The Geometry of Lubin-Tate spaces (Weinstein)
Lecture 1: Formal groups and formal modules

References. Milne's notes on class field theory are an invaluable resource for the subject. The original paper of Lubin and Tate, *Formal Complex Multiplication in Local Fields*, is concise and beautifully written. For an overview of Dieudonné theory, please see Katz' paper *Crystalline Cohomology, Dieudonné Modules, and Jacobi Sums*. We also recommend the notes of Brinon and Conrad on p-adic Hodge theory[1].

Motivation: the local Kronecker-Weber theorem. Already by 1930 a great deal was known about class field theory. By work of Kronecker, Weber, Hilbert, Takagi, Artin, Hasse, and others, one could classify the abelian extensions of a local or global field F, in terms of data which are intrinsic to F. In the case of global fields, abelian extensions correspond to subgroups of ray class groups. In the case of local fields, which is the setting for most of these lectures, abelian extensions correspond to subgroups of $F^\times$. In modern language there exists a *local reciprocity map* (or Artin map)

$$\mathrm{rec}_F \colon F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

which is continuous, has dense image, and has kernel equal to the connected component of the identity in $F^\times$ (the kernel being trivial if F is nonarchimedean).

Nonetheless the situation with local class field theory as of 1930 was not completely satisfying. One reason was that the construction of rec_F was global in nature: it required embedding F into a global field and appealing to the existence of a global Artin map. Another problem was that even though the abelian extensions of F are classified in a simple way, there wasn't any explicit construction of those extensions, save for the unramified ones.

To wit, suppose that F is nonarchimedean and that $\pi \in F$ is a uniformizing element. Then $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is isomorphic via the reciprocity map to $\hat{F}^\times$, the profinite completion of F. Decompose this group as a product: $\hat{F}^\times = \mathcal{O}_F^\times \times \pi^{\hat{\mathbb{Z}}}$. The fixed field of $\mathcal{O}_F^\times$ is nothing more than F^{nr}, the maximal unramified extension of F. (When F has residue characteristic p, this field is generated by all roots of unity of order prime to p.) Let F_π be the fixed field of $\pi^{\hat{\mathbb{Z}}}$; this is an infinite, totally ramified extension of F with $\mathrm{Gal}(F_\pi/F) \cong \mathcal{O}_F^\times$. We have $F^{\mathrm{nr}} F_\pi = F^{\mathrm{ab}}$. All very simple, but what exactly is F_π?

The case of $F = \mathbb{Q}_p$ is a clue: here, by the local Kronecker-Weber theorem, we have $F^{\mathrm{ab}} = F^{\mathrm{cycl}}$ is the maximal cyclotomic extension of F. If one knows the reciprocity law explicitly enough, one finds that (taking $\pi = p$) F_p is the field obtained from $\mathbb{Q}_p$ by adjoining all pth power roots of unity. These roots of unity arise as the p-power torsion in the multiplicative group $\mathbf{G}_{\mathrm{m}}$. The challenge now is replacing $\mathbf{G}_{\mathrm{m}}$ by a suitable object, call it $\mathscr{G}$, when F is a different field. Since the Galois group $\mathrm{Gal}(F_\pi/F)$ needs to be $\mathcal{O}_F^\times$, it stands to reason that $\mathscr{G}$

[1]Available at `math.stanford.edu/~conrad/papers/notes.pdf`.

should admit $\mathcal{O}_F$ as a ring of endomorphisms, and that $\mathscr{G}[\pi^n]$ should be a free $(\mathcal{O}_F/\pi^n\mathcal{O}_F)$-module of rank one. Note the similarity to the scenario of elliptic curves E with complex multiplication by an imaginary quadratic field F: one obtains abelian extensions of F via the action of Galois on the torsion points of E.

Formal groups and formal $\mathcal{O}_F$-modules.

The construction of such objects $\mathscr{G}$ was provided by Lubin and Tate in 1965. This paper has the felicitous property that even though the concepts are original and quite powerful, the proofs are so simple that they could be supplied by the reader. We begin with a definition: A *one-dimensional commutative formal group law*[2] $\mathscr{G}$ over a commutative ring A is a power series $\mathscr{G}(X,Y) \in A[\![X,Y]\!]$ satisfying the properties

- $\mathscr{G}(X,Y) = X + Y +$ higher order terms
- $\mathscr{G}(X,Y) = \mathscr{G}(Y,X)$
- $\mathscr{G}(\mathscr{G}(X,Y),Z) = \mathscr{G}(X,\mathscr{G}(Y,Z))$
- There exists $\iota(X) \in A[\![X]\!]$ with $\mathscr{G}(X,\iota(X)) = 0$

That is, $\mathscr{G}$ behaves like the addition law on an abelian group. (The existence of the inverse actually follows from the other axioms. Also, there is a notion of formal group laws of dimension n; for these, the $\mathscr{G}$ is n power series in two sets of n variables.) To stress the analogy we write $X +_{\mathscr{G}} Y$ for $\mathscr{G}(X,Y)$. The additive formal group law $\hat{\mathbf{G}}_{\mathrm{a}}$ is simply $X+Y$, while the multiplicative formal group law $\hat{\mathbf{G}}_{\mathrm{m}}$ is $X+Y+XY$. (This expression is $(1+X)(1+Y)-1$, and therefore represents multiplication for a parameter centered around 0 rather than 1.) Other formal group laws are much harder to make explicit. One development is given in Silverman's *The Arithmetic of Elliptic Curves*, Chapter IV, where it is shown how to construct a formal group $\hat{E}$ out of the Weierstrass equation for a given elliptic curve E. There is an evident notion of homomorphism $f\colon \mathscr{G} \to \mathscr{G}'$ between formal groups; this is a power series satisfying $f(X +_{\mathscr{G}} Y) = f(X) +_{\mathscr{G}'} f(Y)$. Then $\operatorname{End}\mathscr{G}$ is a (not necessarily commutative) ring. For $n \in \mathbb{Z}$ we write $[n]_{\mathscr{G}}(X)$ for the n-fold addition of X with itself.

In the setting of Lubin-Tate, F is a nonarchimedean local field. Let A be an $\mathcal{O}_F$-algebra with structure map $\iota\colon \mathcal{O}_F \to A$, not presumed injective. A *formal $\mathcal{O}_F$-module law* over A is a formal group law $\mathscr{G}$ over A together with a family of power series $[a]_{\mathscr{G}}$ for $a \in \mathcal{O}_F$ which together represent a homomorphism $\mathcal{O}_F \to \operatorname{End}\mathscr{G}$. It is required that $[a]_{\mathscr{G}}(X) = \iota(a)X + O(X^2)$; that is, the *derivative* of the action of $\mathcal{O}_F$ on $\mathscr{G}$ is just ι.

For example, $\hat{\mathbf{G}}_{\mathrm{a}}$ is an $\mathcal{O}_F$-module over any A. Less trivially, in the case $F = \mathbb{Q}_p$, the multiplicative formal group $\hat{\mathbf{G}}_{\mathrm{m}}$ becomes a formal $\mathcal{O}_F = \mathbb{Z}_p$-module, because for $a \in \mathbb{Z}_p$ we

[2] Also called: one-parameter formal Lie group. All the formal groups we consider will be commutative, so we will be dropping the "commutative" from now on.

have the endomorphism

$$[a]_{\hat{\mathbf{G}}_{\mathrm{m}}}(X) = (1+X)^a - 1 = \sum_{n=1}^{\infty} \binom{a}{n} X^n \in \mathbb{Z}_p[\![X]\!].$$

Multiplication by p is the polynomial $[p]_{\hat{\mathbf{G}}_{\mathrm{m}}}(X) = (1+X)^p - 1$, which has the property that $[p]_{\hat{\mathbf{G}}_{\mathrm{m}}}(X) \equiv X^p \pmod{p}$.

Recall that π is a uniformizing parameter for F. Let $q = p^f$ be the cardinality of the residue field of $\mathcal{O}_F$. Lubin and Tate construct formal $\mathcal{O}_F$-modules over $\mathcal{O}_F$ by starting with a choice of $[\pi]_{\mathscr{G}}$ and constructing $\mathscr{G}$ in the only consistent way possible. Let $f(X) \in \mathcal{O}_F[\![X]\!]$ be any power series satisfying the properties

- $f(X) = \pi X + O(X^2)$
- $f(X) \equiv X^q \pmod{\pi}$.

It is well worth the trouble to work through the proof of the following:

Theorem 1. *There exists a unique formal $\mathcal{O}_F$-module law $\mathscr{G}_f$ over $\mathcal{O}_F$ for which $[\pi]_{\mathscr{G}_f}(X) = f(X)$. Furthermore, if g is another power series satisfying the two criteria above, then $\mathscr{G}_f$ and $\mathscr{G}_g$ are isomorphic.*

In particular, up to isomorphism is there is exactly one formal $\mathcal{O}_F$-module law for which multiplication by π reduces to X^q over the residue field.

The construction of the desired field F_π is now quite easy. Choose an f as above (there's no harm in setting $f(X) = \pi X + X^q$, for instance) and let $\mathscr{G}$ be the corresponding formal $\mathcal{O}_F$-module law as in the theorem. For each $n \geq 1$, let $\mathscr{G}[\pi^n]$ denote the set of $x \in \mathfrak{m}_{F^s}$ (the maximal ideal in the ring of integers of a separable closure of F) satisfying $[\pi^n]_{\mathscr{G}}(x) = 0$.[3] But then the addition law $+_{\mathscr{G}}$ and the power series $[\alpha]_{\mathscr{G}}$ are quickly seen to turn $\mathscr{G}[\pi^n]$ into an $\mathcal{O}_F$-module. Furthermore, we have $[\pi^n]_{\mathscr{G}}(X) \equiv X^{q^n} \pmod{p}$, so that by Weierstrass preparation theorem, the cardinality of $\mathscr{G}[\pi^n]$ is q^n. Finally, $\mathscr{G}[\pi]$ is the kernel of $[\pi]_{\mathscr{G}} \colon \mathscr{G}[\pi^n] \to \mathscr{G}[\pi^n]$. These facts are sufficient to show that $\mathscr{G}[\pi^n]$ is a free $\mathcal{O}_F/\pi^n\mathcal{O}_F$-module of rank 1. Let F_π/F be the field obtained by adjoining $\mathscr{G}[\pi^n]$ for all $n \geq 1$.

Let $T_\pi(\mathscr{G})$ be the *π-adic Tate module*:

$$T_\pi(\mathscr{G}) = \varprojlim \mathscr{G}[\pi^n].$$

Then $T_\pi(\mathscr{G})$ is a free $\mathcal{O}_F$-module of rank 1, admitting a continuous action of Galois:

$$\rho\colon \operatorname{Gal}(F_\pi/F) \to \operatorname{Aut} T_\pi(\mathscr{G}) \cong \mathcal{O}_F^\times.$$

[3]The roots of f and its iterates all lie in an algebraic extension of F by the p-adic Weierstrass preparation theorem, so there is no need to invoke any larger field than $\overline{F}$. What's more, $[\pi^n]_{\mathscr{G}}(X)$ has no repeated roots (exercise), so F^s suffices.

Theorem 2 (Lubin-Tate). *The map ρ as above is an isomorphism. Furthermore, $F_\pi F^{\mathrm{nr}} = F^{\mathrm{ab}}$. The Artin reciprocity map $F^\times \to \mathrm{Gal}(F_\pi/F)$ is the unique map which sends π to 1 and $\alpha \in \mathcal{O}_F^\times$ to $\rho^{-1}(\alpha^{-1})$.*

The invariant differential form, and the logarithm. For a Lie group G, one has a space of *translation-invariant differentials* which are useful for computing the de Rham cohomology spaces. These differential forms are determined completely by their values on the tangent vectors at the origin, and in fact the space of such differentials is dual to $\operatorname{Lie} G$. In this paragraph we consider the infinitesimal version of this picture, namely invariant differentials on formal groups. In arithmetic applications, the differentials lead directly to the theory of Dieudonné modules.

We return to the setting of a formal group law $\mathscr{G}$ over an arbitrary commutative ring A. The space of *differential forms* on $\mathscr{G}$ is the A-module $\Omega^1_{\mathscr{G}/A}$ of formal differentials $P(T)dT$, where $P(T) \in A[\![T]\!]$. This of course has nothing to do with the addition law on $\mathscr{G}$.

Suppose $\Sigma\colon A[\![T]\!] \to A[\![X, Y]\!]$ is the unique adic A-algebra homomorphim sending T to $\mathscr{G}(X, Y)$. Also let $\mathrm{pr}_1, \mathrm{pr}_2\colon A[\![T]\!] \to A[\![X, Y]\!]$ send T to X and Y, respectively. A differential $\omega \in \Omega^1_{\mathscr{G}/A}$ is *translation invariant* if $\Sigma_*(\omega) = (\mathrm{pr}_1)_*(\omega) + (\mathrm{pr}_2)_*(\omega)$.

The module of invariant differentials is a free R-module of rank 1, spanned[4] by $\omega = \mathscr{G}_X(0, T)^{-1}dT$. For the additive formal group this works out to $\omega = dT$. For the multiplicative formal group, this is $\omega = dT/(1 + T)$.

If A is flat over $\mathbb{Z}$, so that A injects into $A \otimes \mathbb{Q}$, we can construct the *formal logarithm* of $\mathscr{G}$ by

$$\log_{\mathscr{G}}(T) = \int \omega \in (A \otimes \mathbb{Q})[\![T]\!],$$

where we choose the antiderivative in such a way that $\log_{\mathscr{G}}(0) = 0$. For instance, $\log_{\hat{\mathbf{G}}_{\mathrm{m}}}(T) = \int (1 + T)^{-1}dT \in \mathbb{Q}[\![T]\!]$ is the series that represents $\log(1 + T)$.

The logarithm $\log_{\mathscr{G}}$ is an isomorphism between $\mathscr{G}_{A\otimes\mathbb{Q}}$ and $\hat{\mathbf{G}}_{\mathrm{a}}$. Thus whenever A is a $\mathbb{Q}$-algebra, all one-dimensional commutative formal groups over A are isomorphic to the additive formal group. This shouldn't come as a surprise. Formal groups are meant to capture the behavior of (honest) Lie groups in a small neighborhood of the origin. Over a field of characteristic zero, a small neighborhood of the origin is isomorphic to the Lie algebra via the logarithm map, and for a commutative Lie group, the Lie algebra isn't terribly interesting. It is only in mixed or positive characteristic settings that we see nontrivial behavior in a commutative formal group.

Exercise. Let R be a commutative ring and $\mathscr{G}$ a formal R-module with invariant differential ω. Show that $[\alpha]^*_{\mathscr{G}}(\omega) = \alpha\omega$ for each $\alpha \in R$.

[4] See Silverman, IV.4.

Exercise. Now suppose R is a DVR with uniformizer π and residue field $k = \mathbf{F}_q$ of characteristic p. Let A be a k-algebra and let $\mathscr{G}$ be a formal R-module over A. Use the previous exercise to show that $[\pi]_{\mathscr{G}}(X) = g(X^p)$ for some $g(X) \in A[\![X]\!]$. In fact more is true: either $[\pi]_{\mathscr{G}}(X) = 0$, or else there exists a unique $h \geq 1$ and power series $g(X)$ with $[\pi]_{\mathscr{G}}(X) = g(X^{q^h})$. See Silverman Prop. 7.2 for a hint. This h is called the *height* of the R-module $\mathscr{G}$.

Exercise. Keep the assumption that R is a DVR, and let K be its fraction field. Let $\mathscr{G}$ be a formal R-module over R itself. Show that as $n \to \infty$, the sequence $\pi^{-n}[\pi^n]_{\mathscr{G}}(X)$ converges coefficientwise in $K[\![X]\!]$ to $\log_{\mathscr{G}}(X)$. Show also that $\log_{\mathscr{G}}(X)$ converges on the open unit disk in $\overline{K}$.

Formal groups: Categorical definition. The definition of formal group law (and formal module law) we have given is a bit stilted in that it relies on a choice of coordinate X. It is better to have an intrinsic definition of formal group (or formal module), which becomes a "law" as soon as a coordinate is chosen. Let A be any ring, and let $\mathcal{C}$ be the category of "adic" A-algebras A' (that is, complete with respect to the I-adic topology, for some ideal $I \subset A'$). Consider the functor $\mathcal{C} \to \mathbf{Sets}$ which sends A' to the set of n-tuples of topologically nilpotent elements of A'. This functor is represented by the power series ring $A[\![X]\!] = A[\![X_1, \ldots, X_n]\!]$. An *$n$-dimensional formal Lie variety* is a functor $V \colon \mathcal{C} \to \mathbf{Sets}$ which is isomorphic to this functor. A choice of isomorphism gives a representation of V by a coordinate ring $A[\![X]\!]$. But there is also an intrinsic coordinate ring, namely the algebra $A(V)$, which consists of all natural transformations from V onto the forgetful functor $\mathcal{C} \to \mathbf{Sets}$. The Lie algebra $\mathrm{Lie}(V)$ is the functor V evaluated on the space of dual numbers $A[X]/X^2$ (with its X-adic topology); this is a free A-module of rank n.

In this language, a *formal group* over A is a group object in the category of formal Lie varieties over A. If R is a commutative ring and A is an R-algebra, then a one-dimensional *formal R-module* over A is a one-dimensional formal group $\mathscr{G}$ together with a ring homomorphism $R \to \operatorname{End} \mathscr{G}$ whose derivative $R \to \operatorname{End} \operatorname{Lie} \mathscr{G} = A$ is the structure map $R \to A$. Once one chooses a coordinate for $\mathscr{G}$, one recovers the definitions of formal group law and formal module law as above.

When p is a prime and $\mathscr{G}/A$ is a formal group, we say that $\mathscr{G}$ is *p-divisible* if the multiplication by p map $[p]_{\mathscr{G}} \colon A(\mathscr{G}) \to A(\mathscr{G})$ presents $A(\mathscr{G})$ as a finite module over itself. If this is the case, then $A(\mathscr{G})$ will be a locally free module over itself of rank p^h for some h; this h is the *height* of $\mathscr{G}$. For instance if $\mathscr{G}$ is the Lubin-Tate formal group over the ring of integers $\mathcal{O}_F$ in a finite extension $F/\mathbb{Q}_p$, then the height of $\mathscr{G}$ is $[F : \mathbb{Q}_p]$ (exercise). In contrast, the formal additive group over $\mathbb{Z}_p$ is not p-divisible.

The de Rham complex, and the module $\mathbf{D}(\mathscr{G}/A)$. For a formal Lie variety V, we have the complex of differentials $\Omega^\bullet_{V/A}$; this is nothing more than the usual de Rham complex of the A-algebra $A(V)$. That is, $\Omega^i_{V/A}$ is the ith exterior power of the A-module of Kähler differentials $\Omega^1_{A(V)/A} \approx A[\![X]\!]dX$. The cohomology of this complex is the de Rham cohomology

$H^\bullet_{\mathrm{dR}}(V/A)$. Right away you should notice that if A is a $\mathbb{Q}$-algebra, then $H^i_{\mathrm{dR}}(V/A) = 0$ for all $i \geq 1$: exact forms can always be integrated. But of course if A is (say) $\mathbb{Z}_p$ or $\mathbf{F}_p$, then there is an obstruction to integrating due to exponents which are $\equiv -1 \pmod p$, and this is precisely what is measured by $H^i_{\mathrm{dR}}(V/A)$.

Now suppose $\mathscr{G}$ is a formal group over A. Let $\mathbf{D}(\mathscr{G}/A)$ be the A-module consisting of those cohomology classes $\omega \in H^1_{\mathrm{dR}}(\mathscr{G}/A)$ which are translation invariant. (This definition involves the group structure of $\mathscr{G}$, whereas that of $H^1_{\mathrm{dR}}(\mathscr{G}/A)$ does not.) Let $\Sigma\colon \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ be the addition law, and let $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ be the projections; then $[\omega] \in H^1_{\mathrm{dR}}(\mathscr{G}/A)$ is translation invariant if and only if $\Sigma^*(\omega) - (\mathrm{pr}_1(\omega) + \mathrm{pr}_2(\omega))$ is an exact differential.

The following proposition gives explicit descriptions of $H^1_{\mathrm{dR}}(\mathscr{G}/A)$ and $\mathbf{D}(\mathscr{G}/A)$ for a one-dimensional formal group $\mathscr{G}$ over a $\mathbb{Z}$-flat ring (the case of higher-dimensional $\mathscr{G}$ is quite similar). Choose a coordinate X for $\mathscr{G}$.

Proposition 1. *Assume that A is a $\mathbb{Z}$-flat ring, and let $K = A \otimes_{\mathbb{Z}} \mathbb{Q}$. We have the diagram*

$$\begin{tikzcd}
0 \arrow[r] & XA[\![X]\!] \arrow[r] & \left\{ f \in XK[\![X]\!] \;\middle|\; df \in A[\![X]\!] \right\} \arrow[r] & H^1_{\mathrm{dR}}(\mathscr{G}/A) \arrow[r] & 0 \\
0 \arrow[r] & XA[\![X]\!] \arrow[r] \arrow[u, equal] & \left\{ f \in XK[\![X]\!] \;\middle|\; df \in A[\![X]\!],\ \partial f \in A[\![X,Y]\!] \right\} \arrow[r] \arrow[u] & \mathbf{D}(\mathscr{G}/A) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Both rows are exact. For $f \in A[\![X]\!]$, we have set $\partial f(X,Y) = f(X +_{\mathscr{G}} Y) - f(X) - f(Y)$. The maps coming out the center objects are both exterior differentiation.

As the above diagram indicates, there is always a map $\omega_{\mathscr{G}} \to \mathbf{D}(\mathscr{G}/A)$. In the case that $\mathscr{G}$ is a formal p-divisible group over a $\mathbb{Z}_p$-algebra A, this map is injective, and there is a *Hodge-Tate exact sequence*

$$0 \to \omega_{\mathscr{G}} \to \mathbf{D}(\mathscr{G}/A) \to \operatorname{Lie} \mathscr{G}^\vee \to 0. \tag{1}$$

Here $\mathscr{G}^\vee$ is the Cartier dual. Note the similarity to the classical Hodge decomposition of the de Rham cohomology of an abelian variety $A/\mathbb{C}$: one has the exact sequence

$$0 \to H^0(A, \Omega^1) \to H^1_{\mathrm{dR}}(A, \mathbb{C}) \to H^1(A, \mathcal{O}_A) \to 0.$$

(The space $H^0(A, \Omega^1)$ is the same as the space of invariant differentials, and $H^1(A, \mathcal{O}_A)$ is isomorphic to $\operatorname{Lie} A^\vee$ for the dual abelian variety $A^\vee$.)

When $\mathscr{G}/A$ is the formal multiplicative group $\hat{\mathbf{G}}_{\mathrm{m}}/\mathbb{Z}_p$, the class of the differential $\omega = dT/(1+T)$ lies in $\mathbf{D}(\hat{\mathbf{G}}_{\mathrm{m}}/\mathbb{Z}_p)$, because ω is already translation-invariant. (Note that translation-invariance in H^1_{dR} is weaker than translation-invariance in Ω^1!) It turns out that the class of

ω spans $\mathbf{D}(\hat{\mathbf{G}}_{\mathrm{m}}/\mathbb{Z}_p)$. What about other formal groups, such as the Lubin-Tate formal group over the ring of integers of a finite extension $F/\mathbb{Q}_p$? The question seems intractable to work out by hand using the above proposition. We shall soon see that the functor $\mathbf{D}$ behaves quite rigidly, carries a fair bit of structure, and can be worked out completely in many cases.

The crystalline nature of H^1_{dR}. Many of the constructions of Dieudonné theory seem to rest on the following lemma and its generalizations:

Lemma 1. *Let A be a $\mathbb{Z}_p$-flat ring. Let $f_1(X), f_2(X) \in A[\![X]\!]$ be power series without constant term, and let $\omega \in A[\![X]\!]dX$ be a differential. If $f_1 \equiv f_2 \pmod{pA}$, then $f_1^*(\omega) - f_2^*(\omega)$ is exact.*

Here's the proof: Suppose $\omega = dg$ for $g(X) \in (A \otimes \mathbb{Q}_p)[\![X]\!]$; this is possible because A is $\mathbb{Z}_p$-flat. Then of course g' is integral; *i.e.*, it has coefficients in A. Write $f_2 = f_1 + p\Delta$. Then

$$\begin{aligned}\int f_2^*(\omega) - f_1^*(\omega) &= g(f_2) - g(f_1)\\ &= g(f_1 + p\Delta) - g(f_1)\\ &= \sum_{n\geq 1} \frac{(p\Delta)^n}{n!} g^{(n)}(f_1)\end{aligned}$$

This is integral because (a) the derivative g' is integral, and hence so are all subsequent derivatives, and (b) for all $n \geq 1$, we have $p^n/n! \in \mathbb{Z}_p$.

The lemma works for power series in any number of variables. Stated categorically, it says that if $f_1, f_2 \colon V' \to V$ are two morphisms between *pointed* formal Lie varieties over A ("V pointed" means "endowed with an origin $0 \in V(A)$") which are congruent mod p, then the f_i induce the same map $H^1_{\mathrm{dR}}(V/A) \to H^1_{\mathrm{dR}}(V'/A)$. Loosely speaking, the formation of H^1_{dR} is insensitive to lifts from A/pA to A: it is "crystalline".

Theorem 3. *Once again, let $\mathscr{G}$ and $\mathscr{G}'$ be formal groups over a $\mathbb{Z}_p$-flat ring A.*

1. *Let $f \colon \mathscr{G}' \to \mathscr{G}$ be a morphism of pointed Lie varieties for which $f \pmod p$ is a group homomorphism. Then f^* maps $\mathbf{D}(\mathscr{G}/A)$ into $\mathbf{D}(\mathscr{G}'/A)$.*

2. *Let $f_1, f_2, f_3 \colon \mathscr{G}' \to \mathscr{G}$ be three morphisms of pointed Lie varieties for which the $f_i \pmod p$ are group homomorphisms satisfying $f_3 \equiv f_1 + f_2 \pmod p$ in the group $\mathrm{Hom}(\mathscr{G}'_{A/pA}, \mathscr{G}_{A/pA})$. Then $f_3^* = f_1^* + f_2^*$ as maps $\mathbf{D}(\mathscr{G}/A) \to \mathbf{D}(\mathscr{G}'/A)$.*

The proof is a diagram chase relying on nothing but Lemma 1. Here it is for part (1): f being a morphism mod p means that $\Sigma \circ (f \times f) \equiv f \circ \Sigma \pmod p$. Now suppose $\omega \in \mathbf{D}(\mathscr{G}/A)$. By the lemma, $(\Sigma \circ (f \times f))^*(\omega) = (f \circ \Sigma)^*(\omega)$ in $H^1_{\mathrm{dR}}(\mathscr{G}/A)$. Similarly, we have $(f \times f)^*(\mathrm{pr}_i^*(\omega)) = \mathrm{pr}_i^*(f^*(\omega))$ (this doesn't require the lemma). Therefore

$$\Sigma^*(f^*(\omega)) - \mathrm{pr}_1^*(f(\omega)) - \mathrm{pr}_2^*(f(\omega)) = (f \times f)^*(\Sigma^*(\omega) - \mathrm{pr}_1^*(\omega) - \mathrm{pr}_2^*(\omega)) = 0$$

and $f^*(\omega)$ is translation-invariant as required.

The action of Frobenius. Here we work out an example of $\mathbf{D}(\mathscr{G}/A)$. Let $h \geq 1$, let $q = p^h$, and suppose $\mathscr{G}$ is a one-dimensional formal group law over $\mathbb{Z}_p$ with $[p]_{\mathscr{G}}(X) = pX + X^q$. (For instance, $\mathscr{G}$ could be the formal group law underlying the unique Lubin-Tate formal $W(\mathbb{F}_q)$-module law for which multiplication by p is $pX + X^q$.) We wish to calculate $\mathbf{D}(\mathscr{G}/\mathbb{Z}_p)$; already we had noted that $\omega_{\mathscr{G}}$, the invariant differential, spans a line in $\mathbf{D}(\mathscr{G}/\mathbb{Z}_p)$. The *Frobenius map* $F\colon \mathscr{G}/\mathbb{F}_p \to \mathscr{G}/\mathbb{F}_p$ given by $F(X) = X^p$ is a group homomorphism. The same map $F\colon \mathscr{G} \to \mathscr{G}$ in characteristic 0 is *not* a group homomorphism, but it is still a pointed morphism of formal Lie varieties over $\mathbb{Z}_p$, and so by Theorem 3 it induces a map $F\colon \mathbf{D}(\mathscr{G}/\mathbb{Z}_p) \to \mathbf{D}(\mathscr{G}/\mathbb{Z}_p)$. One employs a similar diagram chase as in Theorem 3 to show that F is a $\mathbb{Z}_p$-linear map. Now observe that F^h and $[p]_{\mathscr{G}}$ are congruent mod p, so applying the theorem once again shows that as endomorphisms of $\mathbf{D}(\mathscr{G}/\mathbb{Z}_p)$ we have $F^h = p$. (By the first exercise, $[p]_{\mathscr{G}}$ induces p on H^1_{dR}.) In fact $\mathbf{D}(\mathscr{G}/R)$ is spanned by $\omega, F(\omega), \dots, F^{h-1}(\omega)$.

For formal groups over fields larger than $\mathbb{F}_p$, there is a little subtlety that leads to the *semilinear algebra* objects which are ubiquitous in Dieudonné theory. Let k be a perfect field of characteristic p, and let $\mathscr{G}$ be a formal group over $W(k)$. Let $\sigma\colon k \to k$ be the pth power map, and also write $\sigma\colon W(k) \to W(k)$ for its lift.

The Frobenius is no longer a morphism from $\mathscr{G}_k$ to itself, because the pth power map will not fix the coefficients appearing in the addition law for $\mathscr{G}_k$. Rather, the Frobenius is a map $\mathscr{G}_k \to \mathscr{G}_k^{(\sigma)}$, where $\mathscr{G}^{(\sigma)} = \mathscr{G} \otimes_{W(k),\sigma} W(k)$. By Theorem 3, this Frobenius induces a $W(k)$-linear map $\mathbf{D}(\mathscr{G}^{(\sigma)}/W(k)) = \mathbf{D}(\mathscr{G}/W(k)) \otimes_{W(k),\sigma} W(k) \to \mathbf{D}(\mathscr{G}/W(k))$. This is one and the same thing as a σ-*linear* map $F\colon \mathbf{D}(\mathscr{G}/W(k)) \to \mathbf{D}(\mathscr{G}/W(k)$. That is, $F(\alpha v) = \sigma(\alpha)F(v)$ for $\alpha \in W(k)$, $v \in \mathbf{D}(\mathscr{G}/W(k))$.

Exercise. Let k be a perfect field of characteristic p, and $\mathscr{G}$ be a formal group over $W(k)$. Using the presentation of $\mathbf{D}(\mathscr{G}/A)$ in Prop. 1, show that $\sum_{n\geq 1} a_n X^n \mapsto p \sum_{n \geq 1} a_{pn}^{\sigma^{-1}} X^n$ defines a σ^{-1}-linear endomorphism V of $\mathbf{D}(\mathscr{G}/A)$, and that in the endomorphism ring of $\mathbf{D}(\mathscr{G}/A)$ we have $FV = p$.

The Dieudonné module. Theorem 3 can be used to give a quick and easy definition of the Dieudonné module in the case of p-divisible groups.

Let k be a perfect field of characteristic p, and let $\mathscr{G}_0$ be a p-divisible formal group over k of (finite) height h. Choose a lift $\mathscr{G}$ of $\mathscr{G}_0$ to $W(k)$; the existence of such a $\mathscr{G}$ is a result of Lazard. Now define the *Dieudonné module* of $\mathscr{G}$ by

$$M(\mathscr{G}_0) = \mathbf{D}(\mathscr{G}/W(k)).$$

The first order of business is to show that $M(\mathscr{G}_0)$ does not depend on the choice of lift $\mathscr{G}/W(k)$. If $\mathscr{G}'$ is another lift, let $f\colon \mathscr{G}' \to \mathscr{G}$ and $g\colon \mathscr{G} \to \mathscr{G}'$ be isomorphisms of pointed formal Lie varieties which lift the identity on $\mathscr{G}_0$. (These are utterly trivial to construct: the underlying formal Lie varieties of both $\mathscr{G}$ and $\mathscr{G}'$ are represented by $W(k)[\![X]\!]$.) By part

(1) of Theorem 3 we have maps f^* and g^* between $\mathbf{D}(\mathscr{G}/W(k))$ and $\mathbf{D}(\mathscr{G}'/W(k))$ in either direction. Notice that $g \circ f \equiv 1 \pmod p$, so that by part (2) of the same theorem we must have $f^* \circ g^* = 1$ on $\mathbf{D}(\mathscr{G}'/W(k))$ and similarly for $g^* \circ f^*$ (apply the theorem with $f_1 = g \circ f$, $f_2 = 0$ and $f_3 =$ identity). Thus $\mathbf{D}(\mathscr{G}/W(k))$ and $\mathbf{D}(\mathscr{G}'/W(k))$ are isomorphic, and what's more, the theorem also implies that the isomorphism doesn't depend on the choice of map f.

Therefore $\mathscr{G}_0 \mapsto M(\mathscr{G}/W(k))$ is a well-defined functor from the category of formal p-divisible groups over k to the category of $W(k)$-modules. By the previous section, $M(\mathscr{G}/W(k))$ comes equipped with a pair of endomorphisms F and V, with F σ-linear and V σ^{-1}-linear, and $FV = p$.

More on Dieudonné theory (but not quite enough).

It is at this point we admit that a huge part of Dieudonné theory has been omitted from the discussion. For one thing, there are many p-divisible groups besides the formal ones we have given so far. Generally, a *p-divisible group* $\mathscr{G}$ of height h is a family of finite flat group schemes $\mathscr{G}_n$ (over whatever base) which are p^n-torsion of order p^{nh} together with compatible morphisms $\mathscr{G}_n \to \mathscr{G}_{n+1}$ which identify $\mathscr{G}_n$ with $\mathscr{G}_{n+1}[p^n]$. For each n, the *Lie algebra* $\operatorname{Lie} \mathscr{G}_n$ is, as one would expect, the tangent space to $\mathscr{G}_n$ at its zero section; the Lie algebra of $\mathscr{G}$ is $\operatorname{Lie} \mathscr{G} = \varinjlim \operatorname{Lie} \mathscr{G}_n$.

If $\mathscr{G}$ is a *formal* group over a ring A which has the property of being p-divisible, then the multiplication-by-p^n map $A(\mathscr{G}) \to A(\mathscr{G})$ presents $A(\mathscr{G})$ as a finite module over itself. Let I_n be the image under this map of the ideal of topologically nilpotent elements in $A(\mathscr{G})$; then $A(\mathscr{G})/I_n$ is a finite A-algebra, and the family $\mathscr{G}_n = \operatorname{Spec} A(\mathscr{G})/I_n$ constitutes a p-divisible group in the above sense. Over a p-adic ring, a theorem of Grothendieck-Messing implies that the category of formal groups which are p-divisible is the same as the category of *connected* p-divisible groups. More generally, one can begin with a p-divisible group $\mathscr{G}$, and construct its "formal completion" $\hat{\mathscr{G}}$ along its zero section; the p-power torsion of $\hat{\mathscr{G}}$ recovers the connected component of $\mathscr{G}$.

Exercise. Let $\mathscr{G}$ be the Lubin-Tate formal group over the ring of integers $\mathcal{O}_F$ in a p-adic field. Show that $\mathscr{G}_n = \operatorname{Spec} \mathcal{O}_F[\![X]\!]/[p^n]_{\mathscr{G}}(X)$ is a connected group scheme of order p^{nh}, where h is the degree of $F/\mathbb{Q}_p$. Conclude that $\mathscr{G}$ is a connected p-divisible group of height h. Also show that $\operatorname{Lie} \mathscr{G}$ is a free $\mathcal{O}_F$-module of rank 1.

Exercise. At the other end of the spectrum, we have the *constant* p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$, defined to be the injective limit of the constant group schemes $p^{-n}\mathbb{Z}_p/\mathbb{Z}_p$. Show that this really is a p-divisible group (over whatever base) of height 1, with $\operatorname{Lie} \mathbb{Q}_p/\mathbb{Z}_p = 0$.

The category of *finite free Dieudonné modules* over $W(k)$ has objects which are free $W(k)$-modules M of finite rank equipped with a σ-linear endomorphism F and a σ^{-1}-linear endomorphism[5] V satisfying $FV = p$ in $\operatorname{End} M$. Morphisms between objects are $W(k)$-

[5]If one specifies in the definition that $pM \subset F(M)$, then there is no need to mention V at all.

linear maps preserving F and V. If M is such a Dieudonné module, its *dual* is $M^\vee = \mathrm{Hom}_{W(k)}(M, W(k))$; the action of F is by $(F\ell)(v) = \sigma(\ell(Vv))$, where $v \in M$, $\ell \in M^\vee$. (Please do check that this is the right way to define the dual!)

We refer the reader to Fontaine's *Groupes p-divisibles sur les corps locaux* for the following fundamental result.

Theorem 4. *Let k be a perfect field of characteristic p.*

1. *$\mathscr{G} \mapsto M(\mathscr{G})$ is an exact anti-equivalence between the category of p-divisible groups over k and the category of finite free Dieudonné modules.*
2. *If $\mathscr{G}$ has height h, then $M(\mathscr{G})$ is free of rank h.*
3. *There is a functorial isomorphism of k-vector spaces $M(\mathscr{G})/FM(\mathscr{G}) \cong \omega_{\mathscr{G}}$.*
4. *$\mathscr{G}$ is connected if and only if F is topologically nilpotent on $M(\mathscr{G})$. $\mathscr{G}$ is étale (becomes constant after étale base extension) if and only if F is bijective.*
5. *$\mathscr{G} \mapsto M(\mathscr{G})$ commutes with base change in k.*

A few examples. For the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathbb{F}_p$, we have $M(\mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Z}_p$ with $F = 1$ and $V = p$.

The multiplicative formal group $\hat{\mathbf{G}}_{\mathrm{m}}$ over $\mathbb{F}_p$ has $M(\hat{\mathbf{G}}_{\mathrm{m}}) = \mathbb{Z}_p$ with $F = p$ and $V = 1$. Note that this module is dual to the previous one: this manifests the fact that $\hat{\mathbf{G}}_{\mathrm{m}}$ and $\mathbb{Q}_p/\mathbb{Z}_p$ are *Cartier duals* of one another in the category of p-divisible groups.

Suppose $E/\mathbb{Q}_p$ is an unramified degree h extension with residue field k. Let $\mathscr{G}/\mathcal{O}_E$ be the Lubin-Tate formal $\mathcal{O}_E$-module. Then $M = M(\mathscr{G}_k) = \mathbf{D}(\mathscr{G}/\mathcal{O}_E)$ is spanned by $\omega, F(\omega), \dots, F^{h-1}(\omega)$, with $F^h(\omega) = p\omega$, where $\omega = \omega_{\mathscr{G}}$ is the invariant differential of $\mathscr{G}$.

In this example, note that M/FM is the k-line spanned by ω, which makes sense because by part 3 this element is supposed to span the cotangent space of $\mathscr{G}_k$. But beware! Even though the module M (and the endomorphism F) only depend on $\mathscr{G}_k$, the line spanned by $\omega_{\mathscr{G}}$ in M really does depend on the lift $\mathscr{G}$. Any lift of $\mathscr{G}_k$ gives rise via its invariant differential to a line $\mathrm{Fil} \subset M$ having the property that the image of Fil spans M/FM. Different lifts could (and indeed do) give rise to different lines in M having this property. These ideas were made precise by Fontaine, who showed that the set of lifts of $\mathscr{G}_k$ is *canonically the same as* the set of lines $\mathrm{Fil} \subset M$ whose image spans M/FM. This set is a ball of dimension $h-1$. Much more on this topic in the next lecture.

Exercise: Strongly recommended. Let $k = \overline{\mathbb{F}}_p$. Use Theorem 4 to show that there is a unique one-dimensional formal group $\mathscr{G}_0/k$ of height h. (Note that "one-dimensional" means that $\dim_k \omega_{\mathscr{G}_0} = 1$.) Existence has just been shown in the example; here's how the uniqueness part goes. Let $M = M(\mathscr{G}_0)$.

1. Since $\mathscr{G}_0$ is one-dimensional, we have $\dim_k M/FM = 1$. Let $v \in M$ span this space. Show that M is spanned over $W(k)$ by $v, F(v), \dots, F^{h-1}(v)$. (Let N be the space spanned by $F^i(v)$ for all $i \geq 0$; then $K = M/N$ satisfies $K = FK$, but F is topologically nilpotent.)

2. Show that $F^h(M) = pM$.

3. Show there exists $w \in M$ such that M is spanned by $w, F(w), \dots, F^{h-1}(w)$ and $F^h(w) = pw$. This completely determines M. (Starting with v, use induction on n to find a $w \in M$ with $F^h(w) \equiv apw \pmod{F^n M}$ for some $a \in W(k)^*$. Conclude that there must exist w with $F^h w = apw$. Do a final change of variable to get rid of the a.) Thus M is isomorphic to the module described in the example. Notice that to do this, it was required to work in $W(k)$ rather than $\mathbb{Z}_p^{\mathrm{nr}}$!

Exercise. Let $\mathscr{G}_0$ be the one-dimensional formal group of height h from the previous exercise. Show that $\operatorname{End} \mathscr{G}_0$ is the noncommutative $\mathbb{Z}_p$ algebra generated by $W(\mathbb{F}_{p^h})$ and an element Φ which satisfies $\Phi\alpha = \alpha^\sigma \Phi$ for $\alpha \in W(\mathbb{F}_{p^h})$.

Exercise. Generalize the above result: Let F be a p-adic field, and let k be the algebraic closure of the residue field of F. Let $h \geq 1$. Show that up to isomorphism there exists a unique formal $\mathcal{O}_F$-module $\mathscr{G}_0$ over k of height h. (Considered as an abstract formal group, such a module will have height $h[F : \mathbb{Q}_p]$.) Show that in the category of formal $\mathcal{O}_F$-modules we have that $\operatorname{End} \mathscr{G}_0$ is the ring of integers in the (unique) division algebra over F of invariant $1/h$.

The Grothendieck-Messing crystal. See Brinon-Conrad, §12.1. The original source is Mazur-Messing's book *Universal extensions and one dimensional crystalline cohomology*.

It will become necessary to stretch the ideas of the preceding sections a bit further, beyond the context of perfect fields of characteristic p and their Witt rings. Recall Lemma 1: If A is $\mathbb{Z}_p$-flat, then morphisms of pointed Lie varieties over A which agree modulo p induce the same map on H^1_{dR}. In the proof of that lemma, we used the fact that $p^n/n! \in \mathbb{Z}_p$ for all $n \geq 1$. In fact $p^n/n! \in p\mathbb{Z}_p$.

More generally, if A is $\mathbb{Z}$-flat and $I \subset A$ is an ideal, we say that I has *divided powers* (P.D.) if $x^n/n! \in I$ for all $x \in I$, $n \geq 1$. (There is also a notion of an ideal being equipped with a "divided-power structure" which makes sense even if A is not $\mathbb{Z}$-flat.) If I is such an ideal, let $A_0 = A/I$. The machinery of Lemma 1 goes through just the same. It follows that whenever $\mathscr{G}_0$ is a formal group over A_0, and $\mathscr{G}$ is a lift to A, the A-module $\mathbf{D}(\mathscr{G}/A)$ depends only on $\mathscr{G}_0$ and not the lift $\mathscr{G}$.

A similar construction exists for general p-divisible groups. When A_0 is a p-adic ring (separated and complete for the p-adic topology), we say that a surjection of p-adic rings $A \to A_0$ is a *nilpotent P.D. thickening* if its kernel is a topologically nilpotent ideal with divided powers. Let $\mathscr{G}_0$ be a p-divisible group over a ring A_0. Grothendieck and Messing construct a rule (a *crystal*) $\mathbf{D}(\mathscr{G}_0)$ which assigns to every nilpotent P.D. thickening $A \to A_0$ a locally free A-module, which is notated $\mathbf{D}(\mathscr{G}_0)(A \to A_0)$. The rule $\mathbf{D}(\mathscr{G}_0)$ is contravariant in $\mathscr{G}_0$ and commutes with base change in A. Furthermore, when $\mathscr{G}_0$ is a formal p-divisible group, $\mathbf{D}(\mathscr{G}_0)(A_0 \to A_0) = \mathbf{D}(\mathscr{G}_0/A_0)$ is the module we have already constructed. When $\mathscr{G}_0$ is a p-divisible group over a perfect field k of characteristic p, we have $\mathbf{D}(\mathscr{G}_0)(W(k) \to k) = M(\mathscr{G}_0)$.

We'll sum up the construction of $\mathbf{D}(\mathscr{G})(A \to A_0)$ as follows: One lifts $\mathscr{G}_0$ to a p-divisible group $\mathscr{G}/A$ as before, and then forms the *universal vector extension*

$$0 \to V \to E \to \mathscr{G} \to 0$$

of p-divisible groups over A: here V is isomorphic to a product of copies of $\hat{\mathbf{G}}_{\mathrm{a}}$, and this exact sequence is initial in the category of all such exact sequences. Then $\mathbf{D}(\mathscr{G})(A \to A_0)$ is defined as $\operatorname{Lie} E$. One must show that the universal vector extension has a "crystalline nature", and therefore that $\operatorname{Lie} E$ does not depend on the choice of lifting to A.

Exercise. Let $F/\mathbb{Q}_p$ be a finite extension with ramification degree e and residue field k. Show that $\mathcal{O}_F \to k$ is a nilpotent P.D. thickening if and only if $e \leq p-1$.

Lecture 2: The Lubin-Tate tower

References. For the first construction of the Lubin-Tate deformation space (without level structure), see the paper of Lubin-Tate, *Formal moduli for one-parameter formal Lie groups*. Deligne, in his 1973 letter to Piatetski-Shapiro, was the first to draw a connection between the Lubin-Tate tower, the geometry of modular curves, and the local Langlands correspondence[6]. A rigorous treatment was then provided by Carayol (*Sur les représentations ℓ-adiques attachées aux formes modulaires de Hilbert*), who filled in the missing details and treated the difficult case of $p = 2$.

The classical Lubin-Tate story, revisited.

Let $F/\mathbb{Q}_p$ be a finite extension with uniformizer π and residue field $\mathbb{F}_q$. As we have seen, there exists a unique formal $\mathcal{O}_F$-module law $\mathscr{G}/\mathcal{O}_F$ for which $[\pi]_{\mathscr{G}}(X) \equiv X^q \pmod{\pi}$. This $\mathscr{G}$ is important to local class field theory in that the field F_π obtained by adjoining all the π-power torsion to F satisfies the local Kronecker-Weber theorem: $F^{\mathrm{ab}} = F^{\mathrm{nr}} F_\pi$. In this way we have a thoroughly complete construction of all the abelian extensions of F.

This lecture will sketch how formal groups are used to construct *nonabelian* extensions of a p-adic field. Since our formal groups are commutative, it will not be the case that anything nonabelian can be found in the torsion of formal modules. Instead, we will study certain moduli spaces of formal modules (*Lubin-Tate spaces*); the action of Galois on the cohomology of these spaces cuts out interesting nonabelian extensions.

The Lubin-Tate space $\mathcal{M}_0$.

[6] for a transcript, see `http://www.math.ias.edu/ jaredw/DeligneLetterToPiatetskiShapiro.pdf`.

The moduli spaces in question are *deformation spaces*. Naïvely, we consider a formal group $\mathscr{G}_0$ over $\overline{k}$ and consider the moduli space of formal groups which lift $\mathscr{G}_0$. One must be careful because $\mathscr{G}_0$ admits many automorphisms.

Here's the precise definition. Let $k = \overline{\mathbb{F}}_p$, let $W = W(k)$, and let $K = W[1/p]$. Fix an integer $h \geq 1$ and let $\mathscr{G}_0/k$ be the one-dimensional formal group of height h. Let $\mathcal{C}$ be the category of complete local noetherian W-algebras with residue field k. Let M_0 be the functor $\mathcal{C} \to \mathbf{Sets}$ which assigns to A the set of pairs $(\mathscr{G}, \iota)$, where $\mathscr{G}$ is a one-dimensional formal group over A and $\iota\colon \mathscr{G}_0 \to \mathscr{G} \otimes_A k$ is an isomorphism.

Theorem 5. *(Lubin-Tate) M_0 is representable by an adic ring which is (non-canonically) isomorphic to the power series ring $W[\![u_1, \dots, u_{h-1}]\!]$. That is, $M_0 \approx \operatorname{Spf} W[\![u_1, \dots, u_{h_1}]\!]$.*

When $h = 1$ the theorem is interpreted to mean that $\mathcal{M}_0$ is a single point in characteristic 0. That is, there is a unique lift of $\mathscr{G}_0$ (none other than the multiplicative group).

In general, there is a *universal formal group* $\mathscr{G}^{\text{univ}}$ over $W[\![u_1, \dots, u_{h-1}]\!]$, together with an isomorphism ι^{univ} from $\mathscr{G}_0$ onto $\mathscr{G}^{\text{univ}} \otimes k$. Whenever A belongs to $\mathcal{C}$ and $(\mathscr{G}/A, \iota)$ is a pair as above, there are unique elements $x_1, \dots, x_{h-1} \in \mathfrak{m}_A$ (the maximal ideal of A) such that $(\mathscr{G}, \iota)$ is the pull-back of $(\mathscr{G}^{\text{univ}}, \iota^{\text{univ}})$ along the continuous map $W(k)[\![u_1, \dots, u_{h-1}]\!]$ sending u_i to x_i.

But beware: funny things can happen when the ring A is non-noetherian. For instance if $A = \mathcal{O}_{\mathbb{C}_p}$ one can construct a non-continuous map $W[\![u_1, \dots, u_{h-1}]\!] \to \mathcal{O}_{\mathbb{C}_p}$. The pull-back of $(\mathscr{G}^{\text{univ}}, \iota^{\text{univ}})$ along such a map is a perfectly good pair $(\mathscr{G}, \iota)$ which does *not* correspond to an $(h-1)$-tuple of elements of $\mathfrak{m}_{\mathbb{C}_p}$. Later we will use the notion of "isotriviality" to distinguish those pairs arising from continuous maps from those that do not.

Let $\mathcal{M}_0$ be the rigid generic fiber of M_0: this is the p-adic rigid open ball of radius one[7]. The space $\mathcal{M}_0$ is the *Lubin-Tate deformation space at level 0.*

Adding level structures: The spaces $\mathcal{M}_n$. The modular curve $X(1)$ is nothing more complicated than the projective line, but its coverings $X(N) \to X(1)$, corresponding to the addition of level structure in the moduli problem, carry much interesting structure. Drinfeld constructs a rigid space $\mathcal{M}_n$ having the property that for K/K_0 finite, the K-points of $\mathcal{M}$ correspond to triples (G, ι, α), where $(G/\mathcal{O}_K, \iota)$ is as in the previous paragraph, and $\alpha\colon (\mathbb{Z}/p^n\mathbb{Z})^{\oplus h} \tilde{\to} \mathscr{G}[p^n]$ is an isomorphism of $\mathbb{Z}/p^n\mathbb{Z}$-modules.

Drinfeld shows that $\mathcal{M}_n \to \mathcal{M}_0$ is an étale covering of the rigid disk with Galois group $\mathrm{GL}_h(\mathbb{Z}/p^n\mathbb{Z})$. As it turns out, $\mathcal{M}_n$ has exactly $\phi(p^n)$ geometrically connected components: this is a result of M. Strauch. (Does it seem strange that an open disk should admit any connected étale coverings at all? Such is the counterintuitive nature of nonarchimedean

[7] A *rigid space* is the nonarchimedean analogue of a complex analytic manifold. See, *e.g.* Brian Conrad's notes at `http://math.stanford.edu/~conrad/papers/aws.pdf` for an introduction. The most basic sort of rigid space is the rigid closed ball, equal to the maximal spectrum of $F\langle X_1, \dots, X_n\rangle$, F being a p-adic field. The rigid open unit ball is the union of all rigid closed balls of radius < 1.

geometry.) He also constructs a formal model of $\mathcal{M}_n$ over W using the notion of "Drinfeld level structures"–this is the same notion that Katz and Mazur use in their book to make sense of the moduli problems $Y_0(N), Y_1(N), Y(N), \dots$ over schemes where N is not invertible.

Isogenies and quasi-isogenies.

A morphism $f\colon \mathscr{G}' \to \mathscr{G}$ of p-divisible group is an *isogeny* if its kernel is a finite group scheme. If f is a morphism of formal groups, this is the same as saying that f presents $A(\mathscr{G}')$ as a finite $A(\mathscr{G})$-module.

In the category of p-divisible groups, the morphisms between two objects $\mathrm{Hom}(\mathscr{G}', \mathscr{G})$ are naturally a $\mathbb{Z}_p$-module. The category of p-divisible groups *up to isogeny* has the same objects, but morphisms are $\mathrm{Hom}^0(\mathscr{G}', \mathscr{G}) = \mathrm{Hom}(\mathscr{G}', \mathscr{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. An element of such a Hom is a *quasi-isogeny* if it equals $p^{-n}f$, where $n > 0$ and $f \in \mathrm{Hom}(\mathscr{G}', \mathscr{G})$ is an honest isogeny.

Theorem 6. *(The rigidity of quasi-isogenies) Let A be a ring in which p is nilpotent, let $I \subset A$ be a nilpotent ideal, and let $A_0 = A/I$. For p-divisible groups $\mathscr{G}$ and $\mathscr{G}'$ over A, let $\mathscr{G}_0$ and $\mathscr{G}_0'$ be their base change to A_0. Then*

$$\mathrm{Hom}_A(\mathscr{G}', \mathscr{G}) \to \mathrm{Hom}_{A_0}(\mathscr{G}_0', \mathscr{G}_0)$$

is an injection, and becomes an isomorphism when p is inverted.

In this context, then, isogenies on A/I always lift to quasi-isogenies on A.

We apply the theorem in the following context. Suppose K/K_0 is a finite extension, $\mathscr{G}/\mathcal{O}_K$ is a p-divisible group, and $\iota\colon \mathscr{G}_0 \to \mathscr{G} \otimes_{\mathcal{O}_K} k$ is an isomorphism. Now consider the ring $A = \mathcal{O}_K/p\mathcal{O}_K$: this is a k-algebra in which (trivially) p is nilpotent and $I = \mathfrak{m}_K A$ is a nilpotent ideal, with quotient $A/I = k$. Therefore the isomorphism ι may be lifted to a *quasi-isogeny* between $\mathscr{G}_0 \otimes_k A$ and $\mathscr{G}_0 \otimes_{\mathcal{O}_K} A$. This justifies the definition appearing in the next paragraph.

The Lubin-Tate tower, full definition. We would like to produce a space $\mathcal{M}$ which is the inverse limit of the $\mathcal{M}_n$. Such a space doesn't exist in the category of rigid spaces, so we shall have to content ourselves with the following definition on the level of points.

Let $K \supset K_0$ be a field admitting a valuation extending that of K_0, and let $\mathcal{O}_K$ be its ring of integers. Then $\mathcal{M}(K)$ is the set of triples $(\mathscr{G}, \iota, \alpha)$, where

- $\mathscr{G}$ is a formal group over $\mathcal{O}_K$,
- $\iota\colon \mathscr{G}_0 \otimes_k \mathcal{O}_K/p\mathcal{O}_K \to \mathscr{G} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K$ is a quasi-isogeny, and
- $\alpha\colon \mathbb{Q}_p^h \xrightarrow{\sim} V_p(\mathscr{G})$ is an isomorphism of $\mathbb{Q}_p$-vector spaces. The action of $\mathrm{Gal}(\overline{K}/K)$ on $V_p(\mathscr{G})$ is required to be trivial.

Two such triples $(\mathscr{G}, \iota, \alpha)$ and $(\mathscr{G}', \iota', \alpha')$ determine the same point in $\mathcal{M}$ if there is an *isogeny* $\mathscr{G}' \to \mathscr{G}$ translating one set of structures to the other.

Exercise. Let $K_0 \subset K_1 \subset \dots$ be a tower of fields, each of which is finite over K_0. For each n, suppose a triple $(\mathscr{G}_n, \iota_n, \alpha_n) \in \mathcal{M}_n(K_n)$ is given, where $\mathscr{G}_n/\mathcal{O}_{K_n}$ is a p-divisible group, $\iota_n \colon \mathscr{G}_0 \to \mathscr{G} \otimes_{\mathcal{O}_{K_n}} k$ is an isomorphism, and $\alpha_n \colon (\mathbb{Z}/p^n\mathbb{Z})^h \tilde{\to} \mathscr{G}[p^n]$ is a level structure. Suppose these points are compatible in the sense that there are isomorphisms $\mathscr{G}_n \otimes \mathcal{O}_{K_{n+1}} \cong \mathscr{G}_{n+1}$ making all the obvious diagrams commute. Use these data to construct a point of $\mathcal{M}$ over $K = \varinjlim K_n$.

The group actions. The Lubin-Tate tower is quite symmetric, admitting right actions of three important groups:

- $G = \mathrm{GL}_h(\mathbb{Q}_p)$,
- J, the group of units in the division algebra $\operatorname{End} \mathscr{G}_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of dimension h, and
- $W_{\mathbb{Q}_p}$, the Weil group. This is the preimage of $\mathbb{Z}$ in the quotient map $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}}$.

The action of the first two groups on $\mathcal{M}$ is quite simple to describe. An element $g \in G$ sends a triple $(\mathscr{G}, \iota, \alpha)$ to $(\mathscr{G}, \iota, \alpha \circ g)$. Similarly an element $j \in J$ sends a triple $(\mathscr{G}, \iota, \alpha)$ to $(\mathscr{G}, \iota \circ j, \alpha)$.

The action of $W_{\mathbb{Q}_p}$ is a bit more subtle. Let $w \in W_{\mathbb{Q}_p}$ and suppose that w induces σ^n on $\overline{\mathbb{F}}_p/\mathbb{F}_p$, where by definition $n \in \mathbb{Z}$. Let $(\mathscr{G}, \iota, \alpha)$ represent a $\mathbb{C}_p$-point on $\mathcal{M}$. Then one has the p-divisible group $\mathscr{G}^w$ obtained, if you like, by applying w coefficient-wise to a formal group law representing $\mathscr{G}$. We also have the translated level structure α^w on $\mathscr{G}^w$. Applying w to the quasi-isomorphism ι, however, gives $\iota^w \colon \mathscr{G}_0^{(p^n)} \otimes \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \to \mathscr{G}^w \otimes \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$. We declare that w sends the triple $(\mathscr{G}, \iota, \alpha)$ to $(\mathscr{G}^w, \iota^w \circ F^n, \alpha^w)$, where F^n is the p^n-power Frobenius map $\mathscr{G}_0 \to \mathscr{G}_0^{(p^n)}$.

The zero-dimensional Lubin-Tate tower. When $h = 1$, $\mathcal{M}$ has dimension 0. The p-divisible group $\mathscr{G}_0$ is just the multiplicative group, and any lift of $\mathscr{G}_0$ to characteristic 0 is also isomorphic to the multiplicative group (by the uniqueness of lifts of p-divisible groups of height 1). Therefore $\mathcal{M}$ is simply the set of level structures $\alpha \colon \mathbb{Q}_p \tilde{\to} V(\hat{\mathbf{G}}_{\mathrm{m}}) = \mathbb{Q}_p(1)$, which is to say that $\mathcal{M}$ is the set $\mathbb{Q}_p(1)^\times$ of nonzero elements of $\mathbb{Q}_p(1)$. We have that G and J are both $\mathbb{Q}_p^\times$ (but beware: their actions on $\mathcal{M} = \mathbb{Q}_p(1)^\times$ are inverse to one another). The action of $W_{\mathbb{Q}_p}$ on $\mathcal{M} = \mathbb{Q}_p(1)^\times$ is through the reciprocity map $W_{\mathbb{Q}_p}^{\mathrm{ab}} \to \mathbb{Q}_p^\times$.

Nonabelian Lubin-Tate theory. We can now explain why in general the Lubin-Tate tower $\mathcal{M}$ provides a nonabelian analogue to the theorems of local class field theory. For $\ell \neq p$ one can associate to $\mathcal{M}$ the ℓ-adic étale cohomology spaces $H_c^i(\mathcal{M}, \mathbb{Q}_\ell) = \varinjlim H_c^i(\mathcal{M}_n, \mathbb{Q}_\ell)$ (this is nontrivial: one needs Berkovich's theory of vanishing cycles attached to formal schemes). Then $H_c^i(\mathcal{M}, \mathbb{Q}_\ell)$ admits an action of the triple product group $G \times J \times W_{\mathbb{Q}_p}$. In brief, the spaces $H_c^i(\mathcal{M}, \mathbb{Q}_\ell)$ realize canonical correspondences between representations of all three groups G, J, and $W_{\mathbb{Q}_p}$.

It has been known since the early 1980s (Jacquet-Langlands for $h = 2$, Rogawski in general) that there is a canonical bijection JL from irreducible representations of G onto those of J[8].

Theorem 7 (Harris-Taylor, 2002)**.** *There exists a bijection*[9] *between irreducible supercuspidal representations π of G and irreducible h-dimensional representations of $W_{\mathbb{Q}_p}$ having the property that for all irreducible supercuspidal representations π with $\overline{\mathbb{Q}}_\ell$-coefficients we have (up to a sign)*

$$Hom_G\left(\pi, H_c^*(\mathcal{M}, \overline{\mathbb{Q}}_\ell)\right) = \mathrm{JL}(\pi) \otimes \mathrm{rec}(\pi)$$

as (virtual) representations of $J \times W_{\mathbb{Q}_p}$.

The theorem of Harris-Taylor is in fact proved with $\mathbb{Q}_p$ replaced by any p-adic field, using a moduli space of deformations of a formal $\mathcal{O}_F$-module of height h. For a nice overview of the whole subject, see M. Harris' survey article[10]. The proof of Thm. 7 is *global* in nature, requiring techniques from the world of adelic automorphic forms and Shimura varieties. Boyer has a finer theorem than the above which concerns not just the supercuspidal part of the cohomology of $\mathcal{M}$; see his paper "Conjecture de monodromie-poids pour quelques variétés de Shimura unitaires".

A few open questions remain. For instance, does there exist of a purely local proof of Thm. 7? A tiny fraction of this latter question will be treated in Lecture 3.

Exercise. For $n \gg 0$, let $Y(N)$ be the set of isomorphism classes of pairs (E, α), where $E/\mathbb{C}$ is an elliptic curve and $\alpha\colon (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \tilde{\to} E[N](\mathbb{C})$ is an isomorphism. (The notion of an isomorphism between pairs is the evident one.) Let Y_0 be the inverse limit of the sets $Y(N)$. On the other hand, let Y be the set of *isogeny classes* of pairs (E, α), where $E/\mathbb{C}$ is an elliptic curve and $\alpha\colon \mathbf{A}_{\mathbb{Q}}^{\oplus 2} \tilde{\to} V(E) = \varprojlim_N E[N] \otimes_{\hat{\mathbb{Z}}} \mathbf{A}_{\mathbb{Q}}$, is an isomorphism, where $\mathbf{A}_{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adeles. Two pairs (E, α) and (E', α') will be in the same class whenever there is a quasi-isogeny $E' \to E$ making the appropriate diagram commute. There is an obvious map $Y_0 \to Y$. Pass to the top exterior power and show that the Weil pairing gives a map $Y \to V(\mu) = \left(\varprojlim \mu_N\right) \otimes_{\hat{\mathbb{Z}}} \mathbf{A}_{\mathbb{Q}}$, and that Y_0 is the preimage of the set of primitive elements in $T(\mu)$ under this map. Show also that there is a natural action of $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}})$ on Y.

Of course this Y can be given a geometric structure (rather than it just being a set). This is the point of view laid out in Deligne's letter to Piatetski-Shapiro in which he proves nonabelian Lubin-Tate theory for $h = 2$ and $p > 2$: the tower of curves Y, with its action of $\mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}})$, encodes the (global!) Langlands correspondence in its cohomology.

Lecture 3: Period Maps

[8] Of course one has to specify what kinds of representations are in play here. The relevant representations of G are admissible and essentially square-integrable, and the representations of J are smooth. For definitions of these terms, see for instance the book by Bushnell and Henniart.

[9] This bijection is not quite "the" local Langlands correspondence for $\mathrm{GL}(h)$; that bijection differs from this one by an unramified twist which only depends on h.

[10] Available at `http://arxiv.org/pdf/math/0304324v1`

References. The Gross-Hopkins period map appears in the paper of Gross and Hopkins, *Equivariant vector bundles on the Lubin-Tate moduli space*, for which the authors had an application to stable homotopy theory. For the construction of the crystalline period ring A_{cris}, we refer to Fontaine's paper *Le corps des périodes p-adiques*. For the crystalline period mappings associated to the Lubin-Tate tower, we refer to Faltings' terse paper *A relation between two moduli spaces studied by V.G. Drinfeld*, and Fargues' book, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, each chapter of which is available on `arxiv.org`.

The Gross-Hopkins period map. As in the previous lecture, $\mathscr{G}_0$ is the unique p-divisible group of height h over $\overline{\mathbb{F}}_p$. We abbreviate $W = W(\overline{\mathbb{F}}_p)$ and $K_0 = W[1/p]$. Recall that for a p-adic field $K \supset K_0$, the set of K-points of the Lubin-Tate space $\mathcal{M}$ is the set of triples $(\mathscr{G}, \iota, \alpha)$, where $\mathscr{G}/\mathcal{O}_K$ is a p-divisible group, $\alpha\colon \mathscr{G}_0 \otimes \mathcal{O}_K/p\mathcal{O}_K \to \mathscr{G} \otimes \mathcal{O}_K/p\mathcal{O}_K$ is a quasi-isogeny, and $\alpha\colon \mathbb{Q}_p^h \to V_p(\mathscr{G})$ is an isomorphism. The Gross-Hopkins period map is nothing but the quotient of $\mathcal{M}$ by $\mathrm{GL}_h(\mathbb{Q}_p)$, which is to say that it "forgets" the level structure α. We are going to use the technology of the functor $\mathbf{D}$ to describe the quotient.

Suppose a *pair* $(\mathscr{G}, \iota)$ is given (rather than a triple). The module $\mathbf{D}(\mathscr{G}/\mathcal{O}_K) = \mathbf{D}(\mathscr{G})(\mathcal{O}_K \to \mathcal{O}_K)$ sits in the Hodge-Tate exact sequence (from Eq. (7))

$$0 \to \omega_{\mathscr{G}} \to \mathbf{D}(\mathscr{G}/\mathcal{O}_K) \to \mathrm{Lie}\,\mathscr{G}^\vee \to 0.$$

It will be convenient to take duals:

$$0 \to \omega_{\mathscr{G}^\vee} \to \mathbf{D}(\mathscr{G}/\mathcal{O}_K)^\vee \to \mathrm{Lie}\,\mathscr{G} \to 0.$$

Now consider the quasi-isogeny $\iota\colon \mathscr{G}_0 \otimes_k \mathcal{O}_K/p\mathcal{O}_K \to \mathscr{G} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K$. This induces an isomorphism of K-vector spaces

$$\iota_*\colon \mathbf{D}(\mathscr{G}_0 \otimes_k \mathcal{O}_K/p\mathcal{O}_K)(\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K)^\vee[1/p] \xrightarrow{\sim} \mathbf{D}(\mathscr{G} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K)(\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K)^\vee[1/p]$$

which may just as well be rewritten (using the compatibility of the Messing crystal with base change)

$$\iota_*\colon M(\mathscr{G}_0)^\vee \otimes_W K \xrightarrow{\sim} \mathbf{D}(\mathscr{G}/\mathcal{O}_K)^\vee \otimes_{\mathcal{O}_K} K$$

Let $\mathrm{Fil} \subset M(\mathscr{G}_0)^\vee \otimes_W K$ be the preimage of $\omega_{\mathscr{G}^\vee} \otimes_{\mathcal{O}_K} K$ under ι_*:

$$\begin{tikzcd} M(\mathscr{G}_0)^\vee \otimes_W K \arrow[r, "\sim"] & \mathbf{D}(\mathscr{G}/\mathcal{O}_K)^\vee \otimes_{\mathcal{O}_K} K \\ \mathrm{Fil} \arrow[u] \arrow[r, "\sim"] & \omega_{\mathscr{G}^\vee} \otimes_{\mathcal{O}_K} K \arrow[u] \end{tikzcd}$$

Then Fil is a hyperplane in $M(\mathscr{G}_0)^\vee \otimes_W K \approx K^h$; that is, it represents a *point* in projective space $\mathbf{P}^{h-1}(K)$. The association $(\mathscr{G}, \iota, \alpha) \mapsto \mathrm{Fil}$ constitutes a map $\mathcal{M} \to \mathbf{P}^{h-1}$ which does not depend on α. It therefore factors through a map $\mathcal{M}_0 \to \mathbf{P}^{h-1}$.

Theorem 8. ***(Gross-Hopkins)*** *The map $\mathcal{M}_0 \to \mathbf{P}^{h-1}$ is an étale surjection of rigid-analytic spaces. Each fiber consists of a single isogeny class of lifts of $\mathscr{G}_0$.*

Which implies that each $\mathcal{M}_n \to \mathbf{P}^{h-1}$ is an étale surjection as well. The theorem can be expressed as a diagram

$$\begin{tikzcd} \mathcal{M} \arrow[d, "\mathrm{GL}_h(\mathbb{Z}_p)"] \arrow[dd, bend right, "\mathrm{GL}_h(\mathbb{Q}_p)"'] \\ \mathcal{M}_0 \arrow[d] \\ \mathbf{P}^{h-1} \end{tikzcd}$$

This theorem is truly shocking: in the non-archimedean world, *projective space* in dimension $(h-1)$ isn't just non-simply-connected, but its étale fundamental group has no less a group than $\mathrm{SL}_h(\mathbb{Q}_p)$ as a quotient[11]! Nothing like this exists in the algebraic world. Gross and Hopkins even give a formula for their period map (in terms of an appropriate set of coordinates on $\mathcal{M}_0$), which takes the form of a (necessarily transcendental) h-tuple of power series.

One can also show that the Gross-Hopkins map $\mathcal{M} \to \mathbf{P}^{h-1} = \mathbf{P}(M(\mathscr{G}_0))$ is equivariant for the action of $J = \mathrm{Aut}^0 \mathscr{G}_0$.

Fontaine's period ring A_{cris}: motivation. The formalism of Grothendieck-Messing crystals focuses our attention on nilpotent PD thickenings of p-adic rings. Recall that if A_0 is a p-adic ring, then a nilpotent PD thickening of A_0 is a surjection $A \to A_0$ of p-adic rings whose kernel is a topologically nilpotent PD ideal. In the definition of the Gross-Hopkins period map, we evaluated the Grothendieck-Messing crystal on nilpotent PD thickenings of the form $\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$, where K is a p-adic field. This period map only depended on the pair $(\mathscr{G}, \iota)$ and not on the level structure α.

Following Faltings and Fargues, we will contruct a period map out of $\mathcal{M}$ which is in a sense a refinement of the Gross-Hopkins map. To do this, we will evaluate the Grothendieck-Messing crystal on a surjection $A \to A_0$ in which the role of A_0 (not A!) is played by $\mathcal{O}_K$. The role of A will be played by Fontaine's ring of crystalline periods.

The ring A_{cris}. For this we recommend the original source, Fontaine's paper *Le corps des périodes p-adiques*. Fontaine constructs a surjection $\theta\colon A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p}$ which has the following properties:

- A_{cris} is separated and complete for the p-adic topology.
- $\ker\theta$ has divided powers.
- $A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p}$ is the initial object in the category of surjections $A \to \mathcal{O}_{\mathbb{C}_p}$ satisfying the above two properties.

[11] By Strauch's result, there is a surjective map $\mathcal{M} \to \mathbb{Q}_p(1)^\times$ with geometrically connected fibers, each of which is stabilized by $\mathrm{SL}_h(\mathbb{Q}_p)$.

Actually, in Fontaine's formulation, A_{cris} is a *functor* $\mathcal{O} \mapsto A_{\mathrm{cris}}(\mathcal{O})$ on p-adic rings $\mathcal{O}$, defined the same as above with $\mathcal{O}$ replacing $\mathcal{O}_{\mathbb{C}_p}$. This functor is shown to be well-defined so long as $\mathcal{O}/p\mathcal{O}$ is *perfect* (that is, the pth power map is required to be surjective on $\mathcal{O}/p\mathcal{O}$). When we refer to the ring A_{cris}, what we mean is $A_{\mathrm{cris}}(\mathcal{O}_{\mathbb{C}_p})$: by functoriality, this ring admits an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

We refer to Fontaine for the construction of A_{cris}. We only note that in the course of constructing A_{cris} (the functor) one finds that $A_{\mathrm{cris}}(\mathcal{O}) = A_{\mathrm{cris}}(\mathcal{O}/p\mathcal{O})$. A consequence is that the pth power map on $\mathcal{O}/p\mathcal{O}$ induces an endomorphism on A_{cris} which we call ϕ. If $\mathcal{O}$ is a W algebra, then $A_{\mathrm{cris}}(\mathcal{O})$ is a W-algebra as well, and the endomorphism $\phi\colon A_{\mathrm{cris}}(\mathcal{O}) \to A_{\mathrm{cris}}(\mathcal{O})$ is σ-semilinear. *The endomorphism ϕ does not have any obvious relationship with the algebra map $\theta\colon A_{\mathrm{cris}}(\mathcal{O}) \to \mathcal{O}$.*

Finally, we note that if $K/\mathbb{Q}_p$ is unramified then $A_{\mathrm{cris}}(\mathcal{O}_K) = \mathcal{O}_K$. Thus $A_{\mathrm{cris}}(\mathcal{O}_K)$ is only interesting when $K/\mathbb{Q}_p$ is infinitely ramified[12], in which case $A_{\mathrm{cris}}(\mathcal{O}_K)$ is enormous.

Colmez' Finite-Dimensional Vector Spaces. The ring A_{cris} may be unmanageably huge in general, but for the purposes of constructing period maps out of $\mathcal{M}$ it suffices to analyze much smaller pieces of it. Let $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$; this is a K_0-vector space (albeit one of uncountable dimension).

For each $h \geq 1$, the "piece" of B^+_{cris} that we need is $\left(B^+_{\mathrm{cris}}\right)^{\phi^h=p}$, which is a vector space over $E = \mathbb{Q}_{p^h} = W(\mathbb{F}_{p^h})[1/p]$. It turns out we can provide a complete description of $\left(B^+_{\mathrm{cris}}\right)^{\phi^h=p}$ in terms of the (unique) Lubin-Tate formal $\mathcal{O}_E$-module. Let LT be a formal $\mathcal{O}_E$-module law of height 1 (*e.g.*, LT could be the unique such beast for which $[p]_{\mathrm{LT}}(X) = pX + X^{p^h}$).

Let $\mathbf{U}_E$ be the set of sequences $x = (x^{(0)}, x^{(1)}, \dots)$ of elements of $\mathfrak{m}_{\mathbb{C}_p}$ satisfying $[p]_{\mathrm{LT}}(x_i) = x_{i-1}$ for $i \geq 1$. That is,

$$\mathbf{U}_E = \varprojlim_{[p]_{\mathrm{LT}}} \mathfrak{m}_{\mathbb{C}_p}.$$

Then $\mathcal{O}_E$-module structure of LT gives $\mathbf{U}_E$ the structure of an $\mathcal{O}_E$-module. One can also *divide* elements of $\mathbf{U}_E$ by p by the rule

$$\frac{1}{p} \cdot (x^{(0)}, x^{(1)}, \dots) = (x^{(1)}, x^{(2)}, \dots),$$

so that $\mathbf{U}_E$ is naturally an E-vector space.

Define a map $\theta\colon \mathbf{U}_E \to \mathbb{C}_p$ via

$$\theta(x) = \log_{\mathrm{LT}} x^{(0)}.$$

Exercise. Show that θ is a surjective map of E-vector spaces.

[12]If $K/\mathbb{Q}_p$ is finitely ramified, then $\mathcal{O}_K/p\mathcal{O}_K$ cannot be perfect.

Theorem 9. *There is an isomorphism of E-vector spaces $\left(B_{\mathrm{cris}}^{+}\right)^{\phi^h=p} \cong \mathbf{U}_E$ which is compatible with both of the maps θ from either object onto $\mathbb{C}_p$.*

The idea of proof is contained in Colmez' paper *Espaces de Banach de Dimension Finie*, §8.

We ought to mention three little points here:

- The object we have called $\mathbf{U}_E$, much like A_{cris}, is really a functor, rather than a bare vector space.
- The construction of $\mathbf{U}_E$ can be generalized to include the case of $E/\mathbb{Q}_p$ ramified. In that case the theorem above generalizes to an isomorphism of $\mathbf{U}_E$ onto $(B^+_{\mathrm{cris},E})^{\phi_E=\pi_E}$, where $B^+_{\mathrm{cris},E} = A^+_{\mathrm{cris},E}[1/\pi_E]$, and $A_{\mathrm{cris},E}$ is the universal PD thickening of $\mathcal{O}_{\mathbb{C}_p}$ *in the category of π_E-adically complete $\mathcal{O}_E$-algebras* (see Colmez §7 for that construction).
- There is a continuous action of $G_E = \mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$ on the spaces $A_{\mathrm{cris},E}$ and $\mathbf{U}_E$, and the map θ is Galois-equivariant.

Finally, since $\left(B_{\mathrm{cris}}^{+}\right)^{\phi^h=p}$ is visibly stabilized by ϕ, we ought to make precise the corresponding action of ϕ on $\mathbf{U}_E$. If the sequence $(x^{(0)}, x^{(1)}, \dots)$ represents an element $x \in \mathbf{U}_E$, then $\phi(x)$ is the sequence whose ith term is

$$\phi(x)^{(i)} = \lim_{n\to\infty} [p^n]_{\mathrm{LT}}\left(\left(x^{(n+i)}\right)^p\right).$$

Exercise. For any $E/\mathbb{Q}_p$ finite, show that the kernel of $\theta\colon \mathbf{U}_E \to \mathbb{C}_p$ is a one-dimensional E-vector space, spanned by an element $t_E = (0, \lambda_1, \dots)$ where $\lambda_1 \neq 0$. Therefore we have an exact sequence[13] of E-vector spaces

$$0 \longrightarrow Et_E \longrightarrow \mathbf{U}_E \xrightarrow{\theta} \mathbb{C}_p.$$

The action of G_E on the one-dimensional vector space Et_E gives a character $G_E \to E^\times$. What is this character? (Note: when $E = \mathbb{Q}_p$, the element $t = t_E$ is known as the p-adic analogue of $2\pi i$, in that it spans the space of periods for the multiplicative group.)

The crystalline period map out of the Tate module.

Let $\mathscr{G}/\mathcal{O}_{\mathbb{C}_p}$ be a p-divisible group. The Tate module $T(\mathscr{G})$ is quite the same thing as $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathscr{G})$ in the category of p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$, and the rational Tate module $V(\mathscr{G}) = T(\mathscr{G})[1/p]$ is the same thing as $\mathrm{Hom}^0(\mathbb{Q}_p/\mathbb{Z}_p, \mathscr{G})$ in the category of p-divisible groups

[13] In Colmez' notation, $\mathbf{U}_E$ is a "Vector Space" of dimension $(1, h)$.

up to isogeny. When we pass to (covariant) Messing crystals evaluated on the PD thickening $A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p}$, we find a $\mathbb{Q}_p$-linear map

$$V(\mathscr{G}) \to \mathrm{Hom}\left(\mathbf{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee[1/p], \mathbf{D}(\mathscr{G})(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee[1/p]\right).$$

Now $\mathbf{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})$ is identified with A_{cris}, on which Frobenius acts via the action of ϕ. Another way of putting this is that the action of ϕ on 1 is 1. The dual module $\mathbf{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee$ is also identified with A_{cris}, but Frobenius acts via $p\phi$. Since the homomorphisms in the above "Hom" must commute with Frobenius, the image of $1 \in B_{\mathrm{cris}}^+ \cong \mathbf{D}(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee[1/p]$ must lie in $\left(\mathbf{D}(\mathscr{G})(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee[1/p]\right)^{\phi=p}$. We arrive at a *crystalline period map*

$$V(\mathscr{G}) \to \left(\mathbf{D}(\mathscr{G})(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee[1/p]\right)^{\phi=p} \tag{2}$$

Suppose now that $\mathscr{G}$ is isotrivial, and that we are given a quasi-isogeny $\iota\colon \mathscr{G}_0 \otimes_k (\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}) \to \mathscr{G} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{C_p}/p\mathcal{O}_{C_p}$. Then as before, ι induces an isomorphism

$$\iota_*\colon M(\mathscr{G}_0)^\vee \otimes_W A_{\mathrm{cris}} \xrightarrow{\sim} \mathbf{D}(\mathscr{G})(A_{\mathrm{cris}} \to \mathcal{O}_{\mathbb{C}_p})^\vee[1/p]. \tag{3}$$

Combining this with Eq. (2) gives a map

$$V(\mathscr{G}) \to \left(M(\mathscr{G}_0)^\vee \otimes_W A_{\mathrm{cris}}\right)^{\phi=p}.$$

Exercise. Let $E = \mathbb{Q}_{p^h}$. Recall that the Dieudonné module $M(\mathscr{G}_0)$ is spanned by vectors $v, F(v), \dots, F^{h-1}(v)$, with $F^h(v) = pv$. Show that $(M(\mathscr{G}_0)^\vee \otimes_W B_{\mathrm{cris}})^{\phi=p}$ is isomorphic to $\left(B_{\mathrm{cris}}^+\right)^{\phi^h=p}$, which is in turn isomorphic to $\mathbf{U}_E$.

Therefore the crystalline period map may be written

$$\Pi\colon V(\mathscr{G}) \to \left(M(\mathscr{G}_0)^\vee \otimes_W A_{\mathrm{cris}}\right)^{\phi=p} \cong \mathbf{U}_E.$$

Write $\theta\colon M(\mathscr{G}_0)^\vee \otimes_W B_{\mathrm{cris}}^+ \to M(\mathscr{G}_0)^\vee \otimes_W \mathbb{C}_p$ for the base change of $\theta\colon B_{\mathrm{cris}}^+ \to \mathbb{C}_p$ up to $M(\mathscr{G}_0)^\vee \otimes_W B_{\mathrm{cris}}^+$. Recall that to the pair $(\mathscr{G}, \iota)$ we had associated a subspace $\mathrm{Fil} \subset M(\mathscr{G}_0)^\vee \otimes_W \mathbb{C}_p$ of codimension 1.

Proposition 2. $\theta(\Pi(V(\mathscr{G}))) = \mathrm{Fil}$.

Application: Determinants of Galois representations coming from p-divisible groups.

If $x_1, \dots, x_h \in \left(B_{\mathrm{cris}}^+\right)^{\phi^h=p}$, then the determinant $Z = \det(\phi^i(x_j))_{1 \le i,j \le h}$ lies in $\left(B_{\mathrm{cris}}^+\right)^{\phi=-p}$, which is isomorphic (via scaling by any element $w \in W$ with $\sigma(w) = -w$) to $\left(B_{\mathrm{cris}}^+\right)^{\phi=p} \cong \mathbf{U}_{\mathbb{Q}_p}$. We have therefore constructed a $\mathbb{Q}_p$-linear map

$$\bigwedge_{\mathbb{Q}_p}^{h} \mathbf{U}_E \to \mathbf{U}_{\mathbb{Q}_p}$$

Now suppose $\mathscr{G}/\mathcal{O}_{\mathbb{C}_p}$ is an isotrivial p-divisible group, and that $\iota\colon \mathscr{G}_0 \otimes_k (\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}) \to \mathscr{G} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathcal{O}_{C_p}/p\mathcal{O}_{C_p}$ is a quasi-isogeny. Then the period map $\Pi\colon V(\mathscr{G}) \to \mathbf{U}_E$ has determinant

$$\delta\colon \textstyle\bigwedge_{\mathbb{Q}_p}^h V(\mathscr{G}) \xrightarrow{\det \Pi} \bigwedge_{\mathbb{Q}_p}^h \mathbf{U}_E \longrightarrow \mathbf{U}_{\mathbb{Q}_p}$$

By Prop. 2, $\theta(\mathrm{Im}\,(\delta)) = 0$, so that in fact δ is a map from $\bigwedge_{\mathbb{Q}_p}^h V(\mathscr{G})$ onto the kernel of $\theta\colon \mathbf{U}_{\mathbb{Q}_p} \to \mathbb{C}_p$. This kernel is the $\mathbb{Q}_p$-vector space spanned by t, the p-adic analogue of $2\pi i$. One can conclude from this the following

Theorem 10. *Let $K/\mathbb{Q}_p$ be a finite extension, and let $\mathscr{G}/\mathcal{O}_K$ be a one-dimensional p-divisible formal group whose special fiber has height h. There is an isomorphism of $\mathbb{Q}_p[G_K]$-modules between $\bigwedge_{\mathbb{Q}_p}^h V(\mathscr{G})$ and $V(\hat{\mathbf{G}}_m)$.*

In other words, G_K acts on $\bigwedge_{\mathbb{Q}_p}^h V(\mathscr{G})$ through the cyclotomic character.

This theorem is a special case of a result of Raynaud in his paper *Schémas en groups de type* $(p, \dots, p)$. We note the following generalization: Suppose $F/\mathbb{Q}_p$ is a finite extension with uniformizer π, assume K contains F, and let $\mathscr{G}/\mathcal{O}_K$ be a one-dimensional π-divisible formal $\mathcal{O}_F$-module whose special fiber has height h (relative to F). Then there is an isomorphism of $F[G_K]$-modules between $\bigwedge_F^h V(\mathscr{G})$ and $V(\mathrm{LT})$, where LT is the unique Lubin-Tate formal $\mathcal{O}_F$-module over $\mathcal{O}_F$.

If $(\mathscr{G}, \iota, \alpha)$ represents a point of $\mathcal{M}$, the determinant of the level structure α gives a basis for the one-dimensional $\mathbb{Q}_p$-vector space $\bigwedge_{\mathbb{Q}_p}^h V(\mathscr{G})$. The map δ above maps $\bigwedge_{\mathbb{Q}_p}^h V(\mathscr{G})$ into $\mathbb{Q}_p t$. As a result the point $(\mathscr{G}, \iota, \alpha)$ of $\mathcal{M}$ induces a basis for $\mathbb{Q}_p t$. We have constructed a *determinant map*

$$\delta\colon \mathcal{M} \to \mathbb{Q}_p^\times t$$

The map δ transforms the actions of G, J and W_F into the determinant, reduced norm^{-1}, and cyclotomic character respectively.

A description of $\mathcal{M}(\mathbb{C}_p)$ via linear algebra.

Let $(\mathscr{G}, \iota, \alpha)$ represent a $\mathbb{C}_p$-point of $\mathcal{M}$. Let $\mathrm{Fil} \subset M(\mathscr{G}_0)^\vee \otimes_W \mathbb{C}_p$. Composing the level structure $\alpha\colon \mathbb{Q}_p^h \to V(\mathscr{G})$ with the period map Π from above gives an h-tuple of elements $v_1, \dots, v_h \in \left(M(\mathscr{G}_0)^\vee \otimes_W B_{\mathrm{cris}}^+\right)^{\phi=p}$. By Prop. 2, the vectors $\theta(v_1), \dots, \theta(v_h) \in M(\mathscr{G}_0)^\vee \otimes_W \mathbb{C}_p$ span a subspace Fil of codimension 1. In fact such an h-tuple is enough to determine the triple $(\mathscr{G}, \iota, \alpha)$ completely:

Theorem 11. *(adapted from Fargues, Théorème 7.4) The $\mathbb{C}_p$-points of the Lubin-Tate tower $\mathcal{M}$ are naturally identified with the set of h-tuples of elements $v_1, \dots, v_h \in \left(M(\mathscr{G}_0)^\vee \otimes_W B_{\mathrm{cris}}^+\right)^{\phi=p}$ which have the property that the vectors $\theta(v_1), \dots, \theta(v_h) \in M(\mathscr{G}_0) \otimes_W \mathbb{C}_p$ span a space of dimension exactly $h-1$.*

In light of the previous exercise, we may restate the theorem as follows:

Theorem 12. *Let $E = \mathbb{Q}_{p^h}$. The $\mathbb{C}_p$-points of the Lubin-Tate tower $\mathcal{M}$ are naturally identified with h-tuples of elements $x_1, \dots, x_h \in \mathbf{U}_E$ for which the matrix $(\phi^i(x_j))_{1 \le i,j \le h}$ has rank exactly $h-1$.*

We can summarize our description of $\mathcal{M}(\mathbb{C}_p)$ with the following diagram:

$$\begin{tikzcd}
\mathcal{M}(\mathbb{C}_p) \arrow[r, "\Pi"] \arrow[d] \arrow[dd, bend right=60, "\mathrm{GH}"'] & (\mathbf{U}_E)^{\oplus h} \arrow[d, "\Theta"] \\
\mathrm{M}_h(\mathbb{C}_p)^{\mathrm{rank}=h-1} \arrow[r] \arrow[d, "\mathrm{span}"'] & \mathrm{M}_h(\mathbb{C}_p) \\
\mathbf{P}^{h-1}(\mathbb{C}_p) &
\end{tikzcd}$$

The horizontal maps are injections, and the square appearing the diagram is cartesian. The space $\mathbf{P}^{h-1}(\mathbb{C}_p)$ refers to the set of hyperplanes in $\mathbb{C} = M(\mathscr{G}_0)^\vee \otimes_W \mathbb{C}_p$. The arrow labeled "GH" is the Gross-Hopkins period map, and the arrow labeled "span" is the map associating to a matrix M of rank $(h-1)$ the hyperplane in $\mathbb{C}_p^h$ spanned by the columns of M. Finally, the map labeled Θ sends an h-tuple $x_1, \dots, x_h$ to the matrix $(\theta(\phi^i(x_j)))_{1 \le i,j \le h}$.

We remark that there is an analogue of the above theorem for $E/\mathbb{Q}_p$ any finite extension of degree h.

Exercise. In terms of the coordinates $x_1, \dots, x_h$ on $\mathcal{M}$ appearing in Thm. 12, work out the actions of the groups G, J and $W_{\mathbb{Q}_p}$.

The appearance in $\mathcal{M}$ of a Deligne-Lusztig variety over a finite field. We conclude these lectures with an investigation into a "piece" of the Lubin-Tate tower $\mathcal{M}$ which was originally studied in T. Yoshida's paper *On non-abelian Lubin-Tate theory via vanishing cycles*. There, Yoshida finds a direct link between the space $\mathcal{M}_1$ (the deformation space with level p structure) and the *Deligne-Lusztig variety* associated to $\mathrm{GL}_h(\mathbb{F}_p)$. Yoshida uses this link to establish Thm. 7 for a class of supercuspidal representations (those which are *depth zero*.) Using the linear algebra description of $\mathcal{M}$ in Theorem 12, we will sketch how to recover Yoshida's result.

Recall the determinant map $\delta\colon \mathcal{M} \to \mathbb{Q}_p t$, which divides $\mathcal{M}$ into its geometrically connected components. We write $\mathcal{M}^0$ for the preimage of $\mathbb{Z}_p^\times t$ under δ. The stabilizer in $G \times J \times W_{\mathbb{Q}_p}$ in $\mathcal{M}$ is the subgroup $(G \times J \times W_{\mathbb{Q}_p})^0$ "of index $\mathbb{Z}$" consisting of triples (g, j, w) for which $v(\det g) = v(N(j)) + v(\mathrm{rec}^{-1}(w))$, where $N\colon J \to \mathbb{Q}_p^\times$ is the reduced norm and $\mathrm{rec}\colon \mathbb{Q}_p^\times \to W_{\mathbb{Q}_p}^{\mathrm{ab}}$ is the reciprocity map.

By Thm. 12, a point in $\mathcal{M}$ determines an h-tuple $x_1, \dots, x_h \in \mathbf{U}_E$ having the property that the matrix $\theta(\phi^i(x_j))_{1 \le i,j \le h}$ has rank exactly $h-1$. Recall also that $\mathbf{U}_E$ is the E-vector space ($E = \mathbb{Q}_{p^h}$) consisting of sequences $x = (x^{(0)}, x^{(1)}, \dots)$ of elements of $\mathfrak{m}_{\mathbb{C}_p}$ satisfying $[p]_{\mathrm{LT}}(x_i) = x_i^{p^h} + px_i = x_{i-1}$.

Let $\mathcal{M}^{\text{depth }0} \subset \mathcal{M}^0$ denote the subspace characterized by the conditions

$$v(x_i^{(0)}) = \frac{1}{p^h - 1},\ i = 1, \ldots, h.$$

Then the stabilizer of $\mathcal{M}^0$ in $\mathrm{GL}_h(\mathbb{Q}_p) \times J \times W_{\mathbb{Q}_p}$ is the set of triples $(g, j, w) \in \left(G \times J \times W_{\mathbb{Q}_p}\right)^0$ with $g \in \mathbb{Q}_p^\times \mathrm{GL}_h(\mathbb{Z}_p)$ and $j \in \mathbb{Q}_p^\times \operatorname{Aut} \mathscr{G}_0$.

Let $\varpi \in \mathbb{C}_p$ be an element with $\varpi^{p^h - 1} = p$, and make the substitution $x_i^{(0)} = \varpi y_i$, so that the y_i are integral coordinates on $\mathcal{M}^{\text{depth }0}$.

It is not difficult to verify the following congruences (exercise):

$$\begin{aligned} \theta(x_i) &\equiv \varpi(y_i + y_i^{p^h}) \pmod{\varpi^2} \\ \theta(\phi^j(x_i)) &= \varpi^{p^j} y_i^{p^j} \pmod{\varpi^{p^j+1}},\ j = 1, \ldots, h-1 \end{aligned}$$

Since the determinant $\det(\theta(\phi^j(x_i)))_{0 \le i-1, j \le h-1}$ equals 0, we have the congruence

$$\det \begin{pmatrix} y_1 + y_1^{p^h} & \cdots & y_h + y_h^{p^h} \\ y_1^p & \cdots & y_h^p \\ \vdots & \ddots & \vdots \\ y_1^{p^{h-1}} & \cdots & y_h^{p^{h-1}} \end{pmatrix} \equiv 0 \pmod{\mathfrak{m}_{\mathbb{C}_p}}$$

From the fact that our point belongs to $\mathcal{M}^0$ one can deduce that the determinant $D = \det(y_i^{p^j})_{0 \le i-1, j \le h}$ is a p-adic unit; the above congruence is telling us that $D \equiv D^p \pmod{\mathfrak{m}_{\mathbb{C}_p}}$. Therefore the reduction mod $\mathfrak{m}_{\mathbb{C}_p}$ of the tuple $(y_1, \ldots, y_h)$ lies on the affine hypersurface $\mathrm{DL}/\overline{\mathbb{F}}_p$ with equation

$$\det(y_i^{p^j})^{p-1} = 1.$$

We remark that this equation may be rewritten as

$$\prod_a (a_1 y_1 + \cdots + a_h y_h) = 1,$$

where the product is over all nonzero vectors $a = (a_1, \ldots, a_h)$ with components in $\mathbb{F}_p$. DL is an example of a variety appearing in the theory of Deligne and Lusztig on the construction of irreducible representations of reductive groups G over finite fields; in our situation the variety DL admits an action of $\mathrm{GL}_h(\mathbb{F}_p)$, and the cohomology $H_c^*(\mathrm{DL})$ is known to realize the so-called *cuspidal* representations of $\mathrm{GL}(\mathbb{F}_p)$.

To wit, DL admits an action of a second group, namely $\mathbb{F}_{p^h}^\times$ (an element α sends $(y_1, \ldots, y_h)$ to $(\alpha^{-1} y_1, \ldots, \alpha^{-1} y_h)$), and if χ is a character of $\mathbb{F}_{p^h}^\times$ not factoring through any nontrivial norm map, then the χ-eigenspace of $H^*(\mathrm{DL})$ is (up to a sign) an irreducible cuspidal representation π_χ of $\mathrm{GL}_h(\mathbb{F}_p)$. It so happens that once this π_χ is inflated to $\mathrm{GL}_h(\mathbb{Z}_p)$, extended arbitrarily to $\mathbb{Q}_p^\times \mathrm{GL}_h(\mathbb{Z}_p)$, and induced up to $\mathrm{GL}_h(\mathbb{Q}_p)$, the result is a *depth zero supercuspidal* representation of $\mathrm{GL}_h(\mathbb{Q}_p)$, and all such representations appear this way.

The association $(x_i) \mapsto (y_i \pmod{\mathfrak{m}_{\mathbb{C}_p}})$ constitutes a reduction map $\mathcal{M}^{\text{depth } 0} \to \mathrm{DL}$. Passing to cohomology, one finds that $H_c^*(\mathcal{M}^{\text{depth } 0})$ contains a space of cycles, inherited from DL. The contribution of these cycles and their $G \times J \times W_F$-translates inside of $H_c^*(\mathcal{M})$ completely explains the portion of Thm. 7 pertaining to the depth zero supercuspidals.