathrm{Lie}(F) \xrightarrow{\sim} T$$

induces an isomorphism of equivariant vector bundles on X:

$$\theta : \mathcal{E}xt(F, \mathbb{G}_a) \otimes \mathcal{L}ie(F) \xrightarrow{\sim} \mathcal{T}_X$$

PROOF. We must check that θ commutes with the action of $G(A_n)$. Recall that θ takes the class $c \otimes D$, where $c = \{\Delta(X,Y), \delta_a(X)\}$ is a symmetric 2-cocycle and $D = h(X)\partial/\partial X$ is an invariant derivation, to the deformation F_t with

$$\begin{cases} G(X,Y) = \Delta(X,Y) \cdot h(F(X,Y)) \\ g_a(X) = \delta_a(X) \cdot h(X) \end{cases}$$

But we have:

$$b(c) = \{\beta\Delta(k^{-1}X, k^{-1}Y), \beta\delta_a(k^{-1}X)\},$$
$$b(D) = k_*(\beta h(X)\partial/\partial X).$$

One now checks directly that $\theta(b(c) \otimes b(D))$ corresponds to the deformation $F_{b(t)}$ of F given by (17.1). □

COROLLARY 17.3. *We have $\mathcal{T}_X \simeq \mathrm{Hom}(\mathcal{L}ie(F'), \mathcal{L}ie(F))$ as equivariant vector bundles on X.*

PROOF. $\mathrm{Hom}(\mathcal{L}\mathrm{ie}(F'), \mathcal{L}\mathrm{ie}(F)) \simeq \mathcal{L}\mathrm{ie}(F')\check{}\otimes \mathcal{L}\mathrm{ie}(F)$. But $\mathcal{L}\mathrm{ie}(F')\check{}$ is isomorphic to $\omega(F')\check{} = \mathcal{E}\mathrm{xt}(F, \mathbb{G}_a)$. □

Let $\Omega_X^1 = (\mathcal{T}_X)\check{}$ be the cotangent bundle of X and for $i = 0, 1, 2, \ldots, n-1$ let

$$\Omega_X^i = \bigwedge^i \Omega_X^1 = \bigwedge^i \mathcal{T}_X\check{} \tag{17.4}$$

be the bundle of i-forms on X. Then rank $\Omega_X^i = \binom{n-1}{i}$. We have $\Omega_X^0 = \mathcal{O}_X$ and the line bundle

$$\Omega_X^{n-1} = \mathcal{K}_X \tag{17.5}$$

is called the canonical bundle of X.

PROPOSITION 17.6. *For $i = 0, 1, \cdots, n-1$ we have an isomorphism of equivariant bundles on X:*

$$\Omega^i_X \simeq \bigwedge^i \mathcal{L}ie(F') \otimes \varpi(F)^{\otimes i}.$$

Moreover,

$$\mathcal{K}_X \simeq \varpi(F)^{\otimes n}(1).$$

The first claim follows immediately from Corollary 17.3 and the definition of Ω^i_X. Indeed

$$\Omega^1_X \simeq \mathcal{L}\mathrm{ie}(F') \otimes \varpi(F) \tag{17.7}$$

and we have the general formula:

$$\bigwedge^i(\mathcal{M} \otimes \mathcal{L}) = \bigwedge^i \mathcal{M} \otimes \mathcal{L}^{\otimes i} \qquad \text{when} \quad \mathrm{rank}(\mathcal{L}) = 1.$$

The exact sequence of Proposition 16.6 gives, after taking top exterior powers, an isomorphism of equivariant line bundles.

$$\bigwedge^{n-1} \mathcal{L}\mathrm{ie}(F') \otimes \mathcal{L}\mathrm{ie}(F) \simeq \bigwedge^{n} \mathcal{L}\mathrm{ie}(E).$$

In §22 we will show that there is an isomorphism (Proposition 22.4):

$$\bigwedge^n \mathcal{L}\mathrm{ie}(E) \simeq \mathcal{O}_X(1). \tag{17.8}$$

This gives the final formula for $\mathcal{K}_X$.

COROLLARY 17.9. *There is an exact sequence of equivariant bundles on X with trivial central character:*

$$0 \longrightarrow \Omega^1_X \longrightarrow \mathcal{L}ie(E) \otimes \varpi(F) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

PROOF. This is the tensor product of the sequence in Proposition 16.6 with the line bundle $\varpi(F)$. □

The exact sequence of Corollary 17.9 gives a class

$$e = \delta(1) \quad \text{in} \quad H^1_{PG}(X, \Omega^1_X) = \mathrm{Ext}^1_{PG}(\mathcal{O}_X, \Omega^1_X). \tag{17.10}$$

The wedge product of differential forms gives a G-homomorphism

$$\Omega^i_X \otimes \Omega^j_X \longrightarrow \Omega^{i+j}_X, \tag{17.11}$$

which induces a cup-product in cohomology:

$$H^p_{PG}(X, \Omega^i_X) \otimes H^q_{PG}(X, \Omega^j_X) \longrightarrow H^{p+q}_{PG}(X, \Omega^{i+j}_X). \tag{17.12}$$

Using this map, we may define the class

$$e^i \quad \text{in} \quad H^i_{PG}(X, \Omega^i_X) \qquad i = 0, 1, \dots, n-1. \tag{17.13}$$

Part IV. Rigid analytic bundles

18. Rigid analytic spaces

The general fibre $X \otimes K$ of the formal scheme X over A has the structure of a rigid analytic space over K. If R is a flat, local A-algebra we have

$$(X \otimes K)(R \otimes K) = X(R) = P^{n-1}. \tag{18.1}$$

Applying this to $R = \bar{A}$, the integral closure of A in an algebraic closure $\bar{K}$ of K, we have $P = \bar{m} = \{ x \in \bar{K} : |x| < 1 \}$. Here $|x| = q^{-\mathrm{ord}_\pi(x)}$ is the normalized valuation of K, extended uniquely to $\bar{K}$. Hence $X \otimes K$ is the $(n-1)$-dimensional open unit polydisc:

$$(X \otimes K)(\bar{K}) = \{ \underline{x} = (x_1, \cdots, x_{n-1}) \in \bar{K}^{n-1} : \ |x_i| < 1 \ \text{ all } i \}. \tag{18.2}$$

The K-algebra of rigid analytic functions on $X \otimes K$ consists of those power series:

$$\varphi(\underline{u}) = \sum a_{j_1 \cdots j_{n-1}} u_1^{j_1} u_2^{j_2} \cdots u_n^{j_n} = \sum a_J \underline{u}^J \tag{18.3}$$

with coefficients $a_J \in K$ which converge on the open unit polydisc. The condition on the coefficients a_J of φ which is equivalent to convergence is:

$$\lim_{j_1 + j_2 + \cdots + j_{n-1} \to \infty} |a_{j_1 j_2 \cdots j_{n-1}}| \cdot r_1^{j_1} \cdots r_{n-1}^{j_n} = 0 \tag{18.4}$$

whenever r_i are fixed real numbers with $0 \le r_i < 1$ for all i.

We denote the K-algebra of rigid analytic functions on $X \otimes K$ by

$$V = K\{\{\underline{u}\}\}.$$

This contains the A-algebra $M = A[\![\underline{u}]\!]$ of integral power series, which are the formal functions on X. It also contains the K-algebra $M \otimes K$ of power series whose coefficients a_J are bounded in absolute value, and

$$\|\varphi\| = \sup_J \{ |a_J| \} \tag{18.5}$$

defines a norm on the vector space $M \otimes K$. The larger space V has the structure of a topological K-algebra, which we will describe below.

If the series $\varphi(\underline{u})$ lies in V, we may evaluate φ on points $\underline{x} \in (\bar{m})^{n-1}$ in the open polydisc. Fix such a point $\underline{x} = (x_1, \dots, x_{n-1})$ with $|x_i| < 1$ for all i. The resulting homomorphism

$$\rho(\underline{x}) : V \to L = K(\underline{x})$$
$$\varphi \mapsto \varphi(\underline{x})$$

is surjective; its kernel $I(\underline{x})$ is a maximal ideal of V (which depends only on the orbit of $\underline{x}$ under $\mathrm{Aut}_K(\bar{K})$). We now show how the maximal ideals $I(\underline{x})$ can be used to determine membership in M or $M \otimes K$, as well as the units in the K-algebra V.

PROPOSITION 18.6. *Assume* φ *is in* $V = K\{\{\underline{u}\}\}$. *Then*

i) φ *is in* $M = A[\![\underline{u}]\!]$ *iff* $|\varphi(\underline{x})| \leq 1$ *for all* $\underline{x} \in (\bar{m})^{n-1}$.

ii) φ *is in* $M \otimes K$ *iff* $|\varphi(\underline{x})| \leq q^N$ *for all* $\underline{x} \in (\bar{m})^{n-1}$, *where* N *is an integer depending only on* φ.

PROOF. One has the identity

$$\sup_J\{\,|a_J|\,\} = \sup_{\underline{x}}\{\,\varphi(\underline{x})\,\} \qquad \text{in } \mathbb{R}\cup\{\,\infty\,\},$$

where $\varphi(\underline{u}) = \sum a_J \underline{u}^J$. For a proof, see [**BGR84**, 5.1.4]. □

PROPOSITION 18.7. *Assume* $\varphi(\underline{u}) = \sum a_J \underline{u}^J$ *is a non-zero function in* $V = K\{\{\underline{u}\}\}$. *Then the following conditions are all equivalent:*

i) We have $\operatorname{ord}_\pi(a_0) \leq \operatorname{ord}_\pi(a_J)$ *for all* J.

ii) The function $\varphi\cdot\pi^{-N}$ *is a unit in* $M = A[\![\underline{u}]\!]$ *for some integer* N.

iii) We have $|\varphi(\underline{x})| = q^N$ *for all* $\underline{x} \in (\bar{m})^{n-1}$, *where* N *is an integer which depends only on* φ.

iv) The function φ *is a unit in* $M \otimes K$.

v) The function φ *is a unit in* V.

vi) We have $\varphi(\underline{x}) \neq 0$ *for all* $\underline{x} \in (\bar{m})^{n-1}$.

PROOF. If $\operatorname{ord}(a_0) = N \leq \operatorname{ord}(a_J)$ for all J, then φ/π^N is a unit in $A[\![\underline{u}]\!]$ by the formal inverse function theorem. This, in turn, implies that $|\varphi(\underline{x})/\pi^N| = 1$ for all $\underline{x}$, so $|\varphi(\underline{x})| = q^N$ for all $\underline{x}$. By Proposition 18.6, this shows φ is a unit in $M \otimes K$, so it is certainly a unit in V and hence $\varphi(\underline{x}) \neq 0$ for all $\underline{x}$. But if $\operatorname{ord}(a_0) > \operatorname{ord}(a_J)$ for any J, the theory of Newton polygons shows that φ has a zero in $(\bar{m})^{n-1}$. This completes the proof. □

COROLLARY 18.8. *i) Every unit* φ *of the* K*-algebra* $V = K\{\{\underline{u}\}\}$ *has the form* $\pi^k \cdot \varphi_M$, *where* φ_M *is a unit in* $M = A[\![\underline{u}]\!]$, *and* $k = \operatorname{ord}_\pi a_0 = \operatorname{ord}_\pi \varphi(\underline{0})$.

ii) $V = K\{\{\underline{u}\}\}$ *is a faithfully flat* $M \otimes K = A[\![\underline{u}]\!] \otimes K$ *algebra.*

Remark. Although the maximal ideals $I(\underline{x})$ defined in (18) are sufficient to detect membership in M and $M \otimes K$, as well as the units in V, they do not exhaust the set of all maximal ideals of V. For example, take $n = 2$ so $V = K\{\{u_1\}\}$. Let $S = \{\,x_1, x_2, \ldots\,\}$ be an infinite sequence of points in $\bar{m}$ with $\lim_{j\to\infty}|x_j| = 1$, and assume S is stable under $\operatorname{Aut}_K(\bar{K})$. Let I be the ideal of V consisting of functions φ which vanish at all but a finite number of points of S. Then I is non-zero, but is not contained in any maximal ideal $I(\underline{x})$ of the type (18).

We now give V the structure of a Fréchet algebra (= inverse limit of Banach algebras) over K. To do this, we introduce the affinoid subsets of $X \otimes K$:

$$\text{(18.9)}\qquad \begin{aligned} Y(e) &= \{\,\underline{x} \in (\bar{K})^{n-1} : |x_i| \leq q^{-1/e}\,\} \\ &= \{\,\underline{x} \in (\bar{K})^{n-1} : \operatorname{ord}_\pi(x_i) \geq \frac{1}{e}\,\}. \end{aligned}$$

Here $e \geq 1$ is an integer; we have obvious inclusions

$$Y(1) \hookrightarrow Y(2) \hookrightarrow Y(3) \hookrightarrow \cdots$$

and

$$X \otimes K = \varinjlim_e Y(e) = \bigcup_{e\geq 1} Y(e).$$

The K-algebra $V(e)$ of rigid analytic functions on the closed polydisc $Y(e)$ consists of those series $\varphi(\underline{u}) = \sum a_J \underline{u}^J$ which satisfy

$$\text{(18.10)}\qquad \lim_J |a_J|/q^{\frac{j_1+j_2+\cdots+j_{n-1}}{e}} = 0.$$

This is a Banach algebra over K, with norm [**BGR84**, §5.1].

$$\text{(18.11)}\qquad \begin{aligned} \|\varphi\|_e &= \sup_J\{\,|a_J|/q^{\frac{j_1+j_2+\cdots+j_{n-1}}{e}}\,\} \\ &= \sup_{\underline{x}\in Y(e)}\{\,|\varphi(\underline{x})|\,\}. \end{aligned}$$

We let $M(e) = \{\,\varphi \in V(e) : \|\varphi\|_e \leq 1\,\}$ be the unit ball in this Banach space; then $M(e)$ is an algebra over A.

Let $e \geq e'$, so $Y(e') \hookrightarrow Y(e)$. The restriction of rigid analytic functions gives a homomorphism

$$V(e) \longrightarrow V(e')$$

of K-algebras, which is injective and completely continuous with respect to the norms $\|\varphi\|_e$ and $\|\varphi\|_{e'}$ [**Ser62**, p. 185]. Since

$$\text{(18.12)}\qquad \begin{cases} V = \varprojlim_e V(e) = \bigcap_{e\geq 1} V(e) \\ M = \varprojlim_e M(e) = \bigcap_{e\geq 1} M(e) \end{cases}$$

we obtain on V the structure of a Fréchet algebra. Specifically, a sequence $\{\,\varphi_n\,\}$ of functions in V converges to the function φ in V if and only if $\varphi_n \longrightarrow \varphi$ in each $V(e)$. Since the metric topology on $V(e)$ is defined by the sup norm (18.11), this notion of convergence is the usual one of "uniform convergence on compacta". Thus $\varphi_n \to \varphi$ in V iff

$$\text{(18.13)}\qquad \begin{cases} \forall\epsilon > 0, \forall e \geq 1 \exists N(\epsilon, e) : \forall n \geq N(\epsilon, e) \quad \text{and } \underline{x} \in Y(e) \\ \qquad |\varphi_n(\underline{x}) - \varphi(\underline{x})| < \epsilon. \end{cases}$$

Remark. Although V is a complete metric space, the topology of V is not given by a norm. In particular, M is not the unit ball in V. It is the unit ball in the Banach space $M \otimes K$, which is the continuous linear dual of the space of rigid analytic functions on the closed unit polydisc [**Ser62**, p. 172].

19. The group action: continuity

The action of the group scheme G on the formal scheme X extends to an action on its general fibre $X \otimes K$. If $\varphi = \varphi(\underline{u})$ is a rigid function in the space $V_n = V \otimes_K K_n$ and b an element of $G(A_n)$, we have the formula

$$b\varphi(\underline{u}) = \varphi(bu_1, bu_2, \dots, bu_{n-1}). \tag{19.1}$$

The group $\langle\sigma\rangle = \mathrm{Gal}(K_n/K)$ also acts on $V \otimes K_n$ by conjugation, and, as before, this normalizes the action of $G(A_n)$. Hence the group $\underline{G} = G(A_n) \rtimes \langle\sigma\rangle$ acts K-linearly on the left of the K_n-algebra V_n.

Recall that $G(A_n)$ is the group of units in the order

$$B_n = A_n \oplus A_n\varphi \oplus \cdots \oplus A_n\varphi^{n-1} = \mathrm{End}_{k_n}(F \otimes k)$$

For $N \geq 1$ the subgroups $1+\varphi^N B_n$ are normal of finite index in $\underline{G}$, and $\bigcap_{N\geq 1}(1+\varphi^N B_n) = 1$. Taking these subgroups as a basis for the open neighborhoods of the identity gives $\underline{G}$ the structure of a profinite topological group.

PROPOSITION 19.2. *The action of the topological group $\underline{G}$ on the rigid analytic space $X \otimes K_n$ over K is continuous.*

PROOF. This is equivalent to the claim that the map

$$\underline{G} \times V_n \to V_n \qquad (g, \varphi) \mapsto g\varphi$$

of topological spaces is continuous. Since the series bu_i in (19.1) have coefficients in A_n, the action of G on $X \otimes K$ stabilizes the affinoid subdomains $Y(e)$ of (18.9), for all $e \geq 1$. But V is topologized as the inverse limit of the spaces $V(e)$, so we must show the map

$$\underline{G} \times V_n(e) \longrightarrow V_n(e)$$

is continuous for all e, where $V(e)_n = V(e) \otimes_K K_n$. This follows from the following more precise result. □

LEMMA 19.3. *Let $\underline{x} = (x_1, \dots, x_{n-1})$ be a point in $Y(e)(\bar{K})$, so $\mathrm{ord}_\pi(x_i) \geq \frac{1}{e}$ for all i. Assume b lies in the subgroup $1 + \pi^N B_n$ of $\underline{G}$, and write $\underline{y} = \underline{x}b = (y_1, \dots, y_{n-1})$. Then*

$$\mathrm{ord}_\pi(x_i - y_i) \geq \frac{(N+1)}{e} \qquad \text{for all } i.$$

PROOF. We use induction on N. The result is true for $N = 0$, where $1 + \pi^N B_n = B_n^* = G(A_n)$, as $\underline{x}b$ lies in $Y(e)(\bar{K})$. Next assume $N \geq 1$. Let α be an element in $\bar{K}$ with $\mathrm{ord}_\pi(\alpha) = \frac{1}{e}$ and consider the deformations $F_{\underline{x}}$ and $F_{\underline{y}}$ of $F \otimes \bar{k}$ over the local A-algebra $R = \bar{A}/(\alpha)^{N+1}\bar{A}$. We must show they are $\star$-isomorphic, so $\underline{x} \equiv \underline{y} \bmod (\alpha^{N+1}\bar{A})$. This is equivalent, by Proposition 14.13, to showing that $\mathrm{End}_R(F_{\underline{x}})$ contains the subring $A + \pi^N B_n$ of $B_n = \mathrm{End}_{\bar{k}}(F \otimes \bar{k})$.

Let I be the ideal $\alpha^N \bar{A}/(\alpha)^{N+1}\bar{A}$ of R. By induction $\mathrm{End}_{R/I}(F_{\underline{x}})$ contains the subring $A + \pi^{N-1}B_n$ of B_n. But $I^2 = 0$, so [**LT66**, Prop. 2.4] shows that the obstruction to lifting an element $f(X)$ in $\mathrm{End}_{R/I}(F_{\underline{x}})$ to $\mathrm{End}_R(F_{\underline{x}})$ lies in:

$$H^2(F_{\underline{x}} \otimes R, \mathbb{G}_a \otimes I)_s = H^2(F_{\underline{x}} \otimes R/I, \mathbb{G}_a \otimes I)_s. \tag{19.4}$$

Since $\pi I = 0$, the A-module of symmetric 2-cohomology with coefficients in I is annihilated by π. Hence the endomorphisms in $\pi \cdot (\pi^{N-1}B_n) = \pi^N B_n$ lift to R, and $A + \pi^N B_n$ is contained in $\mathrm{End}_R(F_{\underline{x}})$. □

20. Flat bundles; rigid equivariant bundles

Let $\mathcal{M}$ be an equivariant bundle of rank m on X. Then the sections

$$M = H^0(X \otimes A_n, \mathcal{M} \otimes A_n) \tag{20.1}$$

form a free $R = A_n[\![\underline{u}]\!] = H^0(X \otimes A_n, \mathcal{O}_X \otimes A_n)$ module of rank m. The group $\underline{G} = G(A_n) \rtimes \mathrm{Aut}_A(A_n)$ acts A-linearly on the left of M. We say $\mathcal{M}$ is *flat* iff there is an A-linear representation M_0 of $\underline{G}$ on a free A_n-module of rank m and an isomorphism

$$M \simeq R \otimes_{A_n} M_0 \tag{20.2}$$

of $R[\underline{G}]$-modules. (The group $\underline{G}$ acts diagonally on $R \otimes M_0$). Equivalently, $\mathcal{M}$ is flat if M has a basis $\langle e_1, e_2, \cdots, e_m\rangle$ over R such that the A_n-module M_0 spanned by the e_i is $\underline{G}$-stable.

For example, the equivariant line bundle $\mathcal{M} = \mathcal{O}_X(k)$ defined in (15.14) is flat: $M_0 = A_n(k)$ is given by the k^{th} power of the reduced norm character of $G(A_n) = B_n^*$. This example is typical of flat line bundles.

PROPOSITION 20.3. *The flat line bundles on X are determined by the action of $A_n^\times$ on the fibre of the canonical lift. Restriction to this fiber defines an isomorphism between the abelian group of flat line bundles and the subgroup*

$$\mathrm{Hom}\left((1+\pi A)^\times, (1+\pi A)^\times\right) \times \mathrm{Hom}\left((A_n/\pi)^\times, (A_n/\pi)^\times\right) \tag{20.4}$$

of $\mathrm{Hom}(A_n^\times, A_n^\times)$ generated by the reduced norm and the map

$$a \mapsto \langle a\rangle,$$

where

$$\langle a\rangle \equiv a \mod \pi$$

is the "Teichmuller" representative of a modulo π. *For an element* (λ, ρ) *of* (20.4) *we have* $M_0 = A_n(\lambda, \rho) = A_n \cdot e$ *where*

$$\begin{cases} \sigma e = e \\ be = \lambda\left(\frac{\mathrm{N}b}{\langle \mathrm{N}b\rangle}\right)\rho(\bar{b}) \cdot e \qquad b \in G(A_n), \end{cases}$$

with $\bar{b}$ *the reduction of* b *mod* φ. *The corresponding flat line bundle* $\mathcal{O}_X(\lambda, \rho)$ *has central character*

$$\chi(a) = \lambda^n\left(\frac{a}{\langle a\rangle}\right)\rho(\bar{a}).$$

PROOF. If $\mathcal{M}$ is a flat line bundle, then $M_0 = A_n \cdot e$. We may assume e lies in the rank 1 A-module $M_0^{(\sigma)} = H^0(X, \mathcal{M})$. Then $\sigma e = e$ and $be = \alpha(b) \cdot e$ with $\alpha(b) \in A_n^*$. Thus α is a homomorphism $\alpha : B_n^* \longrightarrow A_n^*$. Since the image of α is commutative, α is trivial on the commutator subgroup of B_n^*, which is equal to the intersection of the kernel of the reduced norm with $(1 + \phi B_n)^\times$. The result now follows. □

On the other hand, not all line bundles on X are flat. For example, the equivariant bundle $\mathcal{L}\text{ie}(F)$ is not flat provided $n \geq 2$. Indeed, the central character $\chi_{\mathcal{L}\text{ie}}(a) = a$ of this bundle is not of the form λ^n for $\lambda \in \text{Hom}(A^*, A^*)$ once $n \geq 2$. More generally, the equivariant line bundle $\mathcal{M} = \mathcal{L}\text{ie}(F)^{\otimes k}$ is not flat, provided $n \geq 2$ and $k \neq 0$. Indeed, the torus $T(A_n) = A_n^*$ acts on the fibre $M_{\underline{0}}$ by the character $\alpha \longmapsto \alpha^k$ by Proposition 16.8, but acts by the character $\lambda(\mathrm{N}\alpha)$ on the fibre at $\underline{0}$ of the flat bundle $\mathcal{O}_X(\lambda)$.

We now return to general equivariant bundles $\mathcal{M}$ on X. We let $\mathcal{M} \otimes K$ denote the general fibre of $\mathcal{M}$, which is a rigid equivariant bundle on the rigid analytic G-space $X \otimes K$. The sections

$$V = H^0(X \otimes K_n, \mathcal{M} \otimes K_n) \tag{20.5}$$

form a free $S = K_n\{\{\underline{u}\}\} = H^0(X \otimes K_n, \mathcal{O}_X \otimes K_n)$ module of rank $m = \text{rank}(\mathcal{M})$, and the group $\underline{G}$ acts K-linearly on the left of V. We have

$$V = S \otimes_R M \tag{20.6}$$

where $\underline{G}$ acts diagonally on the tensor product; this isomorphism follows from the fact that X is affine.

Notice that we have changed our notation slightly from the previous section – where $S = K_n\{\{\underline{u}\}\}$ was denoted V_n. The results of §19 shows that V has the structure of a Fréchet space over K_n. For the natural bundles $\mathcal{L}\text{ie}(F)$, $\mathcal{L}\text{ie}(E)$, $\mathcal{T}_X, \Omega^i_X$, etc., considered in §§16-17 one can imitate the method of §19 to show that the action of $\underline{G}$ on V is continuous. However, this is not true in all cases: for example, when $\mathcal{M} = \mathcal{O}_X(\lambda)$ is a flat line bundle, the action of $\underline{G}$ on V is continuous if and only if the character $\lambda : A^* \rightarrow A^*$ is continuous.

We say the bundle $\mathcal{M}$ is *generically flat* if $\mathcal{M} \otimes K$ is flat over $X \otimes K$. By this we mean the following: there is a representation $KM_0 = K_n \otimes M_0$ of $\underline{G}$ on a free K_n-module of rank m and an isomorphism

$$V \simeq S \otimes_{K_n} KM_0 \tag{20.7}$$

of $S[\underline{G}]$-modules. Equivalently, V has a basis $\langle e_1, \ldots, e_m\rangle$ over S such that the K_n-module spanned by the e_i is $\underline{G}$-stable.

A generically flat bundle $\mathcal{M}$ need not be flat: in the next section we will see that the bundle $\mathcal{L}\text{ie}(E)$ of rank n on X gives such an example, once $n \geq 2$. In this case, M_0 is the left regular representation of $G(A_n) = B_n^*$ on $B_n = A_n \oplus \varphi A_n \oplus \cdots \oplus \varphi^{n-1} A_n$. If $\mathcal{M}$ has rank m and is generically flat, let $\langle e_1, \cdots, e_m\rangle$ be a basis for KM_0 over K_n and let $\langle e_1', \ldots, e_m'\rangle$ be a basis for M over R. Since both give bases for V over S, there is a matrix $A \in \text{GL}_m(S)$ which transforms e_i' into e_i. The matrix A is well-determined up to right multiplication by $\text{GL}_m(R)$ and left multiplication by $\text{GL}_m(K_n)$, hence gives a class

$$[A] \in \text{GL}_m(K_n)\backslash\text{GL}_m(S)/\text{GL}_m(R). \tag{20.8}$$

If $[A] = [1]$ in this double coset space, the bundle $\mathcal{M}$ is flat over X. This certainly occurs when $m = 1$, i.e., when $\mathcal{M}$ is a line bundle, as by Corollary 18.8 we have: $S^* = K_n^* \cdot R^* = \pi^{\mathbb{Z}} \times R^*$. Hence

PROPOSITION 20.9. *If* $\mathcal{M}$ *is an equivariant line bundle on* X, *the following are all equivalent:*

i) $\mathcal{M}$ *is flat.*

ii) $\mathcal{M}$ *is generically flat.*

iii) $\mathcal{M} \simeq \mathcal{O}_X(\lambda, \rho)$ *for some element* (λ, ρ) *in the subgroup* (20.4) *of*

$$\text{Hom}(A_n^\times, A_n^\times).$$

21. The bundle $\mathcal{L}\text{ie}(E)$ is generically flat

Let

$$V = S \otimes_R \text{Lie}(E) = \text{Hom}_R(\text{RigExt}(F, \mathbb{G}_a), S) \tag{21.1}$$

be the sections of the equivariant bundle $\mathcal{L}\text{ie}(E) \otimes K_n$ over the space $X \otimes K_n$. In this section we will construct a basis of "flat" sections $\langle c_0, c_1, \ldots, c_{n-1}\rangle$ of V. In the next section we will show that the A_n-submodule spanned by the c_i is $\underline{G}$-stable.

The sections c_i are described by a limit procedure, using the coefficients of quasi-logarithms representing classes in $\text{RigExt}(F, \mathbb{G}_a)$.

PROPOSITION 21.2. *Let F be the universal deformation of height n over*

$$A[\![u_1, \cdots, u_{n-1}]\!],$$

let $g(X) = \sum m_k X^{q^k}$ be a quasi-logarithm of F, and let $i \geq 0$ be a fixed integer. Then the sequence

$$a_k = \pi^k \cdot m_{kn+i} \qquad k = 1, 2, 3, \ldots$$

of elements in $A[\![\underline{u}]\!] \otimes K$ converges to an element $a = \lim_{k\to\infty} a_k$ in $K\{\{\underline{u}\}\}$. The limit a depends only on the class of g in $\mathrm{RigExt}(F, \mathbb{G}_a)$.

PROOF. It suffices to show the sequence $\{a_k\}$ in Cauchy, as $K\{\{\underline{u}\}\}$ is a complete metric space. By our description of the topology on $K\{\{\underline{u}\}\}$ given in §18, we must show the sequence $\{a_k(\underline{x})\}$ is uniformly Cauchy on each affinoid subdomain $\underline{x} \in Y(e)$. Let $N = N(e, q)$ be the smallest non-negative integer such that

$$\frac{q^i}{e} - i + N \geq 0 \qquad \text{for all } i \geq 1. \tag{21.3}$$

If $e \leq q$ we may take $N = 0$; if $e \geq q + 1$ we have $N \geq 1$. We will show that

$$a_k(\underline{x}) \equiv a_{k+1}(\underline{x}) \qquad \bmod \pi^{k-N} A[\underline{x}] \tag{21.4}$$

for all points $\underline{x} \in Y(e)$, thereby establishing the claim.

Assume $\mathrm{char}(A) = p$ for simplicity (the argument is essentially the same in the general case). Write

$$\pi_F(X) = \sum_{k\geq 0} \alpha_k X^{q^k}.$$

Then $\alpha_0 = \pi, \alpha_i \in P = (\pi, u_1, \cdots, u_{n-1})$ for $i \neq n$, and $\alpha_n \equiv 1 \mod P$. Since $g(X)$ is a quasi-logarithm of F, the series

$$-\delta_\pi g = g(\pi_F X) - \pi g(X)$$

is integral (i.e., has coefficients in the subring $A[\![\underline{u}]\!]$ of $A[\![\underline{u}]\!] \otimes K$). But the coefficient of $X^{q^{n+j}}$ is equal to

$$\sum_{i=0}^{n+j} m_i \cdot \alpha_{n+j-i}^{q^i} - \pi m_{n+j}, \tag{21.5}$$

so the function in (21.5) lies in $A[\![\underline{u}]\!]$. We now specialize to an $\underline{x} \in Y(e)$. Since $\pi^i m_i \in A[\![\underline{u}]\!]$ by Proposition 8.12, we have

$$\mathrm{ord}_\pi(m_i(\underline{x})\alpha_{n+j-i}^{q^i}(\underline{x})) \geq \frac{q^i}{e} - i$$

for all $i \neq j$. Hence, by (21.3) we have:

$$m_j(\underline{x})\alpha_n(\underline{x})^{q^j} \equiv \pi m_{n+j}(\underline{x}) \qquad (\bmod \pi^{-N} A[\underline{x}]).$$

But $\alpha_n(\underline{x}) = 1 + \beta$ with $\mathrm{ord}_\pi(\beta) \geq \frac{1}{e}$. Hence

$$m_j(\underline{x})\alpha_n(\underline{x})^{q^j} \equiv m_j(\underline{x}) \qquad (\bmod \pi^{-N} A[\underline{x}]).$$

Combining the previous two congruences, taking $j = kn + i$, and multiplying by π^k we obtain

$$\pi^k m_{kn+i}(\underline{x}) \equiv \pi^{k+1} m_{(k+1)n+i}(\underline{x}) \qquad (\bmod \pi^{k-N} A[\underline{x}]).$$

for all $\underline{x} \in Y(e)$. This gives (21.4). The limit $a = \lim a_k$ depends only on the class of g, as $\lim_{k\to\infty} \pi^k m_{kn+i} = 0$ if $m_{kn+i} \in A[\![\underline{u}]\!]$ for all k. □

We now define the elements c_i in $\mathrm{Hom}_R(\mathrm{RigExt}(F, \mathbb{G}_a), S)$. Let $g(X) = \sum m_k X^{q^k}$ be a quasi-logarithm on F over $R = A_n[\![\underline{u}]\!]$. Then

$$\begin{cases} c_0(g) = \lim_{k\to\infty} \pi^k m_{kn} \\ c_i(g) = \lim_{k\to\infty} \pi^{k+1} m_{kn+i} & i = 1, 2, \cdots, n-1 \end{cases} \tag{21.6}$$

The limits exist in $S = K_n\{\{\underline{u}\}\}$ by Proposition 21.2. The specific constants are chosen to give a simple specialization at $\underline{x} = \underline{0}$. Indeed, if $\langle g_0, g_1, \cdots, g_{n-1}\rangle$ is the basis for $\mathrm{RigExt}(F, \mathbb{G}_a)$ over R given by §9, we have

$$c_i(g_j)(\underline{0}) = \delta_{ij}. \tag{21.7}$$

This follows directly from a calculation of the quasi-logarithms $g_j(\underline{0}) = f_j$ on $F_{\underline{0}}$, given by Proposition 13.8.

PROPOSITION 21.8. *The elements $\langle c_0, c_1, \cdots, c_{n-1}\rangle$ defined by (21.6) give a basis for* $\mathrm{Lie}(E) \otimes S$ *over S.*

PROOF. Let T be the $n \times n$ matrix with entries in S

$$T = (c_i(g_j)) \tag{21.9}$$

where g_j is the standard basis for $\mathrm{RigExt}(F, \mathbb{G}_a)$ over R. The elements c_i form a basis of $\mathrm{Hom}_R(\mathrm{RigExt}(F, \mathbb{G}_a), S)$ if and only if the matrix T is invertible over S. Thus we must show the determinant $\det T$ is a unit in S, or equivalently, by Proposition 18.7, that

$$\det T(\underline{x}) \neq 0 \qquad \text{for all } \underline{x} \in (\bar{m})^{n-1}. \tag{21.10}$$

But $\det T(\underline{x}) \neq 0$ if and only if the linear map

$$\begin{aligned} c(\underline{x}) : \mathrm{RigExt}(F, \mathbb{G}_a) \otimes K &\to K(\underline{x})^n \\ g(\underline{x}) &\mapsto (\cdots, c_i(g(\underline{x})), \ldots) \end{aligned} \tag{21.11}$$

is an isomorphism. Since $c(\underline{x})$ is a map between two $K(\underline{x})$ vector spaces of dimension n, it suffices to show that $c(\underline{x})$ is an injection. This follows from Lemma 21.12 below. □

LEMMA 21.12. *Let $g(\underline{x}) = \sum m_k(\underline{x})X^{q^k}$ be a quasi-logarithm on $F_{\underline{x}}$ with $c_i(g(\underline{x})) = 0$ for all i. Then the coefficients $m_k(\underline{x})$ of $g(\underline{x})$ lie in $A[\underline{x}]$, so $g(\underline{x})$ represents the trivial class in* $\mathrm{RigExt}(F, \mathbb{G}_a)$.

PROOF. Assume $\underline{x} \in Y(e)$ and let $N = N(e, q)$ as defined in (21.3). By the proof of Proposition 21.2 we have the congruence:

$$m_j(\underline{x}) \equiv \pi m_{n+j}(\underline{x}) \qquad \bmod \pi^{-N} A[\underline{x}].$$

If $\mathrm{ord}_\pi(m_j(\underline{x})) < -N$, then this shows that

$$\mathrm{ord}_\pi(m_{n+j}(\underline{x})) = \mathrm{ord}_\pi(m_n(\underline{x})) - 1,$$

and hence $c_i(g(\underline{x})) \neq 0$ for $i \equiv j \mod n$. Hence the hypothesis that $c_i(g(\underline{x})) = 0$ for all i implies that

$$\mathrm{ord}_\pi(m_j(\underline{x})) \geq -N \qquad \text{for all } j.$$

If any coefficient of $g(\underline{x})$ has negative valuation, the above argument shows we may find a coefficient $m_j(\underline{x})$ with *minimal* valuation < 0. But (21.5) then shows that $\mathrm{ord}(\pi m_{n+j}(\underline{x})) = \mathrm{ord}(m_j(\underline{x}))$, which contradicts the minimality. Hence

$$\mathrm{ord}_\pi(m_j(\underline{x})) \geq 0 \qquad \text{for all } j,$$

and $g(\underline{x}) \equiv 0$ in $\mathrm{RigExt}(F, \mathbb{G}_a)$. □

Remark. When $\underline{x}$ lies in $Y(e)$ with $e \leq q$, so the maximal ideal I of $A[\underline{x}]$ has divided powers, one can show that the isomorphism $c(\underline{x})$ of (21.11) identifies the lattice $\mathrm{RigExt}(F, \mathbb{G}_a)$ with the lattice $A[\underline{x}]^n$. When $e \geq q+1$ the elements $c_i(g(\underline{x}))$ may have denominators.

We want to make the functions $c_i(g_j)$ in the matrix T of (21.9) more explicit. For $i = 0, 1, 2, \ldots, n-1$ let

$$\phi_i(\underline{u}) = c_i(g_0) \qquad \text{in } K\{\{\underline{u}\}\}. \tag{21.13}$$

We recall that $g_0 = f(\underline{u})$ is the logarithm of the universal deformation $F = F(\underline{u})$ over X. Since the limits defining the elements c_i commute with the derivations $D_j = \partial/\partial u_j$ used to define the quasi-logarithms $g_j = D_j g_0$, we find

$$D_j\phi_i(\underline{u}) = c_i(g_j) \qquad j = 1, 2, \ldots, n-1. \tag{21.14}$$

Thus the matrix T has the form:

$$T = \begin{pmatrix} \phi_0 & D_1\phi_0 & \cdots & D_{n-1}\phi_0 \\ \phi_1 & D_1\phi_1 & \cdots & D_{n-1}\phi_1 \\ & & \cdots & \\ \phi_{n-1} & D_1\phi_{n-1} & \cdots & D_{n-1}\phi_{n-1} \end{pmatrix}. \tag{21.15}$$

By formula (21.7) we have the specialization

$$T(\underline{0}) = I = \begin{pmatrix} 1\,0 \cdots 0 \\ 0\,1 \cdots 0 \\ \\ 0\,0 \cdots 1 \end{pmatrix}. \tag{21.16}$$

COROLLARY 21.17. *The element $\epsilon = \det T$ is a unit in $A[\![\underline{u}]\!]$ with $\epsilon(\underline{0}) = 1$.*

PROOF. We have already shown that ϵ is a unit in $K\{\{\underline{u}\}\}$, and (21.16) shows that $\epsilon(\underline{0}) = 1$. The fact that ϵ is a unit in $A[\![\underline{u}]\!]$ now follows from Proposition 18.7. □

COROLLARY 21.18. *The wedge product $c = c_0 \wedge c_1 \wedge \cdots \wedge c_{n-1}$ lies in $\bigwedge^n \mathrm{Lie}(E)$ and gives a basis for that rank 1 module over R.*

PROOF. Clearly c lies in $\bigwedge^n \mathrm{Lie}(E) \otimes S$. To verify that it is integral and a basis, we evaluate it on the basis vector $g = g_0 \wedge g_1 \wedge \cdots \wedge g_{n-1}$ of $\bigwedge^n \omega(E)$ over R. But

$$c(g) = \epsilon = \det T$$

is a unit in R. □

22. The group action; crystals and connections

Our aim in this section is to complete the proof that the vector bundle $Lie(E)$ is generically flat, by showing that the submodule

$$M_0 = A_n c_0 \oplus A_n c_1 \oplus \cdots \oplus A_n c_{n-1} \tag{22.1}$$

of V is $\underline{G}$-stable.

Clearly M_0 is stable under the action of $\langle\sigma\rangle = \mathrm{Aut}_A(A_n)$, as the classes c_i are rational over $K\{\{\underline{u}\}\}$. To show M_0 is stable under the action of $G(A_n)$, we need a generalization of Proposition 10.3.

PROPOSITION 22.2. *Let F and F' be A-typical formal A-modules over $R = A_n[\![\underline{u}]\!]$ with $\pi_F \equiv \pi_{F'} \equiv X^{q^n} \mod P$. Let $\varphi : F \to F'$ be a homomorphism of formal A-modules over R and $\psi(X)$ a series in $R[\![X]\!]$ with $\varphi \equiv \psi \mod P$. If g is a quasi-logarithm on F' then*

$$c_i(\varphi^* g) = c_i(g(\psi X)) \qquad \text{in } S,$$

for $i = 0, 1, 2, \ldots, n-1$.

PROOF. We must show that for all $\underline{x} \in (\bar{m})^{n-1}$ we have

$$c_i(g(\varphi))(\underline{x}) = c_i(g(\psi))(\underline{x}).$$

Assume $\underline{x} \in Y(e)$ and let $N = N(e, q)$. The argument in the proof of Proposition 10.3 shows that the series $g_{\underline{x}}(\varphi X) - g_{\underline{x}}(\psi X)$ has coefficients in $\pi^{-N}A[\underline{x}]$. Hence the limits c_i of this series are zero, which establishes the above claim. □

Let b be an element of $G(A_n) = \mathrm{Aut}_{k_n}(F \otimes k)$. By definition of the action on $V = \mathrm{Lie}(E) \otimes S = \mathrm{Hom}_R(\mathrm{RigExt}(F, \mathbb{G}_a), S)$ we have

$$bc_i(g) = \beta(c_i(b^{-1}g)) \qquad \text{in } S$$

where $\beta = \beta(b)$ gives the action on $S = K_n\{\{\underline{u}\}\}$ and g is any quasi-logarithm on F. By (16.7) we have

$$b^{-1}g(X) = \beta^{-1}g(kX)$$

where $k(X)$ is the unique isomorphism from βF to F over R which reduces to b mod P. Thus

$$bc_i(g) = c_i(g(\beta k(X))).$$

Let $k'(X)$ be *any* lifting of the series $b(X)$ to $A_n[\![x]\!]$. Then $k' \equiv \beta k \mod P$, so by Proposition 22.2 we have

$$bc_i(g) = c_i(g(k'X)).$$

Write $g(X) = \sum_{j=0}^{n-1} c_j(g) \cdot f_j(X) + r(X)$, where the series f_j are defined in Proposition 13.8. Since $c_i(f_jX) = \delta_{ij}$, we have $c_i(rX) = 0$ for all i. Since $k'X$ has coefficients in A_n, we also have $c_i(r(k'X)) = 0$ for all i, and hence

$$c_i(g(k'X)) = \sum_{j=0}^{n-1} c_j(g) \cdot c_i(f_j(k'X)).$$

The coefficients $c_i(f_j(k'X))$ lie in A_n; hence

(22.3) $$bc_i = \sum_{j=0}^{n-1} c_i(f_j(k'X)) \cdot c_j$$

lies in $M_0 = \oplus_{j=0}^{n-1} A_n c_j$ as claimed. We have proved

PROPOSITION 22.4. *The equivariant vector bundle $\mathcal{L}ie(E)$ is generically flat over $X \otimes K$, with representation KM_0 given by the left regular representation of $G(A_n) = B_n^*$ on the K_n-vector space $B_n \otimes K$ of dimension n.*

The line bundle $\bigwedge^n \mathcal{L}ie(E)$ is flat, and we have an isomorphism

$$\bigwedge^n \mathcal{L}ie(E) \xrightarrow{\sim} \mathcal{O}_X(1)$$

taking the basis $c_0 \wedge c_1 \wedge \cdots \wedge c_{n-1}$ to 1.

PROOF. To identify the representation of B_n^* on M_0, we restrict to the torus $T(A_n) = \mathrm{Aut}(F_{\underline{0}})$. In this case, we may lift b to an endomorphism $k'(X)$ of F_0 and find:

$$f_j(k'X) = \sigma^j(b) \cdot f_j.$$

Thus $bc_i = \sigma^i(b) \cdot c_i$ for all $b \in T(A_n)$. The unique n-dimensional representation of B_n^* with these eigenvalues is the left regular representation, and its determinant is the reduced norm. □

Remark. The generically flat structure on $\mathcal{L}ie(E)$ may seem a little *ad hoc* as presented here, and we indicate a more conceptual argument when $A = \mathbb{Z}_p$. In this case, the vector bundle $\mathcal{L}ie(E)$ is the covariant Dieudonné module of the p-divisible group F [**Kat79**, Ch. V]. It has a G-invariant integrable connection over X:

(22.5) $$\nabla : \mathcal{L}ie(E) \longrightarrow \mathcal{L}ie(E) \otimes \Omega^1_X.$$

Since $\mathcal{L}ie(E)$ is an "F-crystal", the connection ∇ has what Dwork calls a Frobenius structure. A general result (cf. [**Kat73**, Prop. 3.1]) then implies that the K-vector space

(22.6) $$H^0(X \otimes K, \mathcal{L}ie(E) \otimes K)^{\nabla}$$

of horizontal sections for the associated rigid bundle over $X \otimes K$ has dimension $n = \mathrm{rank}(\mathcal{L}ie(E))$. This space is stable under G, and spanned by our specific elements c_i (which are normalized to be eigenvectors for the torus T).

One reason why we have chosen an explicit construction of the flat sections is that the existence of an integrable connection does *not* suffice to descend from a $K\{\{\underline{u}\}\}$ module to a K-module when $\mathrm{char}(K) = p$. Another reason is that it facilitates computations of the map to projective space, which will be given in the next section.

Remark. Proposition 22.4 only describes the structure of the representation of B_n^* on KM_0. With a bit more work, one can determine the structure of the integral representation of B_n^* on M_0 : it is isomorphic to the representation by left multiplication on the free (right) A_n-module:

(22.7) $$B_n = A_n \oplus \varphi A_n \oplus \cdots \oplus \varphi^{n-1} A_n.$$

The basis elements

$$\langle 1, \varphi, \varphi^2, \ldots, \varphi^{n-1} \rangle$$

correspond to the basis

$$\langle c_0, c_1, \ldots, c_{n-1} \rangle$$

of M_0, and are eigenvectors for the torus. Similarly, one can show that the A_n-module spanned by the dual basis

$$\langle c_0^{\vee}, c_1^{\vee}, \ldots, c_{n-1}^{\vee} \rangle$$

of

$$\omega(E) \otimes S = \mathrm{RigExt}(F, \mathbb{G}_a) \otimes S$$

gives a representation of B_n^* isomorphic to right multiplication (by the inverse) on the module:

$$\varphi^{1-n} B_n = A_n \oplus A_n \frac{\varphi}{\pi} \oplus \cdots \oplus A_n \frac{\varphi^{n-1}}{\pi},$$

which is the inverse different of B_n in $B_n \otimes K$.

The element

$$b = \alpha_0 + \varphi\alpha_1 + \varphi^2\alpha_2 + \cdots + \varphi^{n-1}\alpha_{n-1} \tag{22.8}$$

of B_n^*, with $\alpha_i \in A_n$ and $\alpha_0 \in A_n^*$, acts on M_0 through the matrix:

$$C(b) = \begin{pmatrix} \alpha_0 & \pi\alpha_{n-1}^{\sigma} & \cdots & \pi\alpha_2^{\sigma^{n-2}} & \pi\alpha_1^{\sigma^{n-1}} \\ \alpha_1 & \alpha_0^{\sigma} & \cdots & \alpha_3^{\sigma^{n-2}} & \alpha_2^{\sigma^{n-1}} \\ \alpha_2 & \alpha_1^{\sigma} & \ddots & & \alpha_3^{\sigma^{n-1}} \\ \vdots & \vdots & \ddots & & \vdots \\ \alpha_{n-1} & \alpha_{n-2}^{\sigma} & \cdots & \alpha_1^{\sigma^{n-2}} & \alpha_0^{\sigma^{n-1}} \end{pmatrix} \quad \text{in } \mathrm{GL}_n(A_n) \tag{22.9}$$

with respect to the basis $\langle c_0, \cdots, c_{n-1}\rangle$. Its reduction lies in the parabolic subgroup of $\mathrm{GL}_n(k_n)$ which stabilizes the hyperplane spanned by $\langle \bar{c}_1, \bar{c}_2, \cdots, \bar{c}_{n-1}\rangle$ in $M_0/\pi M_0 = \mathrm{Lie}(E\otimes k)$. This hyperplane is precisely the image of $\mathrm{Lie}(F'\otimes k)$, as it is annihilated by $\bar{c}_0$ in $\omega(F\otimes k)$.

23. The étale map to projective space

As mentioned in the introduction, Corollary 23.21 and Corollary 23.26 are due to Lafaille **[G.L79]**.

In this section, we change notation slightly and let

$$R = A[\![u_1, \cdots, u_{n-1}]\!] = A[\![\underline{u}]\!],$$
$$R_n = A_n[\![\underline{u}]\!]$$

and similarly,

$$S = K\{\{\underline{u}\}\}$$
$$S_n = K_n\{\{\underline{u}\}\}.$$

Let $\langle c_0, c_1, \ldots, c_{n-1}\rangle$ be the flat basis of the free S-module $\mathrm{Lie}(E)\otimes_R S$ which was defined in §21.

We define the rigid sections w_i of the line bundle $\mathcal{L}ie(F)$ over $X\otimes K$ as the image of the elements c_i under the map

$$\begin{aligned} \mathrm{Lie}(E)\otimes S &\to \mathrm{Lie}(F)\otimes S \\ c_i &\mapsto w_i. \end{aligned} \tag{23.1}$$

Let W be the K-subspace of $\mathrm{Lie}(F)\otimes S$ spanned by the sections w_i.

PROPOSITION 23.2. *The sections $\langle w_0, w_1, \ldots, w_{n-1}\rangle$ have no common zeroes on $X\otimes K$, and are linearly independent over K. The subspace $W\otimes K_n$ of $Lie(F)\otimes S_n$ is stable under $\underline{G}$ and gives the left regular representation of $G(A_n)$.*

PROOF. Since the c_i give a basis for $\mathrm{Lie}(E)\otimes S$ over S and the map of free S-modules in (23.1) is surjective, the elements w_i span the S-module $\mathrm{Lie}(F)\otimes S$. Hence they have no common zeroes on $X\otimes K$.

To see that the elements w_i are linearly independent, or equivalently, that dim $W = n$, we assume $\sum_i k_i w_i = 0$ with $k_i \in K$. Since $\mathrm{Lie}(F)\otimes S = \mathrm{Hom}_R(\omega(F), S)$ we have $\sum k_i w_i(\omega) = 0$ for any invariant differential on F. Taking $\omega = dg_0$, where g_0 is the logarithm, we find

$$\sum k_i\phi_i = 0 \qquad \text{in } S.$$

Here $\phi_i = c_i(g_0)$ are the rigid functions on $X\otimes K$ defined in (21.13). Since the k_i are constants, this implies

$$\sum k_i D_j\phi_i = 0 \qquad \text{in } S,$$

where D_j is the derivation $\partial/\partial u_j$ of S. But then the element $\sum k_i c_i$ of $\mathrm{Lie}(E)\otimes S$ is equal to zero, as it annihilates the basis elements $\langle g_0, g_1, \cdots, g_{n-1}\rangle$ of $\omega(E) = \mathrm{RigExt}(F, \mathbb{G}_a)$. Since the c_i are independent over S, we have $k_i = 0$ for all i.

The subspace $W\otimes K_n$ is stable under $\underline{G}$, as the map of (23.1) arises from a map of G-equivariant bundles $\mathcal{L}ie(E) \to \mathcal{L}ie(F)$ on X, and $\underline{G}$ stabilizes the K_n-subspace spanned by the c_i in $\mathrm{Lie}(E)\otimes S_n$. The resulting $G(A_n)$-module is a quotient of the left regular representation on $B_n\otimes K = K_n\oplus\varphi K_n\oplus\cdots\oplus\varphi^{n-1}K_n$. Since it has dimension n, it is isomorphic to the left regular representation. □

Let W' be the dual space $\mathrm{Hom}(W, K)$ and let $\langle w_0', \cdots, w_{n-1}'\rangle$ be the dual basis of W'. Let $\mathbf{P}(W)$ be the projective space of all hyperplanes in W (following Grothendieck), or equivalently, the classical projective space of all lines in W'. Then $\underline{G}$ acts on (the right of) $\mathbf{P}(W)\otimes K_n = \mathbf{P}(W\otimes K_n)$: the subgroup $\langle\sigma\rangle = \mathrm{Gal}(K_n/K)$ acts on the coefficients and the subgroup $G(A_n) = B_n^*$ acts by fractional linear transformations. Indeed, if $C = C(b)$ is the matrix of b acting on $W\otimes K_n$ with respect to the basis $\langle w_i\rangle$, which was described in (22.9), then b acts on the usual homogeneous coordinates $[y_0, y_1, \cdots, y_{n-1}]$ of $\mathbf{P}(W)\otimes K_n$ by right multiplication by C. The calculation at the end of §22 shows that the action of $G(A_n)$ preserves the point $[1, 0, 0, \cdots, 0]$ of the reduction $\mathbf{P}(W)\otimes k_n$.

We define a map of rigid analytic spaces over K

$$\Phi : X\otimes K \longrightarrow \mathbf{P}(W) \tag{23.3}$$

by taking the point $\underline{x}$ in $X\otimes K$ to the hyperplane of sections of W which vanish at $\underline{x}$. Thus

$$\Phi(\underline{x}) = \{w \in W\otimes K(\underline{x}) : w(\underline{x}) = 0\}. \tag{23.4}$$

PROPOSITION 23.5. *The map Φ is an étale rigid analytic morphism. It is $\underline{G}$-equivariant over K_n, and surjective over $\bar{K}$.*

PROOF. We first describe Φ in terms of coordinates on $\mathbb{P}(W)$, using the dual basis w_i'. The section $w = \sum k_i w_i$ vanishes at $\underline{x}$ if and only if

$$\sum k_i \phi_i(\underline{x}) = 0.$$

Hence the line in W' which corresponds to the hyperplane of vanishing sections is spanned by

$$w' = \sum \phi_i(\underline{x}) \cdot w_i'.$$

The map Φ is thus given by the homogeneous coordinates

$$\Phi(\underline{x}) = [\phi_0(\underline{x}), \phi_1(\underline{x}), \dots, \phi_{n-1}(\underline{x})]. \tag{23.6}$$

Since each ϕ_i lies in S, and the ϕ_i have no common zero (by 21.16-21.18), Φ is a well-defined rigid analytic morphism. To see that Φ is étale, we must check that the differential

$$d\phi_{\underline{x}} : T_{\underline{x}}(X \otimes K) \longrightarrow T_{\phi(\underline{x})}(\mathbb{P}(W))$$

is an isomorphism of $K(\underline{x})$ vector spaces, for all $\underline{x} \in X \otimes K$. This is equivalent to showing that the matrix

$$T(\underline{x}) = \begin{pmatrix} \phi_0(\underline{x}) & D_1\phi_0(\underline{x}) & \cdots & D_{n-1}\phi_0(\underline{x}) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \phi_{n-1}(\underline{x}) & D_1\phi_{n-1}(\underline{x}) & \cdots & D_{n-1}\phi_{n-1}(\underline{x}) \end{pmatrix}$$

is invertible. But $\det T(\underline{x}) = \epsilon(\underline{x}) \neq 0$, as $\epsilon = \det T$ is a unit in S by Corollary 21.17.

The $\underline{G}$-equivariance of Φ is immediate. Explicitly, it means that for all $b \in G(A_n)$ we have

$$\Phi(\underline{x}b) = \Phi(\underline{x}) \cdot C(b) \tag{23.7}$$

where $C(b)$ is the matrix of b given in (22.9), acting by right multiplication on the homogeneous coordinates $\Phi(\underline{x}) = [\phi_0(\underline{x}), \dots, \phi_{n-1}(\underline{x})]$. It follows that the action of $\underline{G}$ stabilizes the polydisc:

$$Y = \{\underline{y} = (1, y_1, y_2, \cdots, y_{n-1}) : \operatorname{ord}_\pi(y_i) > 0\} \tag{23.8}$$

of $\mathbb{P}^{n-1}(\bar{K})$, which reduces to the point $(1, 0, 0, \cdots, 0)$ in $\mathbb{P}^{n-1}(\bar{k})$.

The proof of surjectivity will be completed as Corollary 23.21 below. □

DEFINITION 23.9. An *isogeny* between formal A-modules G and G' is an element of $\operatorname{Hom}(G, G') \otimes \mathbb{Q}$ which has a two-sided inverse in $\operatorname{Hom}(G', G) \otimes \mathbb{Q}$.

The group of isogenies of $F \otimes k_n$ is the group $(B_n \otimes K)^\times$. It acts on $\mathbb{P}(W) \otimes K_n$ by fractional linear transformations. The Frobenius element acts via the matrix

$$T = \begin{pmatrix} 0 & \dots & 0 & \pi \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}. \tag{23.10}$$

In particular, the element φ^n acts trivially.

The equivariance of the mapping Φ admits the following generalization.

PROPOSITION 23.11. *Suppose $F_{\underline{x}}$ and $F_{\underline{x}'}$ are two deformations of $F \otimes k$ and that $T : F_{\underline{x}} \to F_{\underline{x}'}$ is an isogeny deforming $b \in (B_n \otimes K)^\times$. Then $\Phi(\underline{x})^b = \Phi(\underline{x}')$.* □

DEFINITION 23.12. A *$\star$-isogeny* between two deformations of $F \otimes k$ is an isogeny deforming the identity map.

We will establish a converse to Proposition 23.11 below (Proposition 23.28). Among other things, this represents the range of the map Φ as the set of $\star$-isogeny classes of deformations of $F \otimes k$.

LEMMA 23.13. *For deformations F and F', the following are equivalent:*

(1) The deformations F and F' are $\star$-isogenous;

(2) There is an isogeny $T : F \to F'$ deforming a multiple of the identity map of $F \otimes k$.

(3) There is an isogeny $T : F \to F'$ deforming a power φ^m of the Frobenius endomorphism of $F \otimes k$, with $m \equiv 0 \bmod n$.

PROOF. This is easy. □

For a point $[\phi_0, \dots, \phi_{n-1}]$ of $\mathbb{P}^{n-1}$, set

$$w_i = \frac{\phi_i}{\phi_0} \qquad 0 \le i \le n-1.$$

Thus the w_i are coordinates on the hyperplane in projective space defined by $\phi_n \neq 0$. By composing with the map Φ, the w_i can also be regarded as "meromorphic" functions on Lubin-Tate space $X \otimes K$.

Let D be the closed polydisk in $X \otimes K$ defined by the inequalities

$$V(u_i) \ge \frac{n-i}{n} \qquad i = 1, \dots, n-1.$$

For a rigid analytic function f on D define the valuation

$$V_D(F) = \inf\{\, V(f(x)) \mid x \in D \,\}.$$

Then, for example

$$V_D(u_i) = \frac{n-i}{n}.$$

LEMMA 23.14. *The functions w_i converge on the domain D. There is an inequality*

$$V_D(w_i - u_i) > V_D(u_i).$$

In particular, the w_i can be taken to be uniformizing parameters on the closed polydisk D.

COROLLARY 23.15. *The mapping Φ restricts to a rigid analytic isomorphism between the polydisk D in Lubin-Tate space, and the polydisk D_W in projective space defined by the inequalities*

$$V(w_i) \geq \frac{n-i}{n} \qquad i = 1, \ldots, n-1.$$

□

PROOF OF LEMMA 23.14. Recall that the log of F is

$$\log_F(x) = x + \sum_{n>0} b_n x^{q^n}.$$

The main step is to establish the inequality

$$V_D(\pi^{k+1} b_{nk+i} - u_i \pi^k b_{nk}^{\sigma^i}) > \frac{n-i}{n} \qquad 1 \leq i \leq n. \tag{23.16}$$

To deduce the Lemma, first set $i = n$ to conclude that

$$V_D(\pi^k b_{nk}^{\sigma^i} - 1) > 0$$
$$V_D(\phi_0 - 1) > 0.$$

It then follows that

$$V_D(w_i - u_i) = V_D(\pi^{k+1} b_{nk+i} - u_i \pi^k b_{nk}^{\sigma^i}),$$

so the result follows from (23.16).

The inequality (23.16) is established by induction on $nk + i$. This is trivial for $k = 0$, $i = 1$, since $b_1 = u_1/\pi$. It follows from the functional equation (5.5) of the log that

$$\pi b_l = \sum_{0<i<n} u_i b_{l-i}^{\sigma^i}.$$

Using this, write

$$\pi^{k+1} b_{nk+i} - u_i \pi^k b_{nk}^{\sigma^i} = \sum_{\substack{0<j<n \\ j \neq i}} u_j \pi^k b_{nk+i-j}^{\sigma^j}.$$

It therefore suffices to show that

$$V(u_j \pi^k b_{nk+i-j}^{\sigma^j}) - (n-i)/n > 0 \qquad 0 < j \neq i < n.$$

First suppose that $j > i$. Using the induction hypothesis calculate

$$\begin{aligned} V(u_j \pi^k b_{nk+i-j}^{\sigma^j}) - \frac{n-i}{n} &= \frac{n-j}{n} + q^j \frac{j-i}{n} - \frac{n-i}{n} \\ &= \frac{(q^j - 1)(j-i)}{n} > 0. \end{aligned}$$

Now suppose that $j < i$. Again using the induction hypothesis calculate

$$\begin{aligned} V(u_j \pi^k b_{nk+i-j}^{\sigma^j}) - \frac{n-i}{n} &= \frac{n-j}{n} + q^j \frac{n-(i-j)}{n} - 1 - \frac{n-i}{n} \\ &= \frac{(q^j - 1)(n-(i-j))}{n} > 0. \end{aligned}$$

This completes the proof. □

Because we are working over K, the formal A-module $F_{\underline{x}}$ is determined by its "physical" groups of points which consists of the maximal ideal of the $\bar{A}$ with the group law given by $F_{\underline{x}}$. This A-module will be written $F_{\underline{x}}(\bar{A})$. The torsion submodule $F_{\underline{x}_{\text{tors}}}(\bar{A})$ of $F_{\underline{x}}(\bar{A})$ is the sub A-module consisting of elements annihilated by a power of π. As an abstract A-module it is isomorphic to $(K/A)^n$. The sub A-module of $F_{\underline{x}}(\bar{A})$ consisting of elements killed by π will be denoted ${}_\pi F_{\underline{x}}$.

DEFINITION 23.17. An element $0 \neq \alpha \in F_{\underline{x}_{\text{tors}}}$ is called *canonical* if

$$0 \neq \beta \in F_{\underline{x}_{\text{tors}}} \implies V(\alpha) \geq V(\beta). \tag{23.18}$$

A sub A-module of $F_{\underline{x}_{\text{tors}}}$ is *canonical* if it is generated by a canonical element.

Remark. i) It is not difficult to show that

$$V([\pi](t)) \geq 1 + V(t).$$

It follows that if $\alpha \in F_{\underline{x}_{\text{tors}}}$ is canonical, then $[\pi](\alpha) = 0$. A canonical sub-module is there for abstractly isomorphic to $A/(\pi)$.

ii) Since, for $i \in A^\times$,

$$[i](x) = ix \cdot (\text{ a unit }),$$

every non-zero element of a canonical subgroup is canonical.

If α is a canonical element of $F_{\underline{x}}$ then there is a natural isogeny

$$T = T_\alpha : F_{\underline{x}} \to F_{\underline{x}}/(\alpha) \tag{23.19}$$

with kernel the sub A-module generated by α. It is given by the following formula of Serre **[Lub67]**

$$\prod_{i \in A/(\pi)} (x \underset{F_{\underline{x}}}{-} [i](\alpha)). \tag{23.20}$$

It follows from (23.20) that $F_{\underline{x}}/(\alpha)$ can be given the structure of a deformation of $F \otimes k$ in such a way that the isogeny (23.19) deforms the Frobenius isogeny $\varphi(x) = x^q$ of $F \otimes k$.

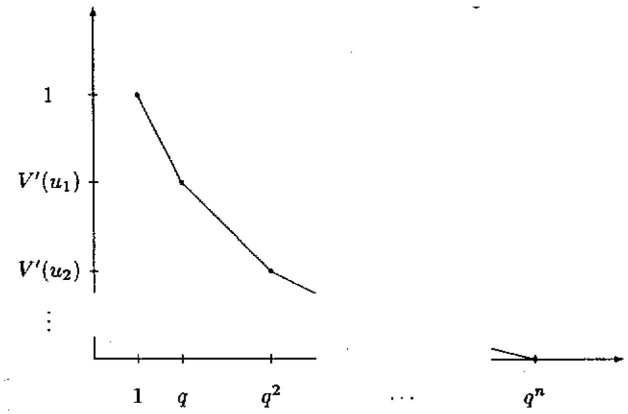


FIGURE 1. The Newton polygon of $[\pi](x)$

COROLLARY 23.21. *The map Φ is surjective over $\bar{K}$.*

PROOF. By Proposition 23.11, if $\underline{w}$ is in the image of the map Φ, then so is $\underline{w} \cdot T$, where T is the matrix (23.10). By Corollary 23.15 every element of D_W is in the image of Φ. It therefore suffices to show that D_W is a "fundamental domain" for T in the sense that the T-translates of D_W cover $\mathbb{P}^{n-1}$. But it is easy to check that if $\underline{w} = [\phi_0, \ldots, \phi_{n-1}]$, and i is chosen so that

$$V(\phi_i) + \frac{i}{n}$$

is minimized, then $\underline{w} \cdot T^i \in D_W$. $\square$

LEMMA 23.22. *Suppose that $\alpha \in F_{\underline{x}_{tors}}$ is a canonical element, and that T is the isogeny given by (23.20). If $0 \neq \beta \in F_{\underline{x}_{tors}}$, then*

$$V(T(\beta)) = qV(\beta). \quad \square$$

The elements killed by π in $F_{\underline{x}}(\bar{A})$ are the roots of the power series $[\pi](x)$. Their valuations can be read off of the Newton polygon of $[\pi](x)$, which turns out to be the convex hull of the collection of points

$$\{(q^i, V(u_i)) \mid i = 0, \ldots, n, \quad u_0 = \pi, \quad u_n = 1\}$$

Let $V'(u_i)$ be y-coordinate of the point on the Newton polygon of $[\pi](x)$ lying above q^i. There is an inequality

$$V(u_i) \geq V'(u_i).$$

Set

$$e_i = V'(u_i) - V'(u_{i-1}) \qquad i = 1, \ldots, n,$$

and define e_i for $i \in \mathbb{Z}$ by requiring that

$$e_i = e_j \quad \text{if } i \equiv j \mod n.$$

It follows from the theory of the Newton polygon that for $1 \leq i \leq n$ there are $q^i - q^{i-1}$ roots of $[\pi(x)]$ with valuation $e_i/(q^i - q^{i-1})$. The canonical elements have valuation $e_1/(q-1)$.

The fact that the Newton polygon is convex translates into the assertion

$$i < j \implies \frac{e_i}{q^i - q^{i-1}} \geq \frac{e_j}{q^j - q^{j-1}}$$

or, equivalently

$$qe_i \geq e_{i+1}.$$

The condition that $F_{\underline{x}}$ be in the domain D is expressed by the series of inequalities

$$e_i + \cdots + e_i \leq \frac{i}{n}, \quad i = 1, \ldots, n.$$

It it useful to introduce

$$s_i = \left(e_1 - \frac{1}{n}\right) + \cdots + \left(e_i - \frac{1}{n}\right) \leq 0, \quad 1i \in \mathbb{Z},$$

and express this condition as

(23.23) $$s_i \leq 0 \quad i = 1, \ldots, n.$$

Since $e_1 + \cdots + e_n = 1$, the value of s_i depends only on i modulo n.

Now suppose that $F_{\underline{x}'}$ is obtained from $F_{\underline{x}}$ by modding out a canonical subgroup, and let

$$T : F_{\underline{x}} \to F_{\underline{x}'}$$

be the map given by (23.20). Write u_i', e_i', s_i' for the moduli and associated invariants of $F_{\underline{x}'}$.

LEMMA 23.24. *If $qe_n > e_1$ then*

$$e_i' = e_{i+1} \qquad i \in \mathbb{Z}.$$

If $qe_n \leq e_1$ then

$$e_n' = qe_n.$$

PROOF. It follows from (23.20) that for $i > 0$, the image in $F_{\underline{x}'}(\bar{A})$ of the $q^{i+1} - q^i$ elements killed by π with valuation $e_i/(q^{i+1} - q^i)$ is a set of $q^i - q^{i-1}$ elements with valuation $e_i/(q^i - q^{i-1})$. This accounts for q^{n-1} of the elements in ${}_\pi F_{\underline{x}'}$. The remaining $q^n - q^{n-1}$ elements are the image under T of the roots of the $q - 1$ equations

$$[\pi](x) - [i](\alpha) = 0 \qquad 0 \neq i \in A/(\pi). \tag{23.25}$$

These equations all have the same Newton polygon. It is the convex hull of the collection of points

$$\left\{ \left(0, \frac{e_1}{q-1}\right), (q^i, V'(u_i)) \mid 1 \leq i \leq n \right\}.$$

To analyze these remaining roots, first note that the line connecting the points $(q^{n-1}, V'(u_{n-1}))$ and $(q^n, V'(u_n)) = (q^n, 0)$ intersects the y-axis at the point $y = qe_n/(q-1)$. There are two cases to consider.

If $qe_n > e_1$, then the common Newton polygon of the equations (23.25) is the line connecting the points

$$\left(\frac{e_1}{q-1}\right) \quad \text{and} \quad (q^n, 0).$$

The q^n roots of each equation all have valuation $e_1/(q^n(q-1))$. Their image in ${}_\pi F_{\underline{x}'}$ is a set of $(q-1)q^{n-1}$ elements with valuation $e_1/(q^n - q^{n-1})$. This accounts for the case $qe_n \geq e_1$ of the lemma.

When $qe_n \leq e_1$ then the common Newton polygon of (23.25) contains the segment connecting $(q^{n-1}, V'(u_{n-1}))$ and $(q^n, 0)$. It follows that the minimum valuation of the elements in ${}_\pi F_{\underline{x}'}$ is $qe_n/(q^n - q^{n-1})$. This gives, then, the equation

$$e_n' = qe_n.$$

This accounts for the case $qe_n \leq e_1$ and completes the proof □

COROLLARY 23.26. *Every deformation $\underline{x} \in X \otimes K(\bar{A})$ is isogenous to one in D, by an isogeny deforming ϕ^m form some m. The resulting deformation satisfies $qe_n \geq e_0$.*

PROOF. By modding out canonical subgroups a few times, if necessary, we can suppose that $qe_n \geq e_1$. Now choose i so that s_i is maximized, and let

$$F_{\underline{x}} \to F_{\underline{x}'}$$

be the isogeny obtained by modding out a canonical subgroup i times. Then $e_j' = e_{j+i}$, and $s_j' = s_{j+i} - s_i$ by Lemma 23.24. Since s_i was chosen to be maximal, we have $s_j' \leq 0$ for all j. This completes the proof. □

COROLLARY 23.27. *If $\Phi(\underline{x}) \in D_W$, then $F_{\underline{x}}$ is $\star$-isogenous to a deformation in D.*

PROOF. For a point $\underline{w} = [\phi_0, \ldots, \phi_n] \in \mathbb{P}^{n-1}(\bar{K})$, set

$$w_0 = \pi$$
$$w_i = \frac{\phi_i}{\phi_0} \quad i = 1, \ldots, n-1$$
$$w_n = 1,$$

and let $V'(w_i)$ be the point on the convex hull of the set

$$(p^i, V(w_i))$$

lying above p^i. Define

$$e_i(\underline{w}) = V'(w_i) - V'(w_{i-1}), \quad i = 1, \ldots n$$
$$e_{i+n}(\underline{w}) = e_i(\underline{w})$$
$$s_i(\underline{w}) = e_1(\underline{w}) + \ldots e_i(\underline{w}) \quad i \in \mathbb{Z}.$$

If $\underline{x} \in D$ then $e_i(\Phi(\underline{x})) = e_i$.

Since $\Phi(\underline{x})) \in D_W$ we have

$$s_i(\Phi(\underline{x})) \leq 0 \quad \text{for all } i.$$

By Lemma 23.26 $F_{\underline{x}}$ is isogenous to a deformation $F_{\underline{x}'}$, with $\underline{x}' \in D$ by an isogeny deforming ϕ^m for some m. Then

$$s_i(\underline{x}') = s_i(\Phi(\underline{x}')) = s_{m+i}(\Phi(\underline{x})) - s_m(\Phi(\underline{x})).$$

Now choose $j > 0$ so that $j + m \equiv 0 \bmod n$, and let $F_{\underline{x}''}$ be obtained from $F_{\underline{x}'}$ by modding out a canonical subgroup j more times. It then follows from Lemma 23.24 that

$$\begin{aligned} s_i(\underline{x}'') &= s_i(\Phi(\underline{x}'')) \\ &= s_{m+i+j}(\Phi(\underline{x})) - s_{m+j}(\Phi(\underline{x})) \\ &= s_i(\Phi(\underline{x})) \leq 0. \end{aligned}$$

This completes the proof. □

PROPOSITION 23.28. *Let $F_{\underline{x}'}$ and $F_{\underline{x}}$ be two deformations of $F \otimes k$ over $\bar{A}$ and $b \in (B_n \otimes K)^*$ an isogeny of $F \otimes k$. Then*

$$\Phi(\underline{x})^b = \Phi(\underline{x}') \qquad \text{in } \mathbb{P}^{n-1}(\bar{K})$$

if and only if there is an isogeny $\varphi : F_{\underline{x}'} \to F_{\underline{x}}$ over $\bar{A}$ with $\varphi \equiv b \pmod{\bar{m}}$. In particular, the stabilizer of $\Phi(\underline{x})$ is the image of $(End(F_{\underline{x}}) \otimes K)^*$ in $(End(F \otimes k) \otimes K)^*$, and $\Phi(\underline{x}) = \Phi(\underline{x}')$ if and only if the formal A-modules $F_{\underline{x}'}$ and $F_{\underline{x}}$ are "$\star$-isogenous" over $\bar{A}$.

PROOF. The "only if" assertion has been proved above. For the "if" assertion, first write $b = \varphi^m b_0$ with $b_0 \in B^\times$. By modding out canonical subgroups m times from $F_{\underline{x}}$ we reduce to the case $b \in B^\times$. By acting on $F_{\underline{x}}$ by b we then reduce to the case $b = 1$. By modding out canonical subgroups from both $F_{\underline{x}}$ and $F_{\underline{x}'}$ and using Corollary 23.26 we reduce to the case

$$\Phi(\underline{x}') = \Phi(\underline{x}) \in D_W.$$

By Corollary 23.27 the groups $F_{\underline{x}}$ and $F_{\underline{x}'}$ are $\star$-isogenous to deformations $F_{\underline{y}}$ and $F_{\underline{y}'}$ respectively, with $\underline{y}, \underline{y}' \in D$. But then $\underline{y} = \underline{y}'$ by Corollary 23.15. This completes the proof. □

The interpretation of points of projective space as $\star$-isogeny classes of deformations over flat, local A-algebras allows one to reconstruct the étale map $\Phi : X \otimes K \longrightarrow \mathbb{P}^{n-1} \otimes K$ as follows. The rigid analytic space $X \otimes K$ has a tower of finite Galois covers $X_m \otimes K$ with Galois groups $\mathrm{GL}_n(A/\pi^m A)$; these are obtained by adjoining the π^m-torsion points on the universal deformation F over $X \otimes K$ [**Car90**, p. 19]. Let

$$X_\infty \otimes K = \varprojlim_m X_m \otimes K \tag{23.29}$$

which is a pro-rigid analytic space over K, with an action of $\mathrm{GL}_n(A)$. The group scheme G also acts on $X_\infty \otimes K$, and one obtains an action of the product group $\mathrm{GL}_n(A) \times B_n^*$ on $X_\infty \otimes K_n$, with the elements $\{(a,a) : a \in A^*\}$ acting trivially.

In fact, Deligne observed that the larger group

$$\{(g,b) \in \mathrm{GL}_n(K) \times (B_n \otimes K)^* : \mathrm{ord}_\pi(\det g) = \mathrm{ord}_\pi(\mathbb{N}b)\} \tag{23.30}$$

acts on $X_\infty \otimes K_n$ ([**Del**], [**Car90**, pp. 20-21]). In particular, the product group

$$\mathrm{GL}_n^0(K) \times B_n^* \qquad \text{acts on } X_\infty \otimes K_n, \tag{23.31}$$

where

$$\mathrm{GL}_n^0(K) = \{g \in \mathrm{GL}_n(K) : \det g \in A^*\}.$$

One can show that, just as $X \otimes K$ is the quotient of $X_\infty \otimes K$ by $\mathrm{GL}_n(A)$, the projective space $\mathbb{P}^{n-1} \otimes K$ arises as the quotient of $X_\infty \otimes K$ by the larger group $\mathrm{GL}_n^0(K)$. The fibres of the map Φ are thus identified with the cosets $\mathrm{GL}_n(A) \backslash \mathrm{GL}_n^0(K)$, which form a subset of the vertices of the building associated to $\mathrm{PGL}_n(K)$.

24. The group action: differentiability

The group $G(A_n) = B_n^*$ has the structure of a K-analytic Lie group of dimension n^2, which is a closed subgroup of $\mathrm{GL}_n(K_n)$ via the representation (22.9). Its Lie algebra $\mathfrak{g} = \mathrm{Lie}(B_n^*)$ is a form of $\mathfrak{gl}_n$ over K:

$$\mathfrak{g} \otimes K_n \simeq \mathfrak{gl}_n(K_n). \tag{24.1}$$

The isomorphism of (24.1) gives rise to a 1-cocycle on $\mathrm{Gal}(K_n/K)$ with values in $\mathrm{Aut}(\mathfrak{g} \otimes K_n) = \mathrm{PGL}_n(K_n)$, taking the generator σ to the automorphism "conjugation by φ". Let $\mathfrak{t}$ be the Lie algebra of the torus $T(A_n) = A_n^*$; then $\mathfrak{t}$ is a Cartan subalgebra of dimension n in $\mathfrak{g}$ which is split over K_n. Finally, let $\mathfrak{z}$ be the 1-dimensional Lie algebra of the center $Z(A_n) = A^*$.

If γ is an element of $\mathfrak{g}$, viewed as an $n \times n$ matrix over K_n via the derivative of the representation (22.9), then the element $1 + \pi^m\gamma$ lies in the group $G(A_n)$ for m sufficiently large. This gives us a working replacement for the exponential map when A has characteristic p. A corollary of the existence of an étale G-map from $X \otimes K$ to projective space is the following

PROPOSITION 24.2. *If φ is a rigid function in $S_n = K_n\{\{\underline{u}\}\}$ and γ is an element of $\mathfrak{g}$, the limit*

$$D_\gamma(\varphi) = \lim_{m\to\infty} \frac{(1+\pi^m\gamma)\varphi - \varphi}{\pi^m}$$

exists in S_n, and defines a K_n-derivation of S_n. The map $\gamma \longmapsto D_\gamma$ defines a representation of Lie algebras

$$\mathfrak{g} \otimes K_n \longrightarrow Der_{K_n}(S_n) = H^0(X \otimes K_n,\ \mathcal{T}_X \otimes K_n)$$

with kernel $= \mathfrak{z} \otimes K_n$ and image isomorphic to $\mathfrak{pgl}_n(K_n)$. The image is stable under the action of $G(A_n)$ on sections of the tangent bundle.

PROOF. The corresponding facts are clear for the derivative of the G-action on the projective space $\mathbb{P}(W)$. Since the mapping $\Phi : X \otimes K \longrightarrow \mathbb{P}(W)$ is étale, there is no obstruction to lifting the resulting vector fields to $X \otimes K$. □

In fact, one can show that the K-Lie subalgebra of $\mathfrak{pg} \otimes K_n$ which acts as K-derivations of the algebra $S = K\{\{\underline{u}\}\}$ is isomorphic to $\mathfrak{pgl}_n$ over K! Indeed, let $\langle c_0, c_1, \cdots, c_{n-1}\rangle$ be the flat basis of $\mathrm{Lie}(E) \otimes S$ described in §21 and let $\langle c_0^\vee, c_1^\vee, \cdots, c_{n-1}^\vee\rangle$ be the dual basis of $\mathrm{RigExt}(F, \mathbb{G}_a) \otimes S = \omega(E) \otimes S$. The K-subspace spanned by the vectors $c_i \otimes c_j^\vee$ in $\mathrm{Lie}(E) \otimes \omega(E) \otimes S$ has dimension n^2; over K_n the group $G(A_n)$ acts via the adjoint representation, and stabilizes the line spanned by $\sum_{i=0}^{n-1} c_i \otimes c_i^\vee$.

Consider the projections $t_{ij} = \beta(c_i \otimes c_j^\vee)$ of these elements to rigid sections of the tangent bundle of X, under the map of equivariant bundles.

$$\beta : \mathcal{L}\mathrm{ie}(E) \otimes \varpi(E) \twoheadrightarrow \mathcal{L}\mathrm{ie}(F) \otimes \varpi(F') \simeq \mathcal{T}_X. \tag{24.3}$$

Then $\sum_{i=0}^{n-1} t_{ii} = 0$ and the vector t_{ij} span a Lie algebra of dimension $n^2 - 1$ over K which is isomorphic to $\mathfrak{pgl}_n$. (The isomorphism takes t_{ij} to the matrix with a single 1 in the $(i+1)^{\text{th}}$ row and $(j+1)^{\text{th}}$ column). Over K_n, the elements t_{ij} are eigenvectors for the torus $T(A_n)$:

$$\alpha(t_{ij}) = \frac{\sigma^i(\alpha)}{\sigma^j(\alpha)} \cdot t_{ij}, \qquad \alpha \in A_n^*. \tag{24.4}$$

Appendix

25. Formulaire when $n = 2$

We give some explicit formulae to illustrate the general theory of the group action on X, in the simplest non-trivial case when $n = 2$. Then $A_2 = A \oplus A\zeta_2$ with ζ_2 a primitive (q^2-1) root of unit. We write

$$\sigma(\alpha) = \bar{\alpha} \qquad \text{for } \alpha \in A_2. \tag{25.1}$$

We have $B_2 = A_2 \oplus A_2\varphi$ with $\varphi^2 = \pi$, $\varphi_\alpha = \bar{\alpha}\varphi$ for $\alpha \in A_2$. The element

$$b = \alpha + \varphi\beta \qquad \text{of } B_2^* \tag{25.2}$$

acts on the basis $\langle c_0, c_1\rangle$ of $\mathrm{Lie}(E) \otimes S$ by the matrix

$$C = C(b) = \begin{pmatrix} \alpha & \pi\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}. \tag{25.3}$$

Let $u = u_1$, so $K\{\{\underline{u}\}\} = K\{\{u\}\}$ is the algebra of rigid functions on the open unit disc. The functions

$$\phi_0(u) = c_0(g_0) \qquad \phi_1(u) = c_1(g_0) \tag{25.4}$$

can be calculated from the coefficients of the universal logarithm $g_0(X)$:

$$g_0(X) = X + \frac{u}{\pi}X^q + \left(\frac{u^{1+q}}{\pi^2} + \frac{1}{\pi}\right)X^{q^2} + \cdots. \tag{25.5}$$

We find

(25.6)

$$\begin{aligned}\phi_0(u) &= 1 + \frac{u^{1+q}}{\pi} + \frac{u^{1+q^3}}{\pi} + \frac{u^{q^2+q^3}}{\pi} + \frac{u^{1+q+q^2+q^3}}{\pi^2} + \cdots \\ &= 1 + \frac{1}{\pi}\sum_{0\le a\le b} u^{q^{2a}+q^{2b+1}} + \frac{1}{\pi^2}\sum_{0\le a\le b\le c\le d} u^{q^{2a}+q^{2b+1}+q^{2c+2}+q^{2d+3}} + \cdots\end{aligned}$$

(25.7)

$$\begin{aligned}\phi_1(u) &= u + u^{q^2} + \frac{u^{1+q+q^2}}{\pi} + \cdots \\ &= \sum_{0\le a} u^{q^{2a}} + \frac{1}{\pi}\sum_{0\le a\le b\le c} u^{q^{2a}+q^{2b+1}+q^{2c+2}} + \cdots\end{aligned}$$

From a consideration of the Newton polygon, we find that ϕ_0 vanishes at:

$$\begin{cases} q+1 \text{ points } \alpha \text{with } \mathrm{ord}_\pi(\alpha) = \frac{1}{q+1} \\ q^2(q+1) \text{ points } \alpha \text{with } \mathrm{ord}_\pi(\alpha) = \frac{1}{q^2(q+1)} \\ \vdots \\ q^4(q+1) \text{ points } \alpha \text{with } \mathrm{ord}_\pi(\alpha) = \frac{1}{q^4(q+1)}. \end{cases} \tag{25.8}$$

Similarly, ϕ_1 vanishes at $\alpha = 0$ as well as

$$\begin{cases} q(q+1) \text{ points } \alpha \text{with } \mathrm{ord}_\pi(\alpha) = \frac{1}{q(q+1)} \\ q^3(q+1) \text{ points } \alpha \text{with } \mathrm{ord}_\pi(\alpha) = \frac{1}{q^3(q+1)} \\ \quad\vdots \end{cases} \tag{25.9}$$

The zeroes α of ϕ_0 and ϕ_1, which are all simple, correspond to "quasi-canonical" liftings F_α of $F \otimes k$, in the sense of **[Gro86]**, with $\mathcal{O} = A_2$. They are the inverse images of the points $[0,1]$ and $[1,0]$ on $\mathbb{P}^1$ under the mapping $\Phi(\underline{x}) = [\phi_0(\underline{x}), \phi_1(\underline{x})] : X \otimes K \longrightarrow \mathbb{P}^1 \otimes K$.

The function $\epsilon(u) = \det T = \phi_0(u)\phi_1'(u) - \phi_1(u)\phi_0'(u)$ has expansion beginning

$$\epsilon(u) = 1 - \frac{q}{\pi}u^{q+1} + q^2u^{q^2-1} + \cdots. \tag{25.10}$$

Its coefficients are all integral, and ϵ has no zeroes on X. When $\mathrm{char}(A) = p$ we have $\epsilon = 1$.

The ratio:

$$w(u) = \phi_1(u)/\phi_0(u) \tag{25.11}$$

is a meromorphic function on X, which is regular, and univalent on the affinoid:

$$Y(q) = \{\, \alpha \in \bar{m} : \mathrm{ord}_\pi(\alpha) \geq \frac{1}{q} \,\}. \tag{25.12}$$

If $b = \alpha + \varphi\beta$ lies in B_2^*, then

$$b(w) = \frac{\bar{\alpha}w + \pi\bar{\beta}}{\beta w + \alpha} \tag{25.13}$$

as rigid analytic parameters on $Y(q)$. Indeed:

$$\begin{aligned}\Phi(\underline{x}^b) = \Phi(\underline{x})^b &= [\phi_0(\underline{x}), \phi_1(\underline{x})] \cdot C \\ &= [\alpha\phi_0(\underline{x}) + \beta\phi_1(\underline{x}), \pi\bar{\beta}\phi_0(\underline{x}) + \bar{\alpha}\phi_1(\underline{x})]\end{aligned}$$

so $bw(\underline{x}) = w(\underline{x}b) = \phi_1(\underline{x}b)/\phi_0(\underline{x}b)$ is given by (25.13).

The derivations:

$$\begin{cases} t_{01} = \dfrac{\phi_0(u)^2}{\epsilon(u)} \cdot \partial/\partial u \\ t_{10} = \dfrac{\phi_1(u)^2}{\epsilon(u)} \cdot \partial/\partial u \\ t_{00} = \dfrac{\phi_0(u)\phi_1(u)}{\epsilon(u)} \cdot \partial/\partial u. \end{cases} \tag{25.14}$$

of $K\{\{u\}\}$ span a 3-dimensional Lie algebra (isomorphic to $\mathfrak{pgl}_2$) over K. They describe the differentiated form of the PG-action on $X \otimes K$, and over K_2 are eigenvectors for elements α in the torus $PT(A_2) = A_2^*/A^*$, with eigenvalues $\alpha/\bar{\alpha}, \bar{\alpha}/\alpha$, and 1 respectively.

Finally, we note that the deformation theory of A-divisible modules of dimension 1 and height 2, which was developed by Serre and Tate when $A = \mathbb{Z}_p$ [**LST64**] and extended in [**Gro86**, p. 326], provides a simple analogous case to the theory discussed above. Let F_0 be a formal A-module of dimension 1 and height 1 over $k = A/\pi A$, and let F be the unique lifting of F_0 to A. Let $G_0 = K/A$ be the étale A-module of height 1 over k with trivial Galois action, and let G be the unique lifting of G_0 to A. We let X denote the formal scheme over A which classifies $\star$-isomorphism classes of deformations of the A-divisible module $F_0 \times G_0$ of height 2 to local A-algebra R.

Since any deformation E of $F_0 \times G_0$ lies in an extension

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0, \tag{25.15}$$

and $\mathrm{Ext}^1_R(G, F) = F(P)$, the A-module of points of F in the maximal ideal P of R, we have

$$X(R) = F(P). \tag{25.16}$$

Thus the rigid analytic space $X \otimes K$ is again isomorphic to the open unit disc, but now has a canonical A-module structure.

The endomorphism ring of $F_0 \times G_0$ is $A \times A$, whereas the universal deformation only has endomorphisms by A (embedded diagonally). Hence the group $A^* \times A^* = \mathrm{Aut}(F_0 \times G_0)$ acts on X, and the diagonal subgroup ΔA^* acts trivially. The element $a = (a, 1) \in A^* \times A^*$ acts on the ring $A[\![u]\!]$ of formal functions on X by the formula:

$$a \circ \varphi(u) = \varphi(a_F u). \tag{25.17}$$

In other words, X is isomorphic to F as a formal scheme with A^*-action over A. The origin of F corresponds to the canonical lifting $F \times G$, and the π^s-torsion points of $F(\bar{m})$ correspond to the quasi-canonical liftings of level s. These give $q^{s-1}(q-1)$ points in $\bar{m}$, each with $\mathrm{ord}_\pi(\alpha) = \frac{1}{q^{s-1}(q-1)}$. Their endomorphism rings are equal to

$$\{\, (a, b) \in A \times A : a \equiv b \pmod{\pi^s} \,\}.$$

The analog of the map Φ to projective space is the A^*-equivariant, étale, surjective homomorphism of rigid analytic groups over K:

$$\begin{aligned} \phi : X \otimes K &\to \mathbf{A}^1 \otimes K \\ x &\mapsto \log_F(x). \end{aligned}$$

Assume F is A-typical, for simplicity. Then the rigid function $\phi(u) = \log_F(u)$ is given by the series

$$\phi(u) = u + \frac{u^q}{\pi} + \frac{u^{q^2}}{\pi^2} + \cdots . \tag{25.18}$$

We have

$$\epsilon(u) = \phi'(u) = 1 + \frac{q}{\pi}u^{q-1} + \frac{q^2}{\pi^2}u^{q^2-1} + \cdots \tag{25.19}$$

integral and invertible on X. The kernel of ϕ consists of all quasi-canonical liftings. The A^*-action on X is converted to the linear action on affine space:

$$a \circ z = az, \qquad z = \phi(u). \tag{25.20}$$

When $A = \mathbb{Z}_p$ and $F_0 = \hat{\mathbb{G}}_m$, a point of X over $\bar{\mathbb{Z}}_p$ corresponds to a unit $q \equiv 1$ mod $\bar{m}$, and the map ϕ takes q to Dwork's parameter $\log q \in \bar{\mathbb{Q}}_p$.

Since the linear action of A^* on affine space is differentiable and ϕ is étale, we may conclude that the group action on $X \otimes K$ is also differentiable. Hence, for any $\gamma \in K = \mathrm{Lie}(A^*)$ and $\varphi \in K\{\{u\}\}$, the limit

$$D_\gamma(\varphi) = \lim_{m \to \infty} \frac{(1 + \pi^m \gamma) \circ \varphi - \varphi}{\pi^m} \tag{25.21}$$

exists in $K\{\{u\}\}$ and defines a derivation D_γ of $K\{\{u\}\}$ over K. Using the limit formula [**Haz78**, p. 217]

$$\log_F(u) = \lim_{m \to \infty} (\pi^{-m} \cdot \pi_F^m(u)) \tag{25.22}$$

we find: $D_\gamma(u) = \gamma \cdot \frac{\phi(u)}{\epsilon(u)}$. Hence the image of $\mathrm{Lie}(A^*)$ in the derivations of $K\{\{u\}\}$ over K is the 1-dimensional Lie algebra spanned by the vector field

$$D_1 = \frac{\phi(u)}{\epsilon(u)} \partial/\partial u. \tag{25.23}$$

References

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-archimedean analysis.*, volume 261 of *Grundlehren der Math. Wiss.* Springer–Verlag, 1984.

[Car90] H. Carayol. Non-abelian lubin-tate theory. In *Automorphic forms, Shimura varieties, and L-functions*, pages 15–39. Academic Press, 1990.

[Del] P. Deligne. Letter to I. Piatetskii-Shapiro, 1973.

[Dri74] V. G. Drinfeld. Elliptic modules. *Math USSR Sbornik*, 23:561–592, 1974.

[Fon77] J.-M. Fontaine. Groupes p-divisibles sur les corps locaux. *Astérisque*, pages 47–48, 1977.

[G.L79] G.Lafaille. Construction de groupes p-divisibles. In *Journées de Géométrie Algébrique de Rennes (III)*, volume 64 of *Astérisque*, pages 103–124, Paris, 1979. Société mathématique de France.

[Gro86] B.H. Gross. On canonical and quasi-canonical liftings. *Invent. math.*, 84:321–326, 1986.

[Haz78] M. Hazewinkel. *Formal groups and applications.* Academic Press, 1978.

[Hea59] R. Heaton. Polynomial 3-cocycles over fields of characteristic *p*. *Duke Math. J.*, 26:269–275, 1959.

[Hon70] T. Honda. On the theory of commutative formal groups. *J. Math. Soc. Japan 22*, pages 213–246, 1970.

[Kat73] N. Katz. Travaux de Dwork. In *Sem. Bourbaki, 1971/1972*, volume 417 of *Lecture Notes in Mathematics*, pages 431–436, Berlin and New York, 1973. Springer–Verlag. exposé 409.

[Kat79] N. Katz. Crystalline cohomology, Dieudonné modules and Jacobi sums. In *Automorphic forms, Representation theory and Arithmetic*, pages 165–246, Bombay, 1979. Tata Institute of Fundamental Research.

[Laz55] M. Lazard. Sur les groupes de Lie formels á une paramètre. *Bull. Soc. Math. France*, 83:251–274, 1955.

[LST64] J. Lubin, J-P. Serre, and J. Tate. Elliptic curves and formal groups. In *Seminar at Woods Hole Institute on algebraic geometry*, 1964.

[LT65] J. Lubin and J. Tate. Formal complex multiplication in local fields. *Annals of Mathematics*, 81:380–387, 1965.

[LT66] J. Lubin and J. Tate. Formal moduli for one parameter formal Lie groups. *Bull. Soc. Math. France*, 94:49–60, 1966.

[Lub67] J. Lubin. Finite subgroups and isogenies of one-parameter formal Lie groups. *Annals of Mathematics*, 85:296–302, 1967.

[Mac50] S. MacLane. Cohomology theory of abelian groups. In *Proc. ICM Cambridge, MA*, pages 8–14, 1950.

[Ser] J-P. Serre. Letter to J. Tate, January 25, 1964.

[Ser62] J-P. Serre. Endomorphisms complètement continus des espaces de banach p-adique. *Publ. Math. IHES*, 12:69–85, 1962.

[Tat] J. Tate. Letter to J-P. Serre,December 16, 1963.

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Constructions of elements in Picard groups

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Abstract. We discuss the first author's Picard groups of stable homotopy. We give a detailed description of the calculation of Pic_1, and go on to describe geometric constructions for lifts of the elements of Pic_1. We also construct a 15 cell complex that localizes to what we speculate is an interesting element of Pic_2. For all n we describe an algebraic approximation to Pic_n using the Adams-Novikov spectral sequence. We also show that the p-adic integers embed in the group Pic_n for all n and p.

1. Introduction and statement of results

We begin with the basic definition. The functor

$$\Sigma^n : X \mapsto S^n \wedge X$$

is an automorphism of the category of spectra, which preserves cofibration sequences and infinite wedges. If T is another such automorphism, then Brown's representability theorem applied to $\pi_*(TX)$ gives a spectrum S_T with

$$TX = S_T \wedge X$$

and

$$S_{T^{-1}} \wedge S_T = S^0.$$

This motivates the following definition.

Definition 1.1. *A spectrum Z is invertible if and only if there is some spectrum W such that*

$$Z \wedge W = S^0.$$

Pic is the group of isomorphism classes of invertible spectra, with multiplication given by smash product. Given an isomorphism class $\lambda \in Pic$ we will write S^λ for a representative spectrum.

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