$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a geometric fundamental group

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1 Introduction

Let p be a prime number. In this article we present a theorem, suggested by Peter Scholze, which states that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the étale fundamental group of certain object Z which is *defined over an algebraically closed field*. As a consequence, p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ correspond to $\mathbb{Q}_p$-local systems on Z.

The precise theorem involves perfectoid spaces, [Sch12]. Let $C/\mathbb{Q}_p$ be complete and algebraically closed. Let D be the open unit disk centered at 1, considered as a rigid space over C, and given the structure of a $\mathbb{Z}_p$-module where the composition law is multiplication, and $a \in \mathbb{Z}_p$ acts by $x \mapsto x^a$. Let

$$\tilde{D} = \varprojlim_{x \mapsto x^p} D.$$

Then $\tilde{D}$ is no longer a classical rigid space, but it does exist in Huber's category of adic spaces, and is in fact a perfectoid space. Note that $\tilde{D}$ has the structure of a $\mathbb{Q}_p$-vector space. Let $\tilde{D}^* = \tilde{D} \backslash \{0\}$; this admits an action of $\mathbb{Q}_p^\times$.

Theorem 1.0.1. *The category of $\mathbb{Q}_p^\times$-equivariant finite étale covers of $\tilde{D}^*$ is equivalent to the category of finite étale $\mathbb{Q}_p$-algebras.*

The object Z of the first paragraph is then the quotient $Z = \tilde{D}^*/\mathbb{Q}_p^\times$. This quotient doesn't belong to the category of adic spaces. Rather, one has a Yoneda-style construction. The category Perf_C of perfectoid spaces over C has a pro-étale topology, [Sch13], and one has a sheaf of sets Z on Perf_C, namely the sheafification of $X \mapsto \mathrm{Hom}(X, \tilde{D}^*)/\mathbb{Q}_p^\times$. Thus Z belongs to the category of sheaves of sets on Perf_C which admit a surjective map from a representable sheaf. In this category, a morphism $\mathcal{F}' \to \mathcal{F}$ is called finite
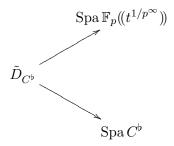
étale if the pull-back to any representable sheaf is finite étale in the usual sense. Then one may define the fundamental group for such objects, and Thm. 1.0.1 asserts that $\pi_1^{\text{ét}}(Z) = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Theorem 1.0.1 can be generalized to a finite extension $E/\mathbb{Q}_p$. Let $\pi \in E$ be a uniformizer, and let H be the corresponding Lubin-Tate formal $\mathcal{O}_E$-module. Then D gets replaced by the generic fiber H_C^{ad}; this is the unit disc centered at 0, considered as an adic space over C. H_C^{ad} is endowed with the $\mathcal{O}_E$-module structure coming from H. Form the *universal cover* $\tilde{H}_C^{\text{ad}} = \varprojlim H_C^{\text{ad}}$, where the inverse limit is taken with respect to multiplication by π in H. Then $\tilde{H}_C^{\text{ad}}$ is an E-vector space object in the category of perfectoid spaces over C. Form the quotient

$$Z_E = (\tilde{H}_C^{\text{ad}} \setminus \{0\})/E^\times.$$

Then the main theorem (Thm. 4.0.10) states that the categories of finite étale covers of Z_E and $\text{Spec}\, E$ are equivalent, so that $\pi_1^{\text{ét}}(Z_E) = \text{Gal}(\overline{E}/E)$.

The proof hinges on a combination of two themes: the *fundamental curve X of p-adic Hodge theory*, due to Fargues-Fontaine, and the *tilting equivalence*, due to Scholze. Let us sketch the proof in the case $E = \mathbb{Q}_p$. Let $C^\flat$ be the tilt of C, a perfectoid field in characteristic p. Consider the punctured open disc $D_{C^\flat}^*$ (with parameter t) and its universal cover $\tilde{D}_{C^\flat}^*$. Then $\tilde{D}_{C^\flat}^*$ is simultaneously a perfectoid space over two fields:

$$
\begin{array}{ccc}
 & & \text{Spa}\,\mathbb{F}_p(\!(t^{1/p^\infty})\!) \\
 & \nearrow & \\
\tilde{D}_{C^\flat} & & \\
 & \searrow & \\
 & & \text{Spa}\,C^\flat
\end{array}
$$

Considered as a perfectoid space over $C^\flat$, $\tilde{D}_{C^\flat}^*$ has an obvious "un-tilt", namely $\tilde{D}_C^*$. The other field $\mathbb{F}_p(\!(t^{1/q^\infty})\!)$ is the tilt of $\hat{\mathbb{Q}}_p(\mu_{p^\infty})$. It turns out that there is a perfectoid space over $\hat{\mathbb{Q}}_p(\mu_{p^\infty})$ whose tilt is also $\tilde{D}_C^*$, and here is where the Fargues-Fontaine curve comes in.

The construction of the Fargues-Fontaine curve X is reviewed in §2. X is an integral noetherian scheme of dimension 1 over $\mathbb{Q}_p$, whose closed points parametrize un-tilts of $C^\flat$ modulo Frobenius. For our purposes we need the adic version X^{ad}, which is the quotient of another adic space Y^{ad} by

a Frobenius automorphism ϕ. The extension of scalars $Y^{\mathrm{ad}}\hat{\otimes}\hat{\mathbb{Q}}_p(\mu_{p^\infty})$ is a perfectoid space; by a direct calculation (Prop. 4.0.8) we show that its tilt is isomorphic to $\tilde{D}^*_{C^\flat}$. In this isomorphism, the action of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ on the field of scalars $\hat{\mathbb{Q}}_p(\mu_{p^\infty})$ corresponds to the geometric action of $\mathbb{Z}_p^\times$ on $\tilde{D}^*_{C^\flat}$, and the automorpism ϕ corresponds (up to absolute Frobenius) to the action of p on $\tilde{D}^*_{C^\flat}$.

Therefore under the tilting equivalence, finite étale covers of $\tilde{D}^*_C$ and $Y^{\mathrm{ad}}\hat{\otimes}\hat{\mathbb{Q}}_p(\mu_{p^\infty})$ are identified. The same goes for $\tilde{D}^*_C/p^{\mathbb{Z}}$ and $X^{\mathrm{ad}}\hat{\otimes}\hat{\mathbb{Q}}_p(\mu_{p^\infty})$. Now we apply the key fact that X^{ad} is geometrically simply connected. The same statement is proved in [FF11] for the algebraic curve X; we have adapted the proof for X^{ad} in Prop. 3.8.4. Thus finite étale covers of $\tilde{D}^*_C/p^{\mathbb{Z}}$ are equivalent to finite étale $\mathbb{Q}_p(\mu_{p^\infty})$-algebras. Now we can descend to $\mathbb{Q}_p$: $\mathbb{Z}_p^\times$-equivariant finite étale covers of $\tilde{D}^*_C/p^{\mathbb{Z}}$ are equivalent to finite étale $\mathbb{Q}_p$-algebras, which is Thm. 1.0.1.

2 The Fargues-Fontaine curve

2.1 The rings B and B^+

Here we review the construction of the Fargues-Fontaine curve. The construction requires the following two inputs:

- A finite extension $E/\mathbb{Q}_p$, with uniformizer π and residue field $\mathbb{F}_q/\mathbb{F}_p$,

- A perfectoid field F of characteristic p containing $\mathbb{F}_q$.

Note that a perfectoid field F of characteristic p is the same as a perfect field of characteristic p which is complete with respect to a nontrivial (rank 1) valuation. Let $x \mapsto |x|$ denote one such valuation.

Let $W(F)$ be the ring of Witt vectors of the perfect field F, and let $W_{\mathcal{O}_E}(F) = W(F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. A typical element of $W_{\mathcal{O}_E}(F)$ is a series

$$x = \sum_{n \gg -\infty} [x_n]\pi^n,$$

where $x_n \in F$. Let $B^b \subset W_{\mathcal{O}_E}(F)[1/\pi]$ denote the subalgebra defined by the condition that $|x_n|$ is bounded as $n \to \infty$. Equivalently, $B^b =$

$W_{\mathcal{O}_E}(\mathcal{O}_F)[1/\pi, 1/[\varpi]]$, where $\varpi \in F$ is any element with $0 < |\varpi| < 1$. Then B^b clearly contains $B^{b,+} = W(\mathcal{O}_F) \otimes_{W(k)} \mathcal{O}_E$ as a subring.

For every $0 < \rho < 1$ we define a norm $|\ |_\rho$ on B^b by

$$|x|_\rho = \sup_{n \in \mathbb{Z}} |x_n| \rho^n.$$

Definition 2.1.1. Let B denote the Fréchet completion of B^b with respect to the family of norms $|\ |_\rho$, $\rho \in (0,1)$.

B can be expressed as the inverse limit $\varprojlim_I B_I$ of Banach E-algebras, where I ranges over closed subintervals of $(0,1)$, and B_I is the completion of B^b with respect to the norm

$$|x|_I = \sup_{\rho \in I} |x|_\rho.$$

(Note that $|x|_\rho$ is continuous in ρ and therefore bounded on I.)

It will be useful to give an explicit description of B_I.

Lemma 2.1.2. *Let $I \subset (0,1)$ be a closed subinterval whose endpoints lie in the value group of F: $I = [|\varpi_1|, |\varpi_2|]$. Let $B_I^{b,\circ}$ be subring of elements $x \in B^b$ with $|x|_I \leq 1$. Then $B_I^{b,\circ} := B^{b,+}\left[\frac{[\varpi_1]}{\pi}, \frac{\pi}{[\varpi_2]}\right]$.*

Proof. It is easy to see that

$$\left|\frac{[\varpi_1]}{\pi}\right|_I = \left|\frac{\pi}{[\varpi_2]}\right| = 1,$$

which proves that $B^{b,+}\left[\frac{[\varpi_1]}{\pi}, \frac{\pi}{[\varpi_2]}\right] \subset B_I^{b,\circ}$.

Conversely, suppose $x = \sum_{n \gg -\infty}[x_n]\pi^n \in B_I^{b,\circ}$. Then $|x_n| \leq |\varpi_1|^{-n}$ for $n < 0$ and $|x_n| \leq |\varpi_2|^{-n}$ for $n \geq 0$. This shows that the "tail term" $\sum_{n<0}[x_n]\pi^n$ lies in $B^{b,+}[[\varpi_1]/\pi]$. It also shows that for $n \geq 0$, each term $[x_n]\pi^n$ lies in $B^{b,+}[\pi/[\varpi_2]]$. Since $x \in B^b$, there exists $C > 0$ with $|x_n| \leq C$ for all n. Let N be large enough so that $|\varpi_2|^N C \leq 1$; then for $n \geq N$ we have $\left|\varpi_2^N x_n\right| \leq 1$. Then the sum of the terms of x of index $\geq N$ are

$$\pi^N([x_N] + [x_{N+1}]\pi + \dots) = \frac{\pi^N}{\varpi_2^N}([\varpi^N x_N] + [\varpi^N x_{N+1}]\pi + \dots),$$

which lies in $B^{b,+}\left[\frac{[\varpi_1]}{\pi}, \frac{\pi}{[\varpi_2]}\right]$. $\qquad\square$

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Lemma 2.1.3. *If B_I° is the closed unit ball in B_I, then B_I° is the π-adic completion of $B_I^{b,\circ}$.*

Proof. B_I is the completion of B^b with respect to the norm $|\ |_I$. This norm induces the π-adic topology on $B_I^{b,\circ}$, so that the π-adic completion of $B_I^{b,\circ}$ is the same as the closed unit ball in B_I. $\square$

The qth power Frobenius automorphism of F induces a map $\phi\colon B^{b,+} \to B^{b,+}$. For $\rho \in (0,1)$ we have $\left|x^\phi\right|_\rho = |x|^q_{\rho^{1/q}}$. As a result, ϕ induces an endomorphism of B. We have $B^{\phi=1} = E$. (See Prop. 7.1 of [FF11].) We put

$$P = \bigoplus_{d \geq 0} B^{\phi=\pi^d},$$

a graded E-algebra.

Definition 2.1.4. Let $X_{E,F} = \operatorname{Proj} P$.

Theorem 2.1.5. *[FF11], Théorème 10.2*

1. $X_{F,E}$ *is an integral noetherian scheme which is regular of dimension 1.*

2. $H^0(X_{F,E}, \mathcal{O}_{X_{F,E}}) = E$. *(E is the field of definition of $X_{F,E}$)*

3. *For a finite extension E'/E there is a canonical isomorphism $X_{F,E'} \cong X_{F,E} \otimes_E E'$.*

3 The adic curve $X_{F,E}^{\mathrm{ad}}$

3.1 Generalities on formal schemes and adic spaces

The category of adic spaces is introduced in [Hub94]. In brief, an adic space is a topological space X equipped with a sheaf of topological rings $\mathcal{O}_X$, which is locally isomorphic to $\operatorname{Spa}(R, R^+)$. Here (R, R^+) is an "affinoid ring", $R^+ \subset R$ is an open and integrally closed subring, and $\operatorname{Spa}(R, R^+)$ is the space of continuous valuations $|\ |$ on R with $|R^+| \leq 1$.

Definition 3.1.1. A topological ring R is *f-adic* if it contains an open subring R_0 whose topology is generated by a finitely generated ideal $I \subset R_0$. An *affinoid ring* is a pair (R, R^+), where R is f-adic and $R^+ \subset R$ is an open and integrally closed subring consisting of power-bounded elements. A morphism of affinoid algebras $(R, R^+) \to (S, S^+)$ is a continuous homomorphism $R \to S$ which sends $R^+ \to S^+$.

R is a *Tate ring* if it contains a topologically nilpotent unit.

Note that if (R, R^+) and R is Tate, say with topologically nilpotent $\pi \in R$, then $R = R^+[1/\pi]$ and $\pi R \subset R$ is open. Note also that if (R, R^+) is an affinoid algebra over $(E, \mathcal{O}_E)$, then R is Tate.

It is important to note that a given affinoid ring (R, R^+) does not necessarily give rise to an adic space $\mathrm{Spa}(R, R^+)$, because the structure sheaf on $\mathrm{Spa}(R, R^+)$ is not necessarily a sheaf. Let us say that (R, R^+) is *sheafy* if the structure presheaf on $\mathrm{Spa}(R, R^+)$ is a sheaf. Huber shows (R, R^+) is a sheaf when R is "strongly noetherian", meaning that $R\langle X_1, \ldots, X_n \rangle$ is noetherian for all n. In §2 of [SW13] we constructed a larger category of "general" adic spaces, whose objects are sheaves on the category of complete affinoid rings (this category can be given the structure of a site in an obvious way). If (R, R^+) is a (not necessarily sheafy) affinoid ring, then $\mathrm{Spa}(R, R^+)$ belongs to this larger category. If X is an adic space in the general sense, let us call X an *honest adic space* if it belongs to the category of adic spaces in the sense of Huber; *i.e.* if it is locally $\mathrm{Spa}(R, R^+)$ for a sheafy (R, R^+).

Now suppose R is an $\mathcal{O}_E$-algebra which is complete with respect to the topology induced by a finitely generated ideal $I \subset R$ which contains π. Then $\mathrm{Spf}\, R$ is a formal scheme over $\mathrm{Spf}\, \mathcal{O}_E$, and (R, R) is an affinoid ring. One can form the (general) adic space $\mathrm{Spa}(R, R)$, which is fibered over the two-point space $\mathrm{Spa}(\mathcal{O}_E, \mathcal{O}_E)$.

By [SW13], Prop. 2.2.1, $\mathrm{Spf}\, R \mapsto \mathrm{Spa}(R, R)$ extends to a functor $M \mapsto M^{\mathrm{ad}}$ from the category of formal schemes over $\mathrm{Spf}\, \mathcal{O}_E$ locally admitting a finitely generated ideal of definition, to the category of (general) adic spaces over $\mathrm{Spa}(\mathcal{O}_E, \mathcal{O}_E)$.

Definition 3.1.2. Let M be a formal scheme over $\mathrm{Spf}\, \mathcal{O}_E$ which locally admits a finitely generated ideal of definition. The *adic generic fiber* M_η^{ad} is the fiber of M^{ad} over the generic point $\eta = \mathrm{Spa}(E, \mathcal{O}_E)$ of $\mathrm{Spa}(\mathcal{O}_E, \mathcal{O}_E)$. We will also notate this as M_E^{ad}.

Lemma 3.1.3. *Let R be a flat $\mathcal{O}_E$-algebra which is complete with respect to the topology induced by a finitely generated ideal I containing π. Let $f_1, \ldots, f_r, \pi$ be generators for I. For $n \geq 1$, let $S_n = R[f_1^n/\pi, \ldots, f_r^n/\pi]^\wedge$ (π-adic completion). Let $R_n = S_n[1/\pi]$, and let R_n^+ be the integral closure of S_n in R_n. There are obvious inclusions $R_{n+1} \hookrightarrow R_n$ and $R_{n+1}^+ \hookrightarrow R_n^+$. Then*

$$M_E^{\mathrm{ad}} = \varinjlim_n \mathrm{Spa}(R_n, R_n^+).$$

Proof. This amounts to showing that whenever (T, T^+) is a complete affinoid

algebra over $(E, \mathcal{O}_E)$, then

$$M_E^{\mathrm{ad}}(T, T^+) = \varinjlim_n \operatorname{Hom}((R_n, R_n^+), (T, T^+)).$$

An element of the left hand side is a continuous $\mathcal{O}_E$-linear homomorphism $g \colon R \to T^+$; we need to produce a corresponding homomorphism $R_n^+ \to T^+$. We have that $g(f_i) \in T^+$ is topologically nilpotent for $i = 1, \ldots, r$, since f_i is. Since πT^+ is open in T, there exists $n \geq$ so that $g(f_i)^n \in \pi T^+$, $i = 1, \ldots, r$. Thus g extends to a map $R^+[f_1^n/\pi, \ldots, f_r^n/\pi] \to T^+$. Passing to the integral closure of the completion gives a map $R_n \to T^+$ as required. $\qquad\square$

Example 3.1.4. The adic generic fiber of $R = \operatorname{Spf} \mathcal{O}_E[\![t]\!]$ is the rigid open disc over E.

3.2 Formal schemes with perfectoid generic fiber

In this section K is a perfectoid field of characteristic 0, and $\varpi \in K$ satisfies $0 < |\varpi| \leq |p|$. Assume that $\varpi = \varpi^\flat$ for some $\varpi^\flat \in \mathcal{O}_{K^\flat}$, so that $\varpi^{1/p^n} \in \mathcal{O}_K$ for all $n \geq 1$.

Definition 3.2.1. Let S be a ring in characteristic p. S is *semiperfect* if the Frobenius map $S \to S$ is surjective. If S is semiperfect, let $S^\flat = \varprojlim_{x \mapsto x^p} S$, a perfect topological ring.

   If R is a topological $\mathcal{O}_K$-algebra with R/ϖ semiperfect, then write $R^\flat = (R/\varpi)^\flat$, a perfect topological $\mathcal{O}_{K^\flat}$-algebra.

   If R is complete with respect to a finitely generated ideal of definition, then so is $R^\flat$.

Proposition 3.2.2. *Let R be an $\mathcal{O}_K$-algebra which is complete with respect to a finitely generated ideal of definition. Assume that R/π is a semiperfect ring. Then $(\operatorname{Spf} R)_K^{\mathrm{ad}}$ and $(\operatorname{Spf} R^\flat)_{K^\flat}^{\mathrm{ad}}$ are a perfectoid spaces over K and $K^\flat$ respectively, so that in particular they are honest adic spaces. There is a natural isomorphism of perfectoid spaces over $K^\flat$:*

$$(\operatorname{Spf} R)_K^{\mathrm{ad},\flat} \cong (\operatorname{Spf} R^\flat)_{K^\flat}^{\mathrm{ad}}$$

Proof. First we show that $(\operatorname{Spf} R^\flat)_{\eta^\flat}^{\mathrm{ad}}$ is a perfectoid space over $K^\flat$. Let $f_1, \ldots, f_r$ be generators for an ideal of definition of $R^\flat$. Then $(\operatorname{Spf} R^\flat)_{\eta^\flat}^{\mathrm{ad}}$

is the direct limit of subspaces $\mathrm{Spa}(R_n^\flat, R_n^{\flat,+})$ as in Lemma 3.1.3. Here $R_n^\flat = R^\flat \langle f_i^n/\varpi^\flat \rangle [1/\varpi^\flat]$. Let

$$S_n^{\flat,\circ} = R^\flat \left\langle \left(\frac{f_1^n}{\varpi^\flat}\right)^{1/p^\infty}, \cdots, \left(\frac{f_r^n}{\varpi^\flat}\right)^{1/p^\infty} \right\rangle \subset R_n^\flat$$

Then $S_n^{\flat,\circ}$ is a $\varpi^\flat$-adically complete flat $\mathcal{O}_{K^\flat}$-algebra. We claim that $S_n^{\flat,\circ a}$ is a perfectoid $\mathcal{O}_{K^\flat}^a$-algebra, which is to say that the Frobenius map $S_n^{\flat,\circ}/\varpi^{1/p} \to S_n^{\flat,\circ}/\varpi$ is an almost isomorphism. For this we refer to [Sch12], proof of Lemma 6.4(i) (case of characteristic p). This shows that $R_n^\flat = S_n^{\flat,\circ}[1/\varpi^\flat]$ is a perfectoid $K^\flat$-algebra, and thus $\mathrm{Spf}(R_n^\flat, R_n^{\flat,+})$ is a perfectoid affinoid, and in particular it is an honest adic space.

To conclude that $(\mathrm{Spf}\, R^\flat)_{K^\flat}^{\mathrm{ad}}$ is a perfectoid space, one needs to show that its structure presheaf is a sheaf. (I thank Kevin Buzzard for pointing out this subtlety.) But since $(\mathrm{Spf}\, R^\flat)_{K^\flat}^{\mathrm{ad}}$ is the direct limit of the $\mathrm{Spa}(R_n^\flat, R_n^{\flat,+})$, its structure presheaf is the inverse limit of the pushforward of the structure presheaves of the $\mathrm{Spa}(R_n^\flat, R_n^{\flat,+})$, which are all sheaves. Now we can use the fact that an arbitrary inverse limit of sheaves is again a sheaf.

We now turn to characteristic 0. Recall the sharp map $f \mapsto f^\sharp$, which is a map of multiplicative monoids $R^\flat \to R$. The elements $f_1^\sharp, \ldots, f_r^\sharp$ generate an ideal of definition of R. Then $(\mathrm{Spf}\, R)_\eta^{\mathrm{ad}}$ is the direct limit of subspaces $\mathrm{Spa}(R_n, R_n^+)$, where $R_n = R \langle f_i^n/\varpi \rangle [1/\varpi]$. The proof of [Sch12], Lemma 6.4(i) (case of characteristic 0) shows that $\mathrm{Spa}(R_n, R_n^+)$ is a perfectoid affinoid, and Lemma 6.4(iii) shows that the tilt of $\mathrm{Spa}(R_n, R_n^+)$ is $\mathrm{Spa}(R_n^\flat, R_n^{\flat,+})$. $\qquad\square$

3.3 The adic spaces $Y_{F,E}^{\mathrm{ad}}$ and $X_{F,E}^{\mathrm{ad}}$

Once again, $E/\mathbb{Q}_p$ is a finite extension.

A scheme X/E of finite type admits a canonical analytification X^{ad}: this is a rigid space together with a morphism $X \to X^{\mathrm{ad}}$ of locally ringed spaces satisfying the appropriate universal property. GAGA holds as well: If X is a projective variety, then the categories of coherent sheaves on X and X^{ad} are equivalent.

The Fargues-Fontaine curve $X = X_{F,E}$, however, is not of finite type, and it is not obvious that such an analytification exists. However, in [Far13] there appears an adic space X^{ad} defined over E, such that X and X^{ad} satisfy a suitable formulation of GAGA.

Definition 3.3.1. Give $B^{b,+} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ the I-adic topology, where $I = ([\varpi], \pi)$ and $\varpi \in \mathcal{O}_F$ is an element with $0 < |\varpi| < 1$. Note that $B^{b,+}$ is I-adically complete. Let

$$Y^{\mathrm{ad}} = (\mathrm{Spf}\, B^{b,+})_E^{\mathrm{ad}} \setminus \{0\},$$

where "0" refers to the valuation on $B^{b,+} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ pulled back from the π-adic valuation on $W_{\mathcal{O}_E}(k_F)$.

Proposition 3.3.2. *As (general) adic spaces we have*

$$Y^{\mathrm{ad}} = \varinjlim_{I} \mathrm{Spa}(B_I, B_I^\circ).$$

This shows that our definition of Y^{ad} agrees with the definition in [Far13], Définition 2.5.

Proof. By Prop. 3.1.3 we have

$$(\mathrm{Spf}\, B^{b,+})_E^{\mathrm{ad}} = \varprojlim_{\varpi_1} \mathrm{Spa}(R_{\varpi_1}, R_{\varpi_1}^+),$$

where ϖ_1 runs over elements of $\mathcal{O}_F$ with $0 < |\varpi| < 1$, $R_{\varpi_1}^+$ is the π-adic completion of $B^{b,+}\left[\frac{[\varpi_1]}{\pi}\right]$, and $R_{\varpi_1} = R_{\varpi_1}^+[1/\pi]$.

By definition, Y^{ad} is the complement in $(\mathrm{Spf}\, B^{b,+})_E^{\mathrm{ad}}$ of the single valuation pulled back from the π-adic valuation on $W_{\mathcal{O}_E}(k)$. Thus if $x \in Y^{\mathrm{ad}}$ we have $|[\varpi](x)| \neq 0$ for all $\varpi \in \mathcal{O}_F \setminus \{0\}$. Thus there exists $\varpi_2 \in \mathcal{O}_F$ with $|\varpi_2(x)| \geq |\pi(x)|$. On the other hand we have $x \in \mathrm{Spa}(R_{\varpi_1}, R_{\varpi_1}^+)$ for some ϖ_1. Thus x belongs to the rational subset $\mathrm{Spa}(B_I, B_I^\circ) \subset \mathrm{Spa}(R_{\varpi_1}, R_{\varpi_1}^+)$, where $I = [|\varpi_1|, |\varpi_2|]$. This shows that the $\mathrm{Spa}(B_I, B_I^\circ)$ cover Y^{ad}. $\square$

The following conjecture is due to Fargues.

Conjecture 3.3.3. *Let $I \subset (0, 1)$ be a closed interval. The Banach algebra B_I is strongly noetherian. That is, for every $n \geq 1$, $B_I \langle T_1, \ldots, T_n \rangle$ is noetherian.*

Despite not being able to prove Conj. 3.3.3, we have the following proposition (Théorème 2.1 of [Far13]), which is proved by extending scalars to a perfectoid field and appealing to results of [Sch12].

Proposition 3.3.4. Y^{ad} *is an honest adic space.*

Definition 3.3.5. The adic space X^{ad} is the quotient of Y^{ad} by the automorphism ϕ.

3.4 The adic Fargues-Fontaine curve in characteristic p

The entire story of the Fargues-Fontaine curve can be retold when E is replaced by a local field of residue field $\mathbb{F}_q$. The construction is very similar, except that $B^{b,+} = W(\mathcal{O}_F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ must be replaced by $\mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E$. One arrives at a curve $X_{F,E}$ defined over E satisfying the same properties as Thm. 2.1.5.

One also gets spaces $X_{F,E}^{\mathrm{ad}}$ and $Y_{F,E}^{\mathrm{ad}}$. Namely,

$$Y_{F,E}^{\mathrm{ad}} = \mathrm{Spf}\left(\mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E\right)_E^{\mathrm{ad}} \setminus \{0\},$$

where "0" refers to the pullback of the valuation on $\mathcal{O}_F$ through the quotient map $\mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E \to \mathcal{O}_F$. As before, $X_{F,E}^{\mathrm{ad}}$ is defined as the quotient of $Y_{F,E}^{\mathrm{ad}}$ by the automorphism ϕ coming from the Frobenius on F.

Since $E \cong \mathbb{F}_q((t))$, we have

$$Y_{F,E}^{\mathrm{ad}} = \mathrm{Spf}\,\mathcal{O}_F[\![t]\!]_F^{\mathrm{ad}} \setminus \{0\}.$$

This is nothing but the punctured rigid open disc D_F^*. We have the quotient $X_{F,E}^{\mathrm{ad}} = Y_{F,E}^{\mathrm{ad}}/\phi^{\mathbb{Z}}$. Note that since ϕ does not act F-linearly, $X_{F,E}^{\mathrm{ad}}$ does not make sense as a rigid space over F.

3.5 Tilts

Suppose once again that E has characteristic 0. Let K be a perfectoid field containing E. Let $Y_{F,E}^{\mathrm{ad}} \hat{\otimes} K$ be the strong completion of the base change to K of Y^{ad}. Then

$$Y_{F,E}^{\mathrm{ad}} \hat{\otimes} K = \mathrm{Spf}\left(W_{\mathcal{O}_E}(\mathcal{O}_F) \hat{\otimes}_{W(\mathbb{F}_q)} \mathcal{O}_K\right)_K^{\mathrm{ad}} \setminus \{0\}.$$

Note that $(W_{\mathcal{O}_E}(\mathcal{O}_F) \hat{\otimes}_{W(\mathbb{F}_q)} \mathcal{O}_K)/\pi = \mathcal{O}_F \otimes_{\mathbb{F}_q} \mathcal{O}_K/\pi$ is semiperfect, and $(W_{\mathcal{O}_E}(\mathcal{O}_F) \hat{\otimes}_{W(\mathbb{F}_q)} \mathcal{O}_K)^{\flat} = \mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_{K^{\flat}}$. By Prop. 3.2.2, $Y_{F,E}^{\mathrm{ad}} \hat{\otimes} K$ is a perfectoid space, and

$$(Y_{F,E}^{\mathrm{ad}} \hat{\otimes} K)^{\flat} \cong \mathrm{Spf}\left(\mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_{K^{\flat}}\right)_{K^{\flat}}^{\mathrm{ad}} \setminus \{0\}. \qquad (3.5.1)$$

As a special case, let E_n be the field obtained by adjoining the π^n-torsion in a Lubin-Tate formal group over E, and let $E_\infty = \bigcup_{n \geq 1} E_n$. Then $\hat{E}_\infty$ is a perfectoid field. Let $L(E)$ be the imperfect field of norms for the extension E_∞/E. As a multiplicative monoid we have

$$L(E) = \varprojlim E_n,$$

where the inverse limit is taken with respect to the norm maps $E_{n+1} \to E_n$. $L(E) \cong \mathbb{F}_q((t))$ is a local field, and $\hat{E}_\infty \cong \mathbb{F}_q((t^{1/q^\infty}))$ is the completed perfection of $L(E)$.

The following proposition follows immediately from Eq. (3.5.1).

Proposition 3.5.1.

$$(Y_{F,E}^{\mathrm{ad}} \hat{\otimes} \hat{E}_\infty)^\flat \cong Y_{F,L(E)}^{\mathrm{ad}} \hat{\otimes} \hat{E}_\infty^\flat.$$

This is Thm. 2.7(2) of [Far13].

3.6 Classification of vector bundles on $X_{F,E}$

In this section we review the results of [FF11] concerning the classification of vector bundles on $X_{F,E}$.

Recall that $X_{F,E} = \mathrm{Proj}\, P$, where P is the graded ring $\bigoplus_{d \geq 0}(B^+)^{\phi = \pi^d}$. For $d \in \mathbb{Z}$, let $P[d]$ be the graded P-module obtained from P by shifting degrees by d, and let $\mathcal{O}_{X_E}(d)$ be the corresponding line bundle on X_E. For $h \geq 1$, let E_h/E be the unramified extension of degree h, let $\pi_h \colon X_{F,E_h} = X_{F,E} \otimes E_h \to X_{F,E}$ be the projection. If $(d,h) = 1$, define $\mathcal{O}_{X_{F,E}}(d/h) = \pi_{h*}\mathcal{O}_{X_{F,E_h}}(d)$, a vector bundle on $\mathcal{O}_{X_{F,E}}$ of rank h. One thus obtains a vector bundle $\mathcal{O}_{X_{F,E}}(\lambda)$ for any $\lambda \in \mathbb{Q}$, which satisfy $\mathcal{O}_{X_{F,E}}(\lambda) \otimes \mathcal{O}_{X_{F,E}}(\lambda') \cong \mathcal{O}_{X_{F,E}}(\lambda + \lambda')$.

Proposition 3.6.1. *Let $\lambda \in \mathbb{Q}$. Then $H^0(\lambda) \neq 0$ if and only if $\lambda \geq 0$.*

Theorem 3.6.2. *Every vector bundle $\mathcal{F}$ on $X_{F,E}$ is isomorphic to one of the form $\bigoplus_{i=1}^s \mathcal{O}_{X_{F,E}}(\lambda_i)$, with $\lambda_i \in \mathbb{Q}$. Furthermore, $\mathcal{F}$ determines the λ_i up to permutation.*

3.7 Classification of vector bundles on $X_{F,E}^{\mathrm{ad}}$

In the absence of the noetherian condition of Conj. 3.3.3, one does not have a good theory of coherent sheaves on $X_{F,E}^{\mathrm{ad}}$ when E has characteristic 0. This frustrates attempts to prove an analogue of Thm. 3.6.2 for $X_{F,E}^{\mathrm{ad}}$. Nonetheless, in [Far13], Fargues gives an ad hoc notion of vector bundle on $X_{F,E}^{\mathrm{ad}}$, and proves a GAGA theorem relating vector bundles on $X_{F,E}$ to those on $X_{F,E}^{\mathrm{ad}}$.

When E has characteristic p, however, $Y_{F,E}^{\mathrm{ad}}$ is isomorphic to the punctured disc $D^* = (\mathrm{Spf}\, \mathcal{O}_F[\![t]\!])_F^{\mathrm{ad}} \setminus \{0\}$ over F. In particular it is a rigid space, which does have a well-behaved theory of coherent sheaves. In this case we

are in the setting of [HP04], which classifies ϕ-equivariant vector bundles $\mathcal{F}$ on the punctured disc $D^* = Y_{F,E}^{\mathrm{ad}}$. In [HP04], these are called σ-bundles, and they admit a Dieudonné-Manin classification along the lines of Thm. 3.6.2. A σ-bundle is one and the same thing as a vector bundle on the quotient $Y_{F,E}^{\mathrm{ad}}/\phi^{\mathbb{Z}} = X_{F,E}^{\mathrm{ad}}$. One gets a vector bundle $\mathcal{O}(\lambda)$ on $X_{F,E}^{\mathrm{ad}}$ for every $\lambda \in \mathbb{Q}$.

For the convenience of the reader we state the main definitions and constructions of [HP04].

A *vector bundle* on D^* is by definition a locally free coherent sheaf of $\mathcal{O}_{D^*}$-modules. The global sections functor is an equivalence between the category of vector bundles on D^* and the category of finitely generated projective modules over $\mathcal{O}_{D^*}(D^*)$. The Frobenius automorphism $x \mapsto x^q$ on F induces an automorphism $\sigma \colon D^* \to D^*$ (the arithmetic Frobenius). Note that the F-points of D^* are $\{x \in F | 0 < |x| < 1\}$, and on this set σ acts as $x \mapsto x^{q^{-1}}$.

Definition 3.7.1. A σ-*bundle* on D^* is a pair $(\mathcal{F}, \tau_{\mathcal{F}})$, where $\mathcal{F}$ is a vector bundle on D^* and $\tau_{\mathcal{F}}$ is an isomorphism $\sigma^*\mathcal{F}\overset{\sim}{\to}\mathcal{F}$. If $(\mathcal{F}, \tau_{\mathcal{F}})$ is a σ-bundle, then a *global section* of $\mathcal{F}$ is a global section of $\mathcal{F}$ which is invariant under $\tau_{\mathcal{F}}$.

We will refer to the pair $(\mathcal{F}, \tau_{\mathcal{F}})$ simply as $\mathcal{F}$. Then the space of global sections of $\mathcal{F}$ will be denoted $H^0(\mathcal{F})$.

The trivial vector bundle $\mathcal{O} = \mathcal{O}_{D^*}$ is a σ-bundle whose global sections $H^0(\mathcal{O})$ are the ring of power series $\sum_{n\in\mathbb{Z}} a_n t^n$ which converge on D^* and which satisfy $a_n^q = a_n$ (thus $a_n \in \mathbb{F}_q$). This ring is easily seen to be the field $\mathbb{F}_q((t))$ (see [HP04], Prop. 2.3).

Let $n \in \mathbb{Z}$. The twisting sheaf $\mathcal{O}(n)$ is a σ-bundle with underlying sheaf $\mathcal{O}$. The isomorphism $\tau_{\mathcal{O}(n)}$ is defined as the composition of $\tau_{\mathcal{O}} \colon \sigma^*\mathcal{O}_{D^*} \to \mathcal{O}_{D^*}$ followed by multiplication by t^{-n}.

For $r \geq 1$, let $[r]$ denote the morphism

$$[r]\colon D^* \to D^*, \ t \mapsto t^r.$$

Then for an integer d relatively prime to r we set

$$\mathcal{F}_{d,r} = [r]_*\mathcal{O}(d),$$

together with the induced isomorphism $\tau_{\mathcal{F}_{d,r}} = [r]_*\tau_{\mathcal{O}(d)}$. Then $\mathcal{F}_{d,r}$ is a σ-bundle of rank r.

In keeping with the notation of [FF11], let us write $\mathcal{O}(d/r) = \mathcal{F}_{d,r}$ whenever d and r are relatively prime integers with $r \geq 1$. Then $\mathcal{O}(\lambda)$ makes sense for any $\lambda \in \mathbb{Q}$. We have $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda') \cong \mathcal{O}(\lambda + \lambda')$, and $\mathcal{O}(\lambda)^* \cong \mathcal{O}(-\lambda)$.

Proposition 3.7.2. *[HP04], Prop. 8.4 Let $\lambda \in \mathbb{Q}$. Then $H^0(\lambda) \neq 0$ if and only if $\lambda > 0$.*

Hartl and Pink give a complete description of the category of σ-bundles. In particular we have the following classification theorem of Dieudonné-Manin type.

Theorem 3.7.3. *[HP04], Thm. 11.1 and Cor. 11.8 Every σ-bundle $\mathcal{F}$ is isomorphic to one of the form $\bigoplus_{i=1}^{s} \mathcal{O}(\lambda_i)$, with $\lambda_i \in \mathbb{Q}$. Furthermore, $\mathcal{F}$ determines the λ_i up to permutation.*

Since $Y_{F,E}^{\mathrm{ad}} \cong D^*$, with ϕ corresponding to σ, vector bundles on $X_{F,E}^{\mathrm{ad}} = Y_{F,E}^{\mathrm{ad}}/\phi^{\mathbb{Z}}$ correspond to σ-bundles, and we have a similar classification of them.

3.8 $X_{F,E}^{\mathrm{ad}}$ is geometrically simply connected

In [FF11], Thm. 18.1, it is shown that the scheme $X = X_{F,E}$ is geometrically connected, which is to say that every finite étale cover of $X_{\overline{E}}$ is split. We need a similar result for the adic curve $X_{F,E}^{\mathrm{ad}}$:

Theorem 3.8.1. *$E' \mapsto X_{F,E}^{\mathrm{ad}} \otimes E'$ is an equivalence between the category of finite étale E-algebras and the category of finite étale covers of $X_{F,E}^{\mathrm{ad}}$.*

Unfortunately, Thm. 3.8.1 cannot be deduced directly from the corresponding theorem about the scheme X. Following a suggestion of Fargues, we can still mimic the proof of the simple-connectedness of X in this adic context, by first translating the problem into characteristic p, and applying the Hartl-Pink classification of σ-bundles (Thm. 3.7.3).

Lemma 3.8.2. *Let $f\colon Y \to X$ be a finite étale morphism of adic spaces of degree d. Then $\mathcal{F} = f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module, and*

$$\left(\bigwedge^d \mathcal{F} \right)^{\otimes 2} \cong \mathcal{O}_X.$$

Proof. If A is a ring, and if B is a finite étale A-algebra, then B is flat and of finite presentation, hence locally free. Furthermore, the trace map

$$\mathrm{tr}_{B/A}\colon B \otimes B \to A$$

is perfect, so that B is self-dual as an A-module. Globalizing, we get that $\mathcal{F}$ is locally free and $\mathcal{F} \cong \mathcal{F}^* = \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$. Taking top exterior powers shows that $\bigwedge^d \mathcal{F} \cong \bigwedge^d \mathcal{F}^* = \left(\bigwedge^d \mathcal{F} \right)^*$, so that the tensor square of $\bigwedge^d \mathcal{F}$ is trivial. $\qquad\square$

The same lemma appears in [FF11], Prop. 4.7.

Let H_E be the Lubin-Tate formal $\mathcal{O}_E$-module corresponding to the uniformizer π, so that $[\pi]_{H_E}(T) \equiv T^q \pmod{\pi}$. Let K be the completion of the field obtained by adjoining the torsion points of H_E to E. Then K is a perfectoid field and $K^\flat = \mathbb{F}_q((t^{1/q^\infty}))$.

Proposition 3.8.3. *Suppose E has characteristic p. Then $X^{\mathrm{ad}}_{F,E}$ is geometrically simply connected. In other words, $E' \mapsto X^{\mathrm{ad}}_{F,E} \otimes E'$ is an equivalence between the category of finite étale E-algebras and finite étale covers of $X^{\mathrm{ad}}_{F,E}$.*

Proof. It suffices to show that if $f \colon Y \to X^{\mathrm{ad}}_{F,E}$ is a finite étale cover of degree n with Y geometrically irreducible, then $n = 1$. Given such a cover, let $\mathcal{F} = f_* \mathcal{O}_Y$. This is a sheaf of $\mathcal{O}_{X^{\mathrm{ad}}_{F,E}}$-algebras, so we have a multiplication morphism $\mu \colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$.

Consider $\mathcal{F}$ as a σ-bundle. By Thm. 3.7.3, $\mathcal{F} \cong \bigoplus_{i=1}^n \mathcal{O}(\lambda_i)$, for a collection of slopes $\lambda_i \in \mathbb{Q}$ (possibly with multiplicity). Assume that $\lambda_1 \geq \cdots \geq \lambda_n$.

The proof now follows that of [FF11], Théorème 18.1. We claim that $\lambda_1 \leq 0$. Assume otherwise, so that $\lambda_1 > 0$. After pulling back $\mathcal{F}$ through some $[d] \colon D^* \to D^*$, we may assume that $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, s$ (see [HP04], Prop. 7.1(a)). The restriction of μ to $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1)$ is the direct sum of morphisms

$$\mu_{1,1,k} \colon \mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to \mathcal{O}(\lambda_k)$$

for $k = 1, \ldots, n$. The morphism $\mu_{1,1,k}$ is tantamount to a global section of

$$\mathrm{Hom}(\mathcal{O}(\lambda_1)^{\otimes 2}, \mathcal{O}(\lambda_k)) = \mathcal{O}(\lambda_k - 2\lambda_1).$$

But since $\lambda_k - 2\lambda_1 < 0$, Prop. 3.7.2 shows that $H^0(\mathcal{O}(\lambda_k - 2\lambda_1)) = 0$. Thus the multiplication map $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to 0$ is 0. This means that $H^0(Y, \mathcal{O}_Y)$ contains zero divisors, which is a contradiction because Y is irreducible.

Thus $\lambda_1 \leq 0$, and thus $\lambda_i \leq 0$ for all i. By Lemma 3.8.2, $(\bigwedge^n \mathcal{F})^{\otimes 2} \cong \mathcal{O}_{D^*}$, from which we deduce $\sum_{i=1}^n \lambda_i = 0$. This shows that $\lambda_i = 0$ for all i, and therefore $\mathcal{F} \cong \mathcal{O}_X^n$. We find that $E' = H^0(Y, \mathcal{O}_Y)$ is an étale E-algebra of degree n. Since Y is geometrically irreducible, $E' \otimes_E E''$ must be a field for every separable field extension E''/E, which implies that $E' = E$ and $n = 1$. $\square$

Proposition 3.8.4. *Suppose E has characteristic 0. Then $X^{\mathrm{ad}}_{F,E}$ is geometrically simply connected.*

Proof. Let C be a complete algebraically closed field containing E. We want to show that $X^{\mathrm{ad}}_{F,E} \hat{\otimes} C$ admits no nontrivial finite étale covers. Such covers are equivalent to covers of its tilt, which is

$$(X^{\mathrm{ad}}_{F,E} \hat{\otimes} C)^\flat \cong X^{\mathrm{ad}}_{F,L(E)} \hat{\otimes} C^\flat.$$

The latter is simply connected, by Prop. 3.8.3 (note that $C^\flat$ is algebraically closed), which shows there are no nontrivial covers. $\square$

4   Proof of the main theorem

As before, H_E is the Lubin-Tate formal $\mathcal{O}_E$-module attached to the uniformizer π. Let us recall the construction of H_E: choose a power series $f(T) \in T\mathcal{O}_E[\![T]\!]$ with $f(T) \equiv T^q \pmod{\pi}$. Then H_E is the unique formal $\mathcal{O}_E$-module satisfying $[\pi]_{H_E}(T) = f(T)$. Let $t = t_1, t_2, \ldots$ be a compatible family of roots of $f(T), f(f(T)), \ldots$, and let $E_n = E(t_n)$. For each $n \geq 1$, $H_E[\pi^n]$ is a free $(\mathcal{O}_E/\pi^n)$-module of rank 1, and the action of Galois induces an isomorphism $\mathrm{Gal}(E_n/E) \cong (\mathcal{O}_E/\pi^n)^\times$.

Let $H_{E,0} = H_E \otimes_{\mathcal{O}_E} \mathbb{F}_q$.

Lemma 4.0.5. *For all $n \geq 2$ we have an isomorphism $H_{E,0}[\pi^{n-1}] \cong \mathrm{Spec}\, \mathcal{O}_{E_n}/t$. The inclusion $H_{E,0}[\pi^{n-1}] \to H_{E,0}[\pi^n]$ corresponds to the qth power Frobenius map $\mathrm{Frob}_q \colon \mathcal{O}_{E_{n+1}}/t \to \mathcal{O}_{E_n}/t$.*

Proof. We have $H_{E,0}[\pi^{n-1}] = \mathrm{Spec}\, \mathbb{F}_q[T]/f^{(n-1)}(T) = \mathrm{Spec}\, \mathbb{F}_q[T]/T^{q^{n-1}}$. Meanwhile $\mathcal{O}_{E_n} = \mathcal{O}_{E_1}[T]/(f^{(n-1)}(T) - t)$, so that $\mathcal{O}_{E_n}/t = \mathbb{F}_q[T]/T^{q-1}$. Thus $H_{E,0}[\pi^{n-1}] \cong \mathrm{Spec}\, \mathcal{O}_{E_n}/t$. The second claim in the lemma follows from $[\pi]_{H_{E,0}}(T) = T^q$. $\square$

Let $\tilde{H}_E$ be the *universal cover*:

$$\tilde{H}_E = \varprojlim_\pi H_E.$$

Then $\tilde{H}_E$ is an E-vector space object in the category of formal schemes over $\mathcal{O}_E$. We will call such an object a *formal E-vector spaces*.

Let $\tilde{H}_{E,0} = \tilde{H}_E \otimes_{\mathcal{O}_E} \mathbb{F}_q$, a formal E-vector space over $\mathbb{F}_q$. Since $\tilde{H}_{E,0} =$

$\operatorname{Spf} \mathbb{F}_q[\![T]\!]$ and $[\pi]_{H_{E,0}}(T) = T^q$, we have

$$
\begin{aligned}
\tilde{H}_{E,0} &= \varprojlim_{\pi} \operatorname{Spf} \mathbb{F}_q[\![T]\!] \\
&= \operatorname{Spf} \left(\varinjlim_{T \mapsto T^q} \mathbb{F}_q[\![T]\!] \right)^{\wedge} \\
&= \operatorname{Spf} \mathbb{F}_q[\![T^{1/q^\infty}]\!].
\end{aligned}
$$

In fact we also have $\tilde{H}_E = \operatorname{Spf} \mathcal{O}_E[\![T^{1/q^\infty}]\!]$, see [Wei14], Prop. 2.4.2(2).

Lemma 4.0.6. *We have an isomorphism of formal E-vector spaces over $\mathbb{F}_q$:*

$$
\tilde{H}_{E,0} = \varinjlim_{\pi} \varprojlim_{n} H_{E,0}[\pi^n].
$$

Proof. For each $n \geq 1$ we have the closed immersion $H_{E,0}[\pi^n] \to H_{E,0}$. Taking inverse limits gives a map $\varprojlim_n H_{E,0}[\pi^n] \to \tilde{H}_{E,0}$, and taking injective limits gives a map $\varinjlim_{\pi} \varprojlim_n H_{E,0}[\pi^n] \to \tilde{H}_{E,0}$. The corresponding homomorphism of topological rings is

$$
\mathbb{F}_q[\![T^{1/q^\infty}]\!] \to \varprojlim_{T \mapsto T^q} \varinjlim_{n} \mathbb{F}_q[T^{1/q^n}]/T = \varprojlim_{T \mapsto T^q} \mathbb{F}_q[T^{1/q^\infty}]/T,
$$

which is an isomorphism. $\square$

Proposition 4.0.7. *There exists an isomorphism of formal schemes over $\mathbb{F}_q$:*

$$
\tilde{H}_{E,0} \cong \operatorname{Spf} \mathcal{O}_{\hat{E}_\infty^\flat}.
$$

This isomorphism is $E^\times$-equivariant, where the action of $E^\times$ on $\mathcal{O}_{\hat{E}_\infty^\flat}$ is defined as follows: $\mathcal{O}_E^\times$ acts through the isomorphism $\mathcal{O}_E^\times \cong \operatorname{Gal}(E_\infty/E)$ of class field theory, and $\pi \in E^\times$ acts as the qth power Frobenius map.

Proof. Combining Lemmas 4.0.5 and 4.0.6, we get

$$
\begin{aligned}
\tilde{H}_{E,0} &\cong \varinjlim_{\pi} \varprojlim_{n} H_{E,0}[\pi^n] \\
&\cong \varinjlim_{\mathrm{Frob}_q} \operatorname{Spec} \mathcal{O}_{E_\infty}/t \\
&\cong \operatorname{Spf} \mathcal{O}_{E_\infty^\flat}.
\end{aligned}
$$

The compatibility of this isomorphism with the $\mathcal{O}_E^\times$ follows from the definition of the isomorphism $\mathcal{O}_E^\times \cong \operatorname{Gal}(E_\infty/E)$ of local class field theory. The compatibility of the action of π follows from the second statement in Lemma 4.0.5. $\square$

Let C be a complete algebraically closed field containing E. Write $\tilde{H}^{\mathrm{ad}}_{E,C}$ for the generic fiber of $\tilde{H}\hat{\otimes}\mathcal{O}_C$. By Prop. 3.2.2, $\tilde{H}^{\mathrm{ad}}_{E,C}$ is a perfectoid space.

Proposition 4.0.8. *There exists an isomorphism of adic spaces*

$$\left(\tilde{H}^{\mathrm{ad}}_{E,C}\setminus\{0\}\right)^{\flat} \cong \left(Y^{\mathrm{ad}}_{C^{\flat},E}\hat{\otimes}\hat{E}_{\infty}\right)^{\flat}$$

equivariant for the action of $\mathcal{O}_E^{\times}$ (which acts on E_{∞} by local class field theory). The action of $\pi \in E^{\times}$ on the left corresponds to the action of $\phi^{-1}\otimes 1$ on the right, up to composition with the absolute Frobenius morphism on $\left(Y_{C^{\flat},E}\hat{\otimes}\hat{E}_{\infty}\right)^{\flat}$.

Proof. By Prop. 4.0.7 we have an isomorphism $\tilde{H}_0 \cong \mathrm{Spf}\,\mathcal{O}_{\hat{E}^{\flat}_{\infty}}$. Extending scalars to $\mathcal{O}_C$, taking the adic generic fiber and applying tilts gives an isomorphism (cf. Prop. 3.2.2)

$$
\begin{aligned}
\left(\tilde{H}^{\mathrm{ad}}_{E,C}\right)^{\flat} &\cong \mathrm{Spf}\left(\mathcal{O}_C\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_{E_{\infty}}\right)^{\flat,\mathrm{ad}}_{C} \\
&\cong \mathrm{Spf}\left(\mathcal{O}_{C^{\flat}}\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_{\hat{E}^{\flat}_{\infty}}\right)^{\mathrm{ad}}_{C^{\flat}}
\end{aligned}
$$

The adic space $\mathrm{Spf}\left(\mathcal{O}_{C^{\flat}}\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_{\hat{E}^{\flat}_{\infty}}\right)^{\mathrm{ad}}_{C^{\flat}}$ has two "Frobenii": one coming from $\mathcal{O}_{C^{\flat}}$ and the other coming from $\mathcal{O}_{\hat{E}^{\flat}_{\infty}}$. Their composition is the absolute qth power Frobenius. The action of π on $\tilde{H}^{\mathrm{ad},\flat}_{C}$ corresponds to the Frobenius on $\mathcal{O}_{\hat{E}^{\flat}_{\infty}}$. Removing the "origin" from both sides of this isomorphism and using Prop. 3.5.1 gives the claimed isomorphism.

$\square$

Lemma 4.0.9. *Let X be a perfectoid space which is fibered over $\mathrm{Spec}\,\mathbb{F}_q$, and suppose $f\colon X \to X$ is an $\mathbb{F}_q$-linear automorphism. Let $\mathrm{Frob}_q\colon X \to X$ be the absolute Frobenius automorphism of X. Then the category of f-equivariant finite étale covers of X is equivalent to the category of $f\circ\mathrm{Frob}_q$-equivariant finite étale covers of X.*

Proof. First observe that a perfectoid algebra in characteristic p is necessarily perfect ([Sch12], Prop. 5.9), which implies that absolute Frobenius is an automorphism of any perfectoid space in characteristic p. Then note that if $Y \to X$ is a finite étale cover, then Y is also perfectoid ([Sch12], Thm. 7.9(iii)), so that Frob_q is an automorphism of Y.

The proof of the lemma is now formal: if $g\colon Y \to X$ is a finite étale cover and $f_Y\colon Y \to Y$ lies over $f\colon X \to X$, then $f_Y\circ\mathrm{Frob}_q\colon Y \to Y$ lies

over $f \circ \mathrm{Frob}_q \colon X \to X$. Since Frob_q is invertible on Y, the functor is invertible. $\square$

We can now prove the main theorem.

Theorem 4.0.10. *There is an equivalence between the category of $E^\times$-equivariant étale covers of $\tilde{H}_C^{\mathrm{ad}} \backslash \{0\}$ and the category of finite étale E-algebras.*

Proof. In the following chain of equivalences, "G-cover of X" is an abbreviation for "G-equivariant finite étale cover of X".

$$
\begin{aligned}
& \left\{ E^\times\text{-covers of } \tilde{H}_C^{\mathrm{ad}} \backslash \{0\} \right\} \\
\cong\ & \left\{ E^\times\text{-covers of } \tilde{H}_C^{\mathrm{ad},\flat} \backslash \{0\} \right\} && \text{[Sch12], Thm. 7.12} \\
\cong\ & \left\{ \mathcal{O}_E^\times \times (\phi^{-1} \circ \mathrm{Frob}_q)^{\mathbb{Z}}\text{-covers of } Y_{C^\flat, L(E)}^{\mathrm{ad}} \otimes \hat{E}_\infty^\flat \right\} && \text{Prop. 4.0.8} \\
\cong\ & \left\{ \mathcal{O}_E^\times \times \phi^{\mathbb{Z}}\text{-covers of } Y_{C^\flat, L(E)}^{\mathrm{ad}} \otimes \hat{E}_\infty^\flat \right\} && \text{Lemma. 4.0.9} \\
\cong\ & \left\{ \mathcal{O}_E^\times\text{-covers of } X_{C^\flat, L(E)}^{\mathrm{ad}} \otimes \hat{E}_\infty^\flat \right\} && \text{Defn. of } X_{C^\flat, L(E)}^{\mathrm{ad}} \\
\cong\ & \left\{ \mathcal{O}_E^\times\text{-equivariant finite étale } \hat{E}_\infty^\flat\text{-algebras} \right\} && \text{Prop. 3.8.3} \\
\cong\ & \left\{ \mathcal{O}_E^\times\text{-equivariant finite étale } \hat{E}_\infty\text{-algebras} \right\} && \text{Theory of norm fields} \\
\cong\ & \left\{ \text{Finite étale } E\text{-algebras} \right\} && L \mapsto L^{\mathcal{O}_E^\times}
\end{aligned}
$$

$\square$

4.1 Functoriality in E.

Let us write
$$
Z_E = (\tilde{H}_{E,C}^{\mathrm{ad}} \backslash \{0\})/E^\times.
$$

As in the introduction, Z_E may be considered as a sheaf on Perf_C, although the reader may also interpret the definition of Z_E purely formally, keeping in mind that a "finite étale cover of Z_E" means the same thing as an $E^\times$-equivariant finite étale cover of $\tilde{H}_{E,C}^{\mathrm{ad}}$.

Let us check that Z_E really only depends on E. The construction depends on the choice of Lubin-Tate formal $\mathcal{O}_E$-module $H = H_E$, which depends in turn on the choice of uniformizer π. If $\pi' \in E$ is a different uniformizer, with corresponding $\mathcal{O}_E$-module H', then H and H' become isomorphic after base extension to $\mathcal{O}_{\hat{E}^{\mathrm{nr}}}$, the ring of integers in the completion of the maximal unramified extension of E. Such an isomorphism is unique up to multiplication by $E^\times$. Thus the adic spaces $\tilde{H}_C^{\mathrm{ad}}$ and $(\tilde{H}')_C^{\mathrm{ad}}$ are isomorphic,

and we get a *canonical* isomorphism $(\tilde{H}_C^{\mathrm{ad}} \setminus \{0\})/E^\times \to (\tilde{H}_C^{\mathrm{ad}} \setminus \{0\})/E^\times$. Thus Z_E only depends on E.

It is also natural to ask whether the formation of Z_E is functorial in E. That is, given a finite extension E'/E of degree d, there ought to be a "norm" morphism $N_{E'/E} \colon Z_{E'} \to Z_E$, which makes the following diagram commute:

$$
\begin{array}{ccc}
\pi_1^{\text{ét}}(Z_{E'}) & \xrightarrow{\;N_{E'/E}\;} & \pi_1^{\text{ét}}(Z_E) \\
\Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\
\mathrm{Gal}(\overline{E}/E') & \longrightarrow & \mathrm{Gal}(\overline{E}/E).
\end{array}
\qquad (4.1.1)
$$

There is indeed such a norm morphism; it is induced from the *determinant* morphism on the level of π-divisible $\mathcal{O}_E$-modules. The existence of exterior powers of such modules is the subject of [Hed10]. Here is the main result we need[1] (Thm. 4.34 of [Hed10]): let G be a π-divisible $\mathcal{O}_E$-module of height h (relative to E) and dimension 1 over a noetherian ring R. Then for all $r \geq 1$ there exists a π-divisible $\mathcal{O}_E$-module $\bigwedge_{\mathcal{O}_E}^r G$ of height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$, together with a morphism $\lambda \colon G^r \to \bigwedge_{\mathcal{O}_E}^r G$ which satisfies the appropriate universal property. In particular the determinant $\bigwedge_{\mathcal{O}_E}^h G$ has height 1 and dimension 1.

Let π' be a uniformizer of E', and let H' be a Lubin-Tate formal $\mathcal{O}_{E'}$-module. Then $H'[(\pi')^\infty]$ is a π'-divisible $\mathcal{O}_{E'}$-module over $\mathcal{O}_{E'}$ of height 1 and dimension 1. By restriction of scalars, it becomes a π-divisible $\mathcal{O}_E$-module $H'[\pi^\infty]$ of height d and dimension 1. Then $\bigwedge_{\mathcal{O}_E}^d H'[\pi^\infty]$ is a π-divisible $\mathcal{O}_E$-module of height 1 and dimension 1, so that it is the π-power torsion in a Lubin-Tate formal $\mathcal{O}_E$-module $\bigwedge^d H'$ defined over $\mathcal{O}_{E'}$. For all $n \geq 1$ we have an $\mathcal{O}_E/\pi^n$-alternating morphism

$$
\lambda \colon H'[\pi^n]^d \to \bigwedge^d H'[\pi^n]
$$

of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_{E'}$. Let $H_0' = H' \otimes \mathcal{O}_{E'}/\pi'$. Reducing mod π', taking inverse limits with respect to n and applying Lemma 4.0.6 gives a morphism

$$
\lambda_0 \colon (\tilde{H}_0')^d \to \overbrace{\bigwedge^d H'}^{d} {}_0
$$

---

[1]This result requires the residue characteristic to be odd, but we strongly suspect this is unnecessary. See [SW13], §6.4 for a construction of the determinant map (on the level of universal covers of formal modules) without any such hypothesis.

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of formal vector spaces over $\mathcal{O}_{E'}/\pi'$. By the crystalline property of formal vector spaces ([SW13], Prop. 3.1.3(ii)), this morphism lifts uniquely to a morphism

$$\tilde{\lambda} \colon (\tilde{H}')^d \to \overset{d}{\overbrace{\bigwedge}} H'$$

of formal vector space over $\mathcal{O}_{E'}$.

Since $\bigwedge^d H'$ and H are both height 1 and dimension 1, they become isomorphic after passing to $\mathcal{O}_C$. Let $\alpha_1, \ldots, \alpha_n$ be a basis for E'/E, and define a morphism of formal schemes

$$
\begin{array}{ccc}
\tilde{H}'_{\mathcal{O}_C} & \to & \overset{d}{\overbrace{\bigwedge}} H'_{\mathcal{O}_{\mathbb{C}}} \cong \tilde{H}_{\mathcal{O}_C} \\
x & \mapsto & \tilde{\lambda}(\alpha_1 x, \ldots, \alpha_d x)
\end{array}
$$

After passing to the generic fiber we get a well-defined map $N_{E'/E} \colon Z_{E'} \to Z_E$ which does not depend on the choice of basis for E'/E.

The commutativity of the diagram in Eq. (4.1.1) is equivalent to the following proposition.

Proposition 4.1.1. *The following diagram commutes:*

$$
\begin{array}{ccc}
\left\{\text{Étale } E\text{-algebras}\right\} & \xrightarrow{A \mapsto A \otimes_E E'} & \left\{\text{Étale } E'\text{-algebras}\right\} \\
\downarrow & & \downarrow \\
\left\{\text{Étale covers of } Z_E\right\} & \longrightarrow & \left\{\text{Étale covers of } Z'_E\right\}.
\end{array}
$$

Here the bottom arrow is pullback via $N_{E'/E}$.

Proof. Ultimately, the proposition will follow from the functoriality of the isomorphism in Lemma 4.0.5. For $n \geq 1$, let E'_n be the field obtained by adjoining the π^n-torsion in H' to E'. (Note that the π^n-torsion is the same as the $(\pi')^{en}$ torsion, where e is the ramification degree of E'/E.) The existence of λ shows that E_n contains the field obtained by adjoining the π^n-torsion in $\bigwedge^d H'$ to E_n. Namely, let $x \in H[\pi^n](\mathcal{O}_{E'_n})$ be a primitive element, in the sense that x generates $H'[\pi^n](\mathcal{O}_{E'_n})$ as an $\mathcal{O}_{E'}/\pi^n$-module. If $\alpha_1, \ldots, \alpha_d$ is a basis for $\mathcal{O}_{E'}/\mathcal{O}_E$ then $\lambda(\alpha_1 x, \ldots, \alpha_d x)$ generates $\bigwedge H'[\pi^n](\mathcal{O}_{E'_n})$ as an $\mathcal{O}_E/\pi^n$-module. Since $\bigwedge^d H'$ and H become isomorphic over $\hat{E}'^{,\text{nr}}$, we get a compatible family of embeddings $\hat{E}_n^{\text{nr}} \hookrightarrow \hat{E}_n'^{,\text{nr}}$.

By construction, these embeddings are compatible with the isomorphisms in Lemma 4.0.5, so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spf}\,\mathcal{O}_{E_n'^{\mathrm{,nr}}/t'} & \xrightarrow{\sim} & \tilde{H}_0'[\pi^{n-1}] \otimes \overline{\mathbb{F}}_q \\ \downarrow & & \downarrow{\scriptstyle N_{E'/E}} \\ \mathrm{Spf}\,\mathcal{O}_{E_n^{\mathrm{nr}}/t} & \xrightarrow{\sim} & \tilde{H}_0[\pi^{n-1}] \otimes \overline{\mathbb{F}}_q \end{array}$$

Here $t' \in \mathcal{O}_{E_1}$ is a uniformizer. From here we get the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Spf}\left(\mathcal{O}_{\hat{E}_\infty'^{,\flat}}\hat{\otimes}_{\mathcal{O}_{E'}/\pi'}\mathcal{O}_{C^\flat}\right) & \longrightarrow & \tilde{H}_{\mathcal{O}_C}'^{,\flat} \\ \downarrow & & \downarrow{\scriptstyle N_{E'/E}} \\ \mathrm{Spf}\left(\mathcal{O}_{\hat{E}_\infty^\flat}\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_{C^\flat}\right) & \longrightarrow & \tilde{H}_{\mathcal{O}_C}^\flat. \end{array}$$

One can now trace this compatibility with the chain of equivalences in the proof of Thm. 4.0.10 to get the proposition. The details are left to the reader. $\qquad\square$

4.2 Descent from C to E

The object $Z_E = (\tilde{H}_{E,C}^{\mathrm{ad}} \setminus \{0\})/E^\times$ is defined over C, but it has an obvious model over E, namely $Z_{E,0} = (\tilde{H}_E^{\mathrm{ad}} \setminus \{0\})/E^\times$. Since the geometric fundamental group of $Z_{E,0}$ is $\mathrm{Gal}(\overline{E}/E)$, we have an exact sequence

$$1 \to \mathrm{Gal}(\overline{E}/E) \to \pi_1^{\text{ét}}(Z_{E,0}) \to \mathrm{Gal}(\overline{E}/E) \to 1.$$

This exact sequence splits as a direct product. In other words:

Proposition 4.2.1. *Let E'/E be a finite extension, and let $Z'_E \to Z_E$ be the corresponding finite étale cover under Thm. 4.0.10. Then Z'_E descends to a finite étale cover $Z'_{E,0} \to Z_{E,0}$.*

Proof. Let us write $R = H^0(\tilde{H}, \mathcal{O}_{\tilde{H}})$ for the coordinate ring of $\tilde{H}$, so that $R \cong \mathcal{O}_E[\![t^{1/q^\infty}]\!]$. Then $R/\pi \cong \mathbb{F}_q[\![t^{1/q^\infty}]\!]$ is a perfect ring and $R = W_{\mathcal{O}_E}(R/\pi)$. By Prop. 4.0.7 we have $R/\pi = \mathcal{O}_{\hat{E}_\infty^\flat}$. Thus $Z_{E,0} = \mathrm{Spf}\,W(\mathcal{O}_{\hat{E}_\infty^\flat})_E^{\mathrm{ad}} \setminus \{0\}$.

Now suppose E'/E is a finite extension. Let $\hat{E}'_\infty = E' \otimes_E \hat{E}_\infty$, a finite étale $\hat{E}_\infty$-algebra. Then $\hat{E}'_\infty$ is a perfectoid $\hat{E}_\infty$-algebra (in fact it a product of perfectoid fields), so that we can form the tilt $\hat{E}_\infty'^{,\flat}$, a perfect ring

containing $\hat{E}^{\flat}_{\infty}$. Let $\mathcal{O}_{\hat{E}'^{\flat}_{\infty}}$ be its subring of power-bounded elements. We put

$$Z'_{E,0} = \operatorname{Spf} W_{\mathcal{O}_E}(\mathcal{O}_{\hat{E}'^{,\flat}})^{\mathrm{ad}}_E \backslash \{0\},$$

a finite étale cover of $Z_{E,0}$.

We claim that $Z'_E = Z_{E,0}\hat{\otimes}C$. It suffices to show that the tilts are the same. We have

$$
\begin{aligned}
\left(Z_{E,0}\hat{\otimes}C\right)^{\flat} &= \operatorname{Spf}\left(W_{\mathcal{O}_E}(\mathcal{O}_{\hat{E}'^{,\flat}}\hat{\otimes}\mathcal{O}_C)^{\flat,\mathrm{ad}}_{C^{\flat}}\backslash\{0\}\right)/E^{\times} \\
&= \left(\operatorname{Spf}\left(\mathcal{O}_{\hat{E}'^{,\flat}}\hat{\otimes}\mathcal{O}_{C^{\flat}}\right)^{\mathrm{ad}}_{C^{\flat}}\backslash\{0\}\right)/E^{\times},
\end{aligned}
$$

which by construction is the tilt of Z'_E. $\qquad\square$

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