On the Kottwitz conjecture for local Shimura varieties

Tasho Kaletha and Jared Weinstein
with an appendix by David Hansen

February 24, 2018

Abstract

Kottwitz's conjecture describes the contribution of a supercuspidal represention to the cohomology of a local Shimura variety in terms of the local Langlands correspondence. Using a Lefschetz-Verdier fixed-point formula, we prove a weakened generalized version of Kottwitz's conjecture. The weakening comes from ignoring the action of the Weil group and only considering the actions of the groups G and J_b up to non-elliptic representations. The generalization is that we allow arbitrary connected reductive groups G and non-minuscule coweights μ.

Contents

T.K. was supported in part by NSF grant DMS-1161489 and a Sloan Fellowship. J.W. was supported by NSF grant DMS-1303312 and a Sloan Fellowship.

1 Introduction

Let F be a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers, and let $\breve{F}$ be the completion of the maximal unramified extension of F, relative to a fixed algebraic closure $\bar{F}$. Let $\sigma \in \mathrm{Aut}(\breve{F}/F)$ be the arithmetic Frobenius element. Let G be a connected reductive group defined over F, $[b] \in B(G)$ a σ-conjugacy class of elements of $G(\breve{F})$, and $\{\mu\}$ a conjugacy class of cocharacters $\mathbf{G}_{\mathrm{m}} \to G$ defined over $\bar{F}$. Assume that $\{\mu\}$ is minuscule and that $[b] \in B(G, \{\mu\})$. The triple $(G, [b], \{\mu\})$ is called a local Shimura datum [RV14, §5]. It is conjectured that there is an associated tower $\mathcal{M}_{G,b,\mu,K}$ of rigid analytic spaces over $\breve{E}$, indexed by open compact subgroups $K \subset G(F)$. Here E is the field of definition of the conjugacy class $\{\mu\}$, a finite extension of F. The isomorphism class of the tower only depends on the classes $[b]$ and $\{\mu\}$. The theory of Rapoport-Zink spaces [RZ96] provides instances in which such a tower exists.

The *Kottwitz conjecture* [Rap95, Conjecture 5.1], [RV14, Conjecture 7.3] relates the cohomology of the $\mathcal{M}_{G,b,\mu,K}$ to the local Langlands correspondence, in the case that $[b]$ is basic. Let us review the precise statement. Let $B(G)_{\mathrm{bas}} \subset B(G)$ be the set of basic σ-conjugacy classes. Assume that $[b] \in B(G)_{\mathrm{bas}}$ and choose a representative $b \in [b]$. Let J_b be the associated inner form of G. Note that since $B(G, \{\mu\})$ contains a unique basic element, $[b]$ is uniquely determined by $\{\mu\}$.

The tower $\mathcal{M}_{G,b,\mu,K}$ receives commuting actions of $J_b(F)$ and $G(F)$. The action of $J_b(F)$ preserves each $\mathcal{M}_{G,b,\mu,K}$, while the action of $g \in G(F)$ sends $\mathcal{M}_{G,b,\mu,K}$ to $\mathcal{M}_{G,b,\mu,gKg^{-1}}$. There is furthermore a *Weil descent datum* on this tower from $\breve{E}$ down to E. It need not be effective.

We have the cohomology

$$R\Gamma_c(G,b,\mu,K) = R\Gamma_c(\mathcal{M}_{G,b,\mu,K} \times_{\breve{E}} \widehat{\overline{E}}, \bar{\mathbf{Q}}_\ell),$$

an object in the derived category of $\bar{\mathbf{Q}}_\ell$-vector spaces equipped with an action of $J_b(F)$. It also comes equipped with a natural action of I_E, which extends to an action of W_E due to the Weil descent datum. The actions of $J_b(F)$ and W_E commute.

Given an irreducible smooth admissible representation ρ of $J_b(F)$ we define

$$H^i(G,b,\mu)[\rho] = \varinjlim_K \operatorname{Ext}^i_{J_b(F)}(R\Gamma_c(G,b,\mu,K),\rho)$$

This is a $\bar{\mathbf{Q}}_l$-vector space with a smooth action of $G(F)\times W_E$. Finally we define the virtual $G(F)\times W_E$-representation

$$H^*(G,b,\mu)[\rho] := \sum_{i\in\mathbf{Z}} (-1)^i H^i(G,b,\mu)[\rho](-d),$$

where $d = \dim \mathcal{M}_{G,b,\mu} = \langle 2\rho_G, \mu\rangle$ and ρ_G is half the sum of the positive roots of G. The isomorphism class of $H^*(G,b,\mu)[\rho]$ only depends on $(G,[b],\{\mu\})$ and ρ.

The Kottwitz conjecture describes $H^*(G,b,\mu)[\rho]$ in terms of the local Langlands correspondence. To state it, fix a quasi-split group G^* and a $G^*(\overline{F})$-conjugacy class Ψ of inner twists $G^*\to G$. The choice of Ψ gives an identification of the (complex) Langlands dual groups of G^*, G, and J_b; we shall denote them all by $\widehat{G}$. The group $\widehat{G}$ carries an action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$; we denote by ${}^LG$ the corresponding L-group. The basic form of the local Langlands conjecture [Kala, Conjecture A] predicts that the set of isomorphism classes of essentially square-integrable representations of $G(F)$ (resp., $J_b(F)$) is partitioned into L-packets $\Pi_\phi(G)$ (resp., $\Pi_\phi(J_b)$), each such packet indexed by a discrete Langlands parameter $\phi : W_F\times \mathrm{SL}_2(\mathbf{C}) \to {}^LG$. When ϕ is trivial on $\mathrm{SL}_2(\mathbf{C})$ it is expected that the packets $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ consist entirely of supercuspidal representations.

Let $S_\phi = \mathrm{Cent}(\phi,\widehat{G})$. For $\lambda\in X^*(Z(\widehat{G})^\Gamma)$, write $\mathrm{Rep}(S_\phi,\lambda)$ for the set of isomorphism classes of algebraic representations of the algebraic group S_ϕ whose restriction to $Z(\widehat{G})^\Gamma$ is λ-isotypic, and write $\mathrm{Irr}(S_\phi,\lambda)$ for the subset of irreducible such representations. The class of b corresponds to a character $\lambda_b : Z(\widehat{G})^\Gamma \to \mathbf{C}^\times$ via the isomorphism $B(G)_{\mathrm{bas}} \to X^*(Z(\widehat{G})^\Gamma)$ of [Kot85, Proposition 5.6]. For any $\pi\in\Pi_\phi(G)$ and $\rho\in\Pi_\phi(J_b)$ there is an element $\delta_{\pi,\rho}\in\mathrm{Rep}(S_\phi,\lambda_b)$, which can be thought of as measuring the relative position of π and ρ, and whose definition will be given in Section 2.

The conjugacy class $\{\mu\}$ of cocharacters gives dually a conjugacy class of weights of $\widehat{G}$ and we denote by $r_{\{\mu\}}$ the irreducible representation of $\widehat{G}$ of highest weight μ. There is a natural extension of $r_{\{\mu\}}$ to ${}^L G_E$, the L-group of the base change of G to E [Kot84a, Lemma 2.1.2].

Let $\mathrm{Groth}(G(F)\times W_E)$ be the Grothendieck group of the category of $(G(F)\times W_E)$-modules over $\overline{\mathbf{Q}}_\ell$ which are admissible as a $G(F)$-module and smooth as a W_E-module.

Conjecture 1.0.1 (Kottwitz). *Let $\phi : W_F \to {}^L G$ be a discrete Langlands parameter. Write $r_{\{\mu\}}\circ\phi_E$ for the representation of $S_\phi\times W_E$ given by*

$$r_{\{\mu\}}\circ\phi_E(s,w) = r_{\{\mu\}}(s\cdot\phi(w)).$$

Given $\rho\in\Pi_\phi(J_b)$, each $H^i(G,b,\mu)[\rho]$ is admissible, and we have the following equality[1] *in* $\mathrm{Groth}(G(F)\times W_E)$*:*

$$H^*(G,b,\mu)[\rho] = (-1)^d \sum_{\pi\in\Pi_\phi(G)} \pi\boxtimes \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_{\{\mu\}}\circ\phi_E)(-\frac{d}{2}). \qquad (1.0.1)$$

Remark 1.0.2. In [RV14], $H^*(G,b,\mu)[\rho]$ is defined as the alternating sum

$$\sum_{i,j\in\mathbf{Z}} (-1)^{i+j} H^{i,j}(G,b,\mu)[\rho],$$

where

$$H^{i,j}(G,b,\mu)[\rho] = \varinjlim_K \mathrm{Ext}^j_{J_b(F)}(R^i\Gamma_c(G,b,\mu,K),\rho).$$

There is a spectral sequence $H^{i,j}(G,b,\mu)[\rho] \implies H^{i+j}(G,b,\mu)[\rho]$, so that if one knew that each $H^{i,j}(G,b,\mu)[\rho]$ were an admissible representation of $J_b(F)$ which is nonzero for only finitely many (i,j), then the admissibility of $H^{i+j}(G,b,\mu)[\rho]$ would follow; in that case the two definitions of $H^*(G,b,\mu)[\rho]$ are consistent.

[RV14, Proposition 6.1] proves the admissibility of $H^{i,j}(G,b,\mu)[\rho]$ under an assumption (Properties 5.3(iii)) that $\mathcal{M}_{G,b,\mu,K}$ admits a covering by $J_b(F)$-translates of an open subset U obeying a certain condition which guarantees [Hub98a, Theorem 3.3] that the compactly supported ℓ-adic cohomology of U is finite-dimensional. We do not prove this assumption, nor do we prove that $H^{i,j}(G,b,\mu)[\rho]$ is admissible for general (G,b,μ).

[1] [RV14, Conjecture 7.3] omits the sign $(-1)^d$.

In this article we prove a version of this conjecture that is both weaker and more general. The weakening comes from ignoring the Weil-group action and thus working in $\mathrm{Groth}(G(F))$ instead of $\mathrm{Groth}(G(F) \times W_E)$. Moreover, we only detect the behavior of representations on the set of elliptic conjugacy classes in $G(F)$. This means that, while we identify the right hand side as contributing to the left hand side, we are not able to exclude potential contributions to the left hand side of non-elliptic representations (meaning those whose distribution characters are supported away from the locus of regular elliptic elements in $G(F)$).

The generalization is that we remove two conditions that are present in the formulations of Kottwitz's conjecture in [Rap95] and [RV14]. One of them is that G is a B-inner form of its quasi-split inner form G^*. This condition, reviewed in Subsection 2.2, has the effect of making the definition of $\delta_{\pi,\rho}$ straightforward. To remove it, we use the formulation of the refined local Langlands correspondence [Kala, Conjecture G] based on the cohomology sets $H^1(u \to W, Z \to G)$ of [Kalb]. The definition of $\delta_{\pi,\rho}$ in this setting is a bit more involved and is given in Subsection 2.3, see Definition 2.3.2.

The second condition that we remove is that the conjugacy class $\{\mu\}$ consists of minuscule cocharacters. When μ is not minuscule, then the $\mathcal{M}_{G,b,\mu,K}$ are not rigid spaces; rather they belong to Scholze's category of diamonds.

The proof of our theorem is based on the following assumptions, each of which is currently being addressed by work in progress due to various authors.

Assumptions 1.0.3.

1. We assume the refined local Langlands correspondence for supercuspidal L-parameters, in the formulation of [Kala, Conjecture G]. Some of it is reviewed in Section 2.

2. We assume that the geometric Satake equivalence holds for Scholze's mixed-characteristic affine Grassmannian, see Subsection 4.2.

3. The final assumption concerns the moduli stack Bun_G of G-bundles on the Fargues-Fontaine curve [Far]. We need to assume that certain ℓ-adic sheaves on Bun_G are *reflexive*, meaning that they are isomorphic to their double Verdier duals. This is part of forthcoming work of Scholze on the "automorphic to Galois" direction of the local Langlands correspondence, which in turn is modeled on V. Lafforgue's work [Laf02] which accomplishes the same direction for function fields. Assumption 3 is stated in Subsection 4.10.

Theorem 1.0.4. *We work under assumptions (1)-(3). Let* $\phi\colon W_F \to {}^LG$ *be a discrete Langlands parameter. Let* $\mathrm{Groth}(G(F))^{\mathrm{ell}}$ *be the quotient of* $\mathrm{Groth}(G(F))$ *by the subgroup generated by non-elliptic representations. Then each* $H^i(G,b,\mu)[\rho]$ *is an admissible representation of* $G(F)$*. Furthermore,* (1.0.1) *is true in* $\mathrm{Groth}(G(F))^{\mathrm{ell}}$*. That is, the following equation holds in* $\mathrm{Groth}(G(F))^{\mathrm{ell}}$*:*

$$H^*(G,b,\mu)[\rho] = (-1)^d \sum_{\pi\in\Pi_\phi(G)} \left[\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_{\{\mu\}})\right] \pi$$

1.1 Remarks on the proof, and relation with prior work

Theorem 1.0.4 is proved by an application of a Lefschetz-Verdier fixed-point formula. Let us illustrate the idea in the Lubin-Tate case, when $G = \mathrm{GL}_n$, $\mu = (1,0,\dots,0)$, and b is basic of slope $1/n$. In this case $J_b(F) = D^\times$, where D/F is the division algebra of invariant $1/n$, and the spaces $\mathcal{M}_{G,b,\mu,K}$ are known as the Lubin-Tate tower. Atop the tower sits the infinite-level Lubin-Tate space $\mathcal{M}_\infty$ as described in [SW13]. $\mathcal{M}_\infty$ is a pre-perfectoid space admitting an action of $\mathrm{GL}_n(F)\times D^\times$. The Hodge-Tate period map exhibits $\mathcal{M}_\infty$ as a pro-étale cover of Drinfeld's upper half-space Ω^{n-1} (the complement in $\mathbf{P}^{n-1}_F$ of all F-rational hyperplanes).

Now suppose $g \in \mathrm{GL}_n(F)$ is a regular elliptic element, and let C/F be a complete algebraically closed field. Then g has exactly n fixed points on Ω^{n-1}_C. For each such fixed point x, g acts on the fiber $\mathcal{M}_{\infty,x}$.

Key observation. The action of g on $\mathcal{M}_{\infty,x}$ agrees with the action of an element $g' \in D^\times$, where g and g' are related (meaning they become conjugate over $\overline{F}$).

Suppose that ρ is an admissible representation of $D^\times$. There is a corresponding ℓ-adic local system $\mathcal{L}_\rho$ on $\Omega^{n-1}_{C,\text{ét}}$.

A naïve form of the Lefschetz trace formula would predict:

$$\mathrm{tr}\left(g|H^*_c(\Omega^{n-1}_C,\mathcal{L}_\rho)\right) = \sum_{x\in(\Omega^{n-1}_C)^g} \mathrm{tr}(g|\mathcal{L}_{\rho,x}),$$

where $\mathbf{P}^{n-1}(C)^g$ is the set of g-fixed points. For each such point x, the key observation above gives $\mathrm{tr}(g|\mathcal{L}_{\rho,x}) = \mathrm{tr}\,\rho(g')$, where g and g' are related. By the Jacquet-Langlands correspondence, there exists a discrete series representation π of $\mathrm{GL}_n(F)$ satisfying $\mathrm{tr}\,\pi(g) = (-1)^{n-1}\,\mathrm{tr}\,\rho(g')$ (here $\mathrm{tr}\,\pi(g)$ is interpreted as a Harish-Chandra character). Therefore in $\mathrm{Groth}(\mathrm{GL}_n(F))^{\mathrm{ell}}$ we have an equality

$$H_c^*(\Omega_C^{n-1}, \mathcal{L}_\rho) = (-1)^{n-1} n\pi.$$

The virtual $\mathrm{GL}_n(F) \times W_F$-representation $H(G, [b], \{\mu\})[\rho]$ is dual to the Euler characteristic $H_c^*(\Omega_C^{n-1}, \mathcal{L}_{\rho^\vee})$, where $\rho^\vee$ is the smooth dual; thus the above is in accord with Theorem 1.0.4.

This argument goes back at least to the 1990s, as discussed in [Har15, Chap. 9], and as far as we know first appears in [Fal94]. The difficulty lies in proving the validity of the Lefschetz formula. Prior work of Strauch and Mieda proved Theorem 1.0.4 in the case of the Lubin-Tate tower [Str05], [Str08], [Mie12], [Mie14a] and also in the case of a basic Rapoport-Zink space for GSp(4) [Mie]. These results are unconditional. In each of these cases the local Langlands correspondence was already known in sufficient detail (no Assumption 1 was necessary), and also the cocharacter μ was minuscule, so the relevant period space (generally a mixed-characteristic affine Grassmannian) is simply a flag variety (and thus no Assumption 2 was necessary).

In applying a Lefschetz formula to a non-proper rigid space, care must be taken to treat the boundary. For instance, if D is the closed unit disc $\{|T| \leq 1\}$ in the adic space $\mathbf{A}^1$, then the automorphism $T \mapsto T+1$ has Euler characteristic 1 on D, despite having no fixed points. The culprit is that this automorphism fixes the single boundary point in $\overline{D}\backslash D$. Mieda [Mie14b] proves a Lefschetz formula for an operator on a rigid space, under an assumption that the operator has no topological fixed points on a compactification. Now, in all of the above cases, $\mathcal{M}_{G,b,\mu,K}$ admits a *cellular decomposition*. This means (approximately) that $\mathcal{M}_{G,b,\mu,K}$ contains a compact open subset, whose translates by Hecke operators cover all of $\mathcal{M}_{G,b,\mu,K}$. This is enough to establish the "topological fixed point" hypothesis necessary to apply Mieda's Lefschetz formula. Shen [She14] constructs a cellular decomposition for a basic Rapoport-Zink space attached to the group $U(1, n-1)$, which paves the way for an unconditional proof of Theorem 1.0.4 in this case as well.

For general (G, b, μ), the $\mathcal{M}_{G,b,\mu,K}$ do not admit a cellular decomposition, and so there is probably no hope of applying [Mie14b]. This is where Assumption 3 comes in: it is precisely the input necessary to prove the Lefschetz formula we need.

1.2 Overview of the article

In Section 2, we review the refined local Langlands conjectures of [Kala] and then give the construction of $\delta_{\pi,\rho}$ without assuming that G is a B-inner form of G^*.

In Section 3, we review Scholze's theory of diamonds [SW13], and prove a Lefschetz-Verdier fixed-point formula for their cohomology, along the lines of [Var07]. This formula applies to those sheaves which are *reflexive*, meaning that they are isomorphic to their double Verdier dual.

In Section 4, we review Scholze's mixed-characteristic Grassmannian Gr_G (also called the B^+_{dR}-Grassmannian). It is expected that the geometric Satake equivalence, linking representations of $\widehat{G}$ and equivariant perverse sheaves on Gr_G, holds as it does in the classical case [MV07]. This is our Assumption 2. We also define the diamonds $\mathcal{M}_{G,b,\mu,K}$ and the representations $H^i(G,b,\mu)[\pi]$. We introduce Assumption 3, which implies that each $H^i(G,b,\mu)[\pi]$ is an admissible representation of $J_b(F)$.

In Section 5, we apply our Lefschetz-Verdier fixed point formula to a bounded Grassmannian $\mathrm{Gr}_{G,\leq\mu}$ to prove Theorem 1.0.4.

1.3 Acknowledgements

We owe a large debt of gratitude to Peter Scholze, who explained to us much of his work in progress, and especially helped us with Section 3. We also thank Laurent Fargues and David Hansen for many helpful conversations.

2 Review of the local Langlands correspondence

2.1 Basic notions

Recall that we have fixed a quasi-split group G^* and a $G^*(\bar{F})$-conjugacy Ψ class of inner twists $\psi : G^* \to G$. Given an element $b \in G(\breve{F})$, there is an associated inner form J_b of a Levi subgroup of G^* as described in [Kot97, §3.3,§3.4]. Its group of F-points is given by

$$J_b(F) \cong \left\{ g \in G(\breve{F}) \middle| \mathrm{Ad}(b)\sigma(g) = g \right\}.$$

Up to isomorphism the group J_b depends only on the σ-conjugacy class $[b]$. It will be convenient to choose b to be decent [RZ96, Definition 1.8]. Then there exists a finite unramified extension F'/F such that $b \in G(F')$. This allows us to replace $\breve{F}$ by F' in the above formula. The slope morphism $\nu : \mathbf{D} \to G_{\breve{F}}$ of b, [Kot85, §4], is also defined over F'. The centralizer $G_{F',\nu}$

of ν in $G_{F'}$ is a Levi subgroup of $G_{F'}$. The $G(F')$-conjugacy class of ν is defined over F, and then so is the $G(F')$-conjugacy class of $G_{F',\nu}$. There is a Levi subgroup M^* of G^* defined over F and $\psi \in \Psi$ that restricts to an inner twist $\psi : M^* \to J_b$, see [Kot97, §4.3].

From now on assume that b is basic. This is equivalent to $M^* = G^*$, so that J_b is in fact an inner form of G^* and of G.

2.2 The case that G is a B-inner form of G^*

The assumption that G is a B-inner form of G^* means that some $\psi \in \Psi$ can be equipped with a decent basic $b^* \in G^*(F^{\mathrm{nr}})$ such that ψ is an isomorphism $G^*_{F^{\mathrm{nr}}} \to G_{F^{\mathrm{nr}}}$ satisfying $\psi^{-1}\sigma(\psi) = \mathrm{Ad}(b^*)$. In other words, ψ becomes an isomorphism over F from the group J_{b^*}, now construced relative to G^* and b^*, and G. Under this assumption, and after choosing a Whittaker datum $\mathfrak{w}$ for G^*, the isocrystal formulation of the refined local Langlands correspondence [Kala, Conjecture F] predicts the existence of bijections

$$\begin{aligned} \Pi_\phi(G) &\cong \mathrm{Irr}(S_\phi, \lambda_{b*}) \\ \Pi_\phi(J_b) &\cong \mathrm{Irr}(S_\phi, \lambda_{b^*} + \lambda_b) \end{aligned}$$

where we have used the isomorphisms $B(G)_{\mathrm{bas}} \cong X^*(Z(\widehat{G})^\Gamma) \cong B(G^*)_{\mathrm{bas}}$ of [Kot85, Proposition 5.6] to obtain from $[b] \in B(G)_{\mathrm{bas}}$ and $[b^*] \in B(G^*)_{\mathrm{bas}}$ characters λ_b and λ_{b^*} of $Z(\widehat{G})^\Gamma$.

These bijections are uniquely characterized by the endoscopic character identities which are part of [Kala, Conjecture F]. Write $\pi \mapsto \tau_{b^*,\mathfrak{w},\pi}$, $\rho \mapsto \tau_{b^*,\mathfrak{w},\rho}$ for these bijections, and $\tau \mapsto \pi_{b^*,\mathfrak{w},\tau}$, $\tau \mapsto \rho_{b^*,\mathfrak{w},\tau}$ for their inverses and define

$$\delta_{\pi,\rho} := \check{\tau}_{b^*,\mathfrak{w},\pi} \otimes \tau_{b^*,\mathfrak{w},\rho}.$$

While all of these bijections depend on the choice of Whittaker datum $\mathfrak{w}$ and the choice of b^*, we will argue in Subsection 2.3 that for any pair π and ρ the representation $\delta_{\pi,\rho}$ is independent of these choices. Of course it does depends on b, but this we take as part of the given data.

2.3 The general case

We now drop the assumption that G is a B-inner form of G^*. Because of this, we no longer have the isocrystal formulation of the refined local Langlands correspondence. However, we do have the formulation based on rigid inner twists [Kala, Conjecture G]. What this means with regards to the Kottwitz conjecture is that neither π nor ρ correspond to representations of S_ϕ. Rather, they correspond to representations τ_π and τ_ρ of a different group $\pi_0(S_\phi^+)$. Nonetheless it will turn out that $\check{\tau}_\pi \otimes \tau_\rho$ provides in a natural way a representation $\delta_{\pi,\rho}$ of S_ϕ.

In order to make this precise we will need the material of [Kalb] and [Kalc], some of which is summarized in [Kala]. First, we will need the cohomology set $H^1(u \to W, Z \to G^*)$ defined in [Kalb, §3] for any finite central subgroup $Z \subset G^*$ defined over F. As in [Kalc, §3.2] it will be convenient to package these sets for varying Z into the single set

$$H^1(u \to W, Z(G^*) \to G^*) := \varinjlim H^1(u \to W, Z \to G^*).$$

The transition maps on the right are injective, so the colimit can be seen as an increasing union.

Next, we will need the reinterpretation, given in [Kot], of $B(G)$ as the set of cohomology classes of algebraic 1-cocycles of a certain Galois gerbe $1 \to \mathbf{D}(\bar{F}) \to \mathcal{E} \to \Gamma \to 1$. This reinterpretation is also reviewed in [Kalc, §3.1]. Let us make it explicit for basic decent elements $b' \in G(F^{\mathrm{nr}})$. There is a uniquely determined 1-cocycle $\mathbf{Z} \cong \langle \sigma \rangle \to G(F^{\mathrm{nr}})$ whose value at σ is equal to b'. By inflation we obtain a 1-cocycle of W_F in $G(\bar{F})$. Since b' is basic and decent, for some finite Galois extension K/F splitting G the restriction of this 1-cocycle to W_K factors through $K^\times$ and is a homomorphism $K^\times \to Z(G)(K)$. Moreover, this homomorphism is algebraic and is in fact given by a multiple of the slope morphism $\nu : \mathbf{D} \to G$. In this way we obtain a 1-cocycle valued in $G(\bar{F})$ of the extension $1 \to K^\times \to W_{K/F} \to \Gamma_{K/F} \to 1$, which we can pull-back along $\Gamma \to \Gamma_{K/F}$ and then combine with ν to obtain a 1-cocycle of $\mathcal{E}$ valued in $G(\bar{F})$, that is algebraic in the sense that its restriction to $\mathbf{D}(\bar{F})$ is given by a morphism of algebraic groups, namely ν. The reader is referred to [Kot97, §8 and App B] for further details.

Finally, we will need the comparison map

$$B(G)_{\mathrm{bas}} \to H^1(u \to W, Z(G) \to G)$$

of [Kalc, §3.3]. In fact, this comparison map is already defined on the level of cocycles, via pull-back along the diagram [Kalc, (3.13)], and takes the form

$$G(F^{\mathrm{nr}})_{\mathrm{d,bas}} \to Z^1(u \to W, Z(G) \to G) \tag{2.3.1}$$

where on the left we have the decent basic elements in $G(F^{\mathrm{nr}})$.

After this short review we turn to the construction of $\delta_{\pi,\rho} \in \mathrm{Rep}(S_\phi, \lambda_b)$. Choose any inner twist $\psi \in \Psi$ and let $\bar{z}_\sigma := \psi^{-1}\sigma(\psi) \in G^*_{\mathrm{ad}}(\bar{F})$. Then $\bar{z} \in Z^1(F, G^*_{\mathrm{ad}})$ and the surjectivity of the natural map $H^1(u \to W, Z(G^*) \to$

$G^*) \to H^1(F, G^*_{\mathrm{ad}})$ asserted in [Kalb, Corollary 3.8] allows us to choose $z \in Z^1(u \to W, Z(G^*) \to G^*)$ lifting $\bar{z}$. Then $(\psi, z) : G^* \to G$ is a rigid inner twist. Let $z_b \in Z^1(u \to W, Z(G) \to G)$ be the image of b under (2.3.1). For psychological reasons, let $\xi : G_{F'} \to J_{b,F'}$ denote the identity map. Then $(\xi \circ \psi, \psi^{-1}(z) \cdot z_b) : G^* \to J_b$ is also a rigid inner twist.

The L-packets $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ are now parameterized by representations of a certain cover S_ϕ^+ of S_ϕ. While [Kala, Conjecture G] is formulated in terms of a finite cover depending on an auxiliary choice of a finite central subgroup $Z \subset G^*$, we will adopt here the point of view of [Kalc] and work with a canonical infinite cover. Following [Kalc, §3.3] we let $Z_n \subset Z(G)$ be the subgroup of those elements whose image in $Z(G)/Z(G_{\mathrm{der}})$ is n-torsion, and let $G_n = G/Z_n$. Then G_n has adjoint derived subgroup and connected center. More precisely, $G_n = G_{\mathrm{ad}} \times C_n$, where $C_n = C_1/C_1[n]$ and $C_1 = Z(G)/Z(G_{\mathrm{der}})$. It is convenient to identify $C_n = C_1$ as algebraic tori and take the m/n-power map $C_1 \to C_1$ as the transition map $C_n \to C_m$ for $n|m$. The isogeny $G \to G_n$ dualizes to $\widehat{G}_n \to \widehat{G}$ and we have $\widehat{G}_n = \widehat{G}_{\mathrm{sc}} \times \widehat{C}_1$. Note that $\widehat{C}_1 = Z(\widehat{G})^\circ$. The transition map $\widehat{G}_m \to \widehat{G}_n$ is then the identity on $\widehat{G}_{\mathrm{sc}}$ and the m/n-power map on $\widehat{C}_1$. Set $\widehat{\bar{G}} = \varprojlim \widehat{G}_n = \widehat{G}_{\mathrm{sc}} \times \widehat{C}_\infty$, where $\widehat{C}_\infty = \varprojlim \widehat{C}_n$. Elements of $\widehat{\bar{G}}$ can be written as $(a, (b_n)_n)$, where $a \in \widehat{G}_{\mathrm{sc}}$ and $(b_n)_n$ is a sequence of elements $b_n \in \widehat{C}_1$ satisfying $b_n = b_m^{\frac{m}{n}}$ for $n|m$. In this presentation, the natural map $\widehat{\bar{G}} \to \widehat{G}$ sends $(a, (b_n))$ to $a_{\mathrm{der}} \cdot b_1$, where $a_{\mathrm{der}} \in \widehat{G}_{\mathrm{der}}$ is the image of $a \in \widehat{G}_{\mathrm{sc}}$ under the natural map $\widehat{G}_{\mathrm{sc}} \to \widehat{G}_{\mathrm{der}}$.

Definition 2.3.1. Let $Z(\widehat{\bar{G}})^+ \subset S_\phi^+ \subset \widehat{\bar{G}}$ be the preimages of $Z(\widehat{G})^\Gamma \subset S_\phi \subset \widehat{G}$ under $\widehat{\bar{G}} \to \widehat{G}$.

Given a character $\lambda : \pi_0(Z(\widehat{\bar{G}})^+) \to \mathbf{C}^\times$ (which we will always assume trivial on the kernel of $Z(\widehat{\bar{G}})^+ \to \widehat{G}_n$ for some n) let $\mathrm{Rep}(\pi_0(S_\phi^+), \lambda)$ denote the set of isomorphism classes of representations of $\pi_0(S_\phi^+)$ whose pull-back to $\pi_0(Z(\widehat{\bar{G}})^+)$ is λ-isotypic, and let $\mathrm{Irr}(\pi_0(S_\phi^+), \lambda)$ be the (finite) subset of irreducible representations. Let λ_z be the character corresponding to the class of z under the Tate-Nakayama isomorphism

$$H^1(u \to W, Z(G^*) \to G^*) \to \pi_0(Z(\widehat{\bar{G}})^+)^*$$

of [Kalb, Corollary 5.4], and let λ_{z_b} be the character corresponding to the class of z_b in $H^1(u \to W, Z(G) \to G)$. Then according to [Kala, Conjecture

G], upon fixing a Whittaker datum $\mathfrak{w}$ for G^* there are bijections

$$\begin{aligned}\Pi_\phi(G) &\cong \mathrm{Irr}(\pi_0(S_\phi^+),\lambda_z)\\ \Pi_\phi(J_b) &\cong \mathrm{Irr}(\pi_0(S_\phi^+),\lambda_z+\lambda_{z_b})\end{aligned}$$

again uniquely determined by the endoscopic character identities. We write $\pi\mapsto\tau_{z,\mathfrak{w},\pi}$, $\rho\mapsto\tau_{z,\mathfrak{w},\rho}$ for these bijections, and $\tau\mapsto\pi_{z,\mathfrak{w},\tau}$, $\tau\mapsto\rho_{z,\mathfrak{w},\tau}$ for their inverses. We form the representation $\check{\tau}_{z,\mathfrak{w},\pi}\otimes\tau_{z,\mathfrak{w},\rho}\in\mathrm{Rep}(\pi_0(S_\phi^+),\lambda_{z_b})$.

Recall the map [Kalc, (4.7)]

$$S_\phi^+\to S_\phi,\qquad (a,(b_n))\mapsto\frac{a_{\mathrm{der}}\cdot b_1}{N_{E/F}(b_{[E:F]})}. \tag{2.3.2}$$

Here $a_{\mathrm{der}}\in\widehat{G}_{\mathrm{der}}$ is the image of $a\in\widehat{G}_{\mathrm{sc}}$ under the natural map $\widehat{G}_{\mathrm{sc}}\to\widehat{G}_{\mathrm{der}}$ and E/F is a sufficiently large finite Galois extension. This map is independent of the choice of E/F. According to [Kalc, Lemma 4.1] pulling back along this map sets up a bijection $\mathrm{Irr}(\pi_0(S_\phi^+),\lambda_{z_b})\to\mathrm{Irr}(S_\phi,\lambda_b)$. Note that since ϕ is discrete the group $S_\phi^\natural$ defined in loc. cit. is equal to S_ϕ. The lemma remains valid, with the same proof, if we remove the requirement of the representations being irreducible, and we obtain the bijection $\mathrm{Rep}(\pi_0(S_\phi^+),\lambda_{z_b})\to\mathrm{Rep}(S_\phi,\lambda_b)$.

Definition 2.3.2. Let $\delta_{\pi,\rho}$ be the image of $\check{\tau}_{z,\mathfrak{w},\pi}\otimes\tau_{z,\mathfrak{w},\rho}$ under the bijection $\mathrm{Rep}(\pi_0(S_\phi^+),\lambda_{z_b})\to\mathrm{Rep}(S_\phi,\lambda_b)$.

In the situation when G is a B-inner form of G^*, this definition of $\delta_{\pi,\rho}$ agrees with the one of Subsection 2.2, because then we can take z to be the image of b^* under (2.3.1) and then $\tau_{z,\mathfrak{w},\pi}$ and $\tau_{b^*,\mathfrak{w},\pi}$ are related via (2.3.2), and so are $\tau_{z,\mathfrak{w},\rho}$ and $\tau_{b^*,\mathfrak{w},\rho}$, see [Kalc, §4.2].

Lemma 2.3.3. *The representation $\delta_{\pi,\rho}$ is independent of the choices of Whittaker datum $\mathfrak{w}$ and of a rigidifying 1-cocycle $z\in Z^1(u\to W,Z(G^*)\to G^*)$.*

Proof. Both of these statements follow from [Kala, Conjecture G]. For the independence of Whittaker datum, one can prove that the validity of this conjecture implies that if $\mathfrak{w}$ is replaced by another choice $\mathfrak{w}'$ then there is an explicitly constructed character $(\mathfrak{w},\mathfrak{w}')$ of $\pi_0(S_\phi/Z(\widehat{G})^\Gamma)$ whose inflation to $\pi_0(S_\phi^+)$ satisfies $\tau_{z,\mathfrak{w},\sigma}=\tau_{z,\mathfrak{w}',\sigma}\otimes(\mathfrak{w},\mathfrak{w}')$ for any $\sigma\in\Pi_\phi(G)\cup\Pi_\phi(J_b)$. See §4 and in particular Theorem 4.3 of [Kal13], the proof of which is valid for a general G that satisfies [Kala, Conjecture G], bearing in mind that the transfer factor we use here is related to the one used there by $s\mapsto s^{-1}$. The independence of z follows from the same type of argument, but now using [Kalc, Lemma 6.2]. $\square$

2.4 Endoscopic character relations

We recall here the endoscopic character identities, which are part of the refined local Langlands correspondence, following the formulation of [Kalb, §5.4], also recalled in [Kala, §4.2]. They will be an important ingredient in the proof of our main result.

We summarize the notation established so far.

- $F/\mathbf{Q}_p$ is a finite extension.
- G is a connected reductive group defined over F.
- G^* is a quasi-split connected reductive group defined over F.
- Ψ is a G^*-conjugacy class of inner twists $\psi\colon G^* \to G$.
- $\bar{z}_\sigma = \psi^{-1}\sigma(\psi) \in G^*_{\mathrm{ad}}$, so that $\bar{z} \in Z^1(F, G^*_{\mathrm{ad}})$.
- $z \in Z^1(u \to W, Z(G^*) \to G^*)$ is a lift of $\bar{z}$.
- $b \in G(F^{\mathrm{nr}})$ is a decent basic element.
- J_b is the corresponding inner form of G.
- $\xi\colon G_{F^{\mathrm{nr}}} \to J_{b,F^{\mathrm{nr}}}$ is the identity map.
- $z_b \in Z^1(u \to W, Z(G) \to G)$ is the image of b under (2.3.1).
- $\mathfrak{w}$ is a Whittaker datum for G^*.
- $\phi\colon W_F \to {}^LG$ is a discrete parameter.
- $S_\phi = \mathrm{Cent}(\phi, \widehat{G})$.
- S_ϕ^+ is the group defined in Definition 2.3.1.

Associated to ψ are the L-packets $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ and the bijections

$$\Pi_\phi(G) \to \mathrm{Irr}(\pi_0(S_\phi^+), \lambda_z), \qquad \Pi_\phi(J_b) \to \mathrm{Irr}(\pi_0(S_\phi^+), \lambda_z + \lambda_{z_b})$$

denoted by $\pi \mapsto \tau_{z,\mathfrak{w},\pi}$ and $\rho \mapsto \tau_{z,\mathfrak{w},\rho}$.

We now choose a semi-simple element $s \in S_\phi$ and an element $\dot{s} \in S_\phi^+$ which lifts s. Let $e(G)$ and $e(J_b)$ be the Kottwitz signs of the groups G and J_b, as defined in [Kot83]. Consider the virtual characters

$$e(G) \sum_{\pi \in \Pi_\phi(G)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\pi}(\dot{s}) \cdot \Theta_\pi \qquad \text{and} \qquad e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\rho}(\dot{s}) \cdot \Theta_\rho.$$

The endoscopic character identities are equations which relate these two virtual characters to virtual characters on an endoscopic group H_1 of G and J_b. From the pair $(\phi, \dot{s})$ one obtains a refined elliptic endoscopic datum

$$\dot{\mathfrak{e}} = (H, \mathcal{H}, \dot{s}, \eta) \tag{2.4.1}$$

in the sense of [Kalb, §5.3] as follows. Let $\widehat{H} = \mathrm{Cent}(s, \widehat{G})^\circ$. The image of ϕ is contained in $\mathrm{Cent}(s, \widehat{G})$, which in turns acts by conjugation on its connected component $\widehat{H}$. This gives a homomorphism $W_F \to \mathrm{Aut}(\widehat{H})$. Letting $\Psi_0(\widehat{H})$ be the based root datum of $\widehat{H}$ [Kot84b, §1.1] and $\Psi_0^\vee(\widehat{H})$ its dual, we obtain the homomorphism

$$W_F \to \mathrm{Aut}(\widehat{H}) \to \mathrm{Out}(\widehat{H}) = \mathrm{Aut}(\Psi_0(\widehat{H})) = \mathrm{Aut}(\Psi_0(\widehat{H})^\vee).$$

Since the target is finite, this homomorphism extends to Γ_F and we obtain a based root datum with Galois action, hence a quasi-split connected reductive group H defined over F. Its dual group is by construction equal to $\widehat{H}$. We let $\mathcal{H} = \widehat{H} \cdot \phi(W_F)$, noting that the right factor normalizes the left so their product $\mathcal{H}$ is a subgroup of ${}^LG$. Finally, we let $\eta : \mathcal{H} \to {}^LG$ be the natural inclusion. Note that by construction ϕ takes image in $\mathcal{H}$, i.e. it factors through η.

We can realize the L-group of H as ${}^LH = \widehat{H} \rtimes W_F$, but we caution the reader that W_F does not act on $\widehat{H}$ via the map $W_F \to \mathrm{Aut}(\widehat{H})$ given by ϕ as above. Rather, we have to modify this action to ensure that it preserves a pinning of $\widehat{H}$. More precisely, after fixing an arbitrary pinning of $\widehat{H}$ we obtain a splitting $\mathrm{Out}(\widehat{H}) \to \mathrm{Aut}(\widehat{H})$ and the action of W_F on $\widehat{H}$ we use to form ${}^LH$ is given by composing the above map $W_F \to \mathrm{Out}(\widehat{H})$ with this splitting.

Both ${}^LH$ and $\mathcal{H}$ are thus extensions of W_F by $\widehat{H}$, but they need not be isomorphic. If they are, we fix arbitrarily an isomorphism $\eta_1 : \mathcal{H} \to {}^LH$ of extensions. Then $\phi^s = \eta_1 \circ \phi$ is a supercuspidal parameter for H.

In the general case we need to introduce a z-pair $\mathfrak{z} = (H_1, \eta_1)$ as in [KS99, §2]. It consists of a z-extension $H_1 \to H$ (recall this means that H_1 has a simply connected derived subgroup and the kernel of $H_1 \to H$ is an induced torus) and $\eta_1 : \mathcal{H} \to {}^LH_1$ is an L-embedding that extends the natural embedding $\widehat{H} \to \widehat{H}_1$. As is shown in [KS99, §2.2], such a z-pair always exists. Again we set $\phi^s = \eta_1 \circ \phi$ and obtain a supercuspidal parameter for H_1. In the situation where an isomorphism $\eta_1 : \mathcal{H} \to {}^LH$ does exist, we will allows ourselves to take $H = H_1$ and so regard $\mathfrak{z} = (H, \eta_1)$ as a z-pair, even though in general H will not have a simply connected derived subgroup.

The virtual character on H_1 that the above virtual characters on G and J_b are to be related to is

$$S\Theta_{\phi^s} := \sum_{\pi^s \in \Pi_{\phi^s}(H_1)} \dim(\tau_{\pi^s})\Theta_{\pi^s}.$$

Here $\pi^s \mapsto \tau_{\pi^s}$ is a bijection $\Pi_{\phi^s}(H_1) \to \mathrm{Irr}(\pi_0(\mathrm{Cent}(\phi^s, \widehat{H}_1)/Z(\widehat{H}_1)^\Gamma))$ determined by an arbitrary choice of Whittaker datum for H_1. The argument in the proof of Lemma 2.3.3 shows the independence of $\dim(\tau_{\pi^s})$ of the choice of a Whittaker datum for H_1.

The relationship between the virtual characters on G, J_b, and H_1, is expressed in terms of the Langlands-Shelstad transfer factor $\Delta'_{\mathrm{abs}}[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\psi, z)]$ for the pair of groups (H_1, G) and the corresponding Langlands-Shelstad transfer factor $\Delta'_{\mathrm{abs}}[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi \circ \psi, \psi^{-1}(z_b) \cdot z)]$ for the pair of groups (H_1, J_b), both of which are defined by [Kalb, (5.10)]. We will abbreviate both of them to just Δ. It is a simple consequence of the Weyl integration formula that the character relation [Kalb, (5.11)] can be restated in terms of character functions (rather than character distributions) as

$$e(G) \sum_{\pi \in \Pi_\phi(G)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\pi}(\dot{s})\Theta_\pi(g) = \sum_{h_1 \in H_1(F)/\mathrm{st.}} \Delta(h_1, g) S\Theta_{\phi^s}(h_1) \qquad (2.4.2)$$

for any strongly regular semi-simple element $g \in G(F)$. The sum on the right runs over stable conjugacy classes of strongly regular semi-simple elements of $H_1(F)$. We also have the analogous identity for J_b:

$$e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\rho}(\dot{s})\Theta_\rho(j) = \sum_{h_1 \in H_1(F)/\mathrm{st.}} \Delta(h_1, j) S\Theta_{\phi^s}(h_1). \qquad (2.4.3)$$

We are only interested in the right hand sides of these two equations as a bridge between their left-hand sides. Essential for this bridge is a certain compatibility between the transfer factors appearing on both right-hand sides.

Definition 2.4.1. Two strongly regular semi-simple elements $g \in G(F)$ and $j \in J_b(F)$ are called *stably conjugate*, or *related*, if there exists $y \in G(G^{\mathrm{nr}})$ such that $j = \xi(ygy^{-1})$. In that case, letting $T = \mathrm{Cent}(g, G)$ the element $y^{-1}by^\sigma$ belongs to $T(F^{\mathrm{nr}})$ and its image in $B(T)$ is called $\mathrm{inv}[b](g, j)$.

Note that, according to Steinberg's theorem, the existence of $y \in G(F^{\mathrm{nr}})$ with $j = \xi(ygy^{-1})$ is equivalent to the existence of $y \in G(\overline{F})$ with the same property. We work here with $G(F^{\mathrm{nr}})$ to facilitate the definition of the invariant. It is straightforward to check that the image in $B(T)$ of $y^{-1}by^{\sigma}$ is independent of the choice of y.

The compatibility satisfied by the transfer factors is then the following.

Lemma 2.4.2.

$$\Delta(h_1, j) = \Delta(h_1, g) \cdot \langle \mathrm{inv}[b](g, j), s_{h,g} \rangle. \tag{2.4.4}$$

We need to explain the second factor. Given maximal tori $T_H \subset H$ and $T \subset G$, there is a notion of an admissible isomorphism $T_H \to T$, for which we refer the reader to [Kala, §1.3]. Two strongly regular semi-simple elements $h \in H(\mathbf{Q}_p)$ and $g \in G(\mathbf{Q}_p)$ are called *related* if there exists an admissible isomorphism $T_h \to T_g$ between their centralizers mapping h to g. If such an isomorphism exists, it is unique, and in particular defined over F, and shall be called $\varphi_{h,g}$. An element $h_1 \in H_1(F)$ is called related to $g \in G(F)$ if and only if its image $h \in H(F)$ is so. Since g and j are stably conjugate, an element $h_1 \in H_1(F)$ is related to g if and only if it is related to j. If that is not the case, both $\Delta(h_1, j)$ and $\Delta(h_1, g)$ are zero and (2.4.4) is trivially true. Thus assume that h_1 is related to both g and j. Let $s^\natural \in S_\phi$ be the image of $\dot{s}$ under (2.3.2). Note that $s^\natural \in s \cdot Z(\widehat{G})^{\circ,\Gamma}$ and hence the preimage of $s^\natural$ under η belongs to $Z(\widehat{H})^\Gamma$, which in turns embeds naturally into $\widehat{T}_h^\Gamma$. Using the admissible isomorphism $\varphi_{h,g}$ we transport $s^\natural$ into $\widehat{T}_g^\Gamma$ and denote it by $s_{h,g}$. It is then paired with $\mathrm{inv}[b](g, j)$ via the isomorphism $B(T_g) \cong X^*(\widehat{T}_g^\Gamma)$ of [Kot85, §2.4].

Proof. For every finite subgroup $Z \subset Z(G) \subset T_g$ one obtains from $\varphi_{h,g}$ an isomorphism $T_h/\varphi_{h,g}^{-1}(Z) \to T_g/Z$. Using the subgroups Z_n from the previous subsection we form the quotients $T_{h,n} = T_h/\varphi_{h,g}^{-1}(Z_n)$ and $T_{g,n} = T_g/Z_n$. From $\varphi_{h,g}$ we obtain an isomorphism

$$\widehat{\bar{T}_h} \to \widehat{\bar{T}_g}$$

between the limits over n of the tori dual to $T_{h,n}$ and $T_{g,n}$. Let $\dot{s}_{h,g} \in [\widehat{\bar{T}_g}]^+$ be the image of $\dot{s}$ under this isomorphism. Let $\mathrm{inv}[z_b](g, j) \in H^1(u \to W, Z(G) \to T_g)$ be the invariant defined in [Kalb, §5.1]. If we replace $\langle \mathrm{inv}[b](g, j), s_{h,g} \rangle$ by $\langle \mathrm{inv}[z_b](g, j), \dot{s}_{h,g} \rangle$ then the lemma follows immediately from the defining formula [Kalb, (5.10)] of the transfer factors. The lemma follows from the equality $\langle \mathrm{inv}[b](g, j), s_{h,g} \rangle = \langle \mathrm{inv}[z_b](g, j), \dot{s}_{h,g} \rangle$ proved in [Kalc, §4.2]. $\square$

3 Geometric Preparations

3.1 Diamonds

We give a brief review of Scholze's theory of diamonds [Sch17]. We will work extensively with perfectoid rings and spaces which have no specified field of scalars, as in [Fon13] or [SW14, Definition 7.1.2].

Definition 3.1.1. A morphism $f\colon X \to Y$ of perfectoid spaces is *pro-étale* if it is locally (on the source) of the form $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$, where A and A_∞ are perfectoid algebras, and

$$(A_\infty, A_\infty^+) = \left[\varinjlim (A_i, A_i^+)\right]^\wedge$$

is a filtered colimit of affinoid perfectoid algebras (A_i, A_i^+), such that

$$\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$$

is étale for each i.

Example 3.1.2. Fix an algebraically closed perfectoid field C. For a profinite topological space S, let $\underline{S} = \mathrm{Spa}(A, A^\circ)$, where A is the ring of continuous functions $S \to C$. Then $\underline{S}$ is a perfectoid space whose underlying topological space is S, and for which the structure morphism $\underline{S} \to \mathrm{Spa}\, C$ is pro-étale. By gluing, one can define $\underline{S}$ for any locally profinite topological space. Note that for any perfectoid space X over C, a C-morphism $X \to \underline{S}$ is the same thing as a continuous map $|X| \to S$. In the case that $S = G$ is a locally profinite *group*, it makes sense to talk about a $\underline{G}$-action on a perfectoid space X: this is a morphism $\underline{G} \times X \to X$ satisfying the appropriate axioms.

Definition 3.1.3. Let (Perf) be the category of perfectoid spaces. A collection of morphisms $\{f_i\colon X_i \to X\}_{i\in I}$ is a *pro-étale covering* if all the f_i are pro-étale, and if for all quasi-compact open $U \subset X$, there exists a finite subset $I_U \subset I$ and a quasi-compact open $U_i \subset X_i$ for each $i \in I_U$, such that $U = \cup_{i\in I_U} f_i(U_i)$. We endow (Perf) with the structure of the site generated by pro-étale covers.

For an object X of (Perf), define a pre-sheaf h_X on (Perf) by $h_X(Y) = \mathrm{Hom}(Y, X)$. For a perfectoid Huber pair (R, R^+), we use the abbreviation $\mathrm{Spd}(R, R^+)$ (or just $\mathrm{Spd}\, R$ if $R^+ = R^\circ$) for $h_{\mathrm{Spa}(R^\flat, R^{\flat +})}$.

Proposition 3.1.4 ([SW14, Proposition 8.2.7])**.** *The pre-sheaf h_X is a sheaf.*

We remark that in the category of sheaves on (Perf), morphisms $h_X \to \mathcal{G}$ correspond to elements of $\mathcal{G}(X)$. That is, the functor $X \mapsto h_X$ from (Perf) into the category of sheaves on (Perf) is fully faithful. Therefore we are justified in referring to the representable sheaf h_X simply as X.

Definition 3.1.5.

1. If $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on (Perf), say that f is *étale* if for all objects X of (Perf) and all morphisms $X \to \mathcal{G}$, the pull-back $X \times_{\mathcal{G}} \mathcal{F}$ is representable by an object Y of (Perf), and $Y \to X$ is étale.

2. A *diamond* is a sheaf X on (Perf) of the form X'/R, where X' and R are perfectoid spaces and $R \to X' \times X'$ is an equivalence relation for which both projections $R \to X'$ are pro-étale. The *underlying topological space* of X is defined as the quotient of $|X'|$ by the equivalence relation $|R| \to |X'| \times |X'|$.

3. A morphism $X' \to X$ of diamonds is an *open immersion* if for all objects U of (Perf) mapping to X, the pullback $X' \times_X U \to U$ is representable by an open immersion of perfectoid spaces. In this case we say X' is an *open sub-diamond* of X.

4. A diamond X is *quasi-separated* if for all quasi-compact $U, V \to X$, the fiber product $U \times_X V$ is quasi-compact. X is *spatial* if it is quasi-compact and quasi-separated, and if $|X|$ admits a basis of open subsets of the form $|U|$, where $U \to X$ is a quasi-compact open immersion. X is *locally spatial* if it admits an open cover by spatial sub-diamonds.

5. A morphism $X \to Y$ of diamonds is *proper* if it is quasi-compact, quasi-separated, and universally closed.

6. Let X be a locally spatial diamond. Define $X_{\text{ét}}$ to be the category of étale morphisms $X' \to X$, endowed with the topology where covers are defined as jointly surjective maps.

7. The *v-topology* on (Perf) is the topology where a cover consists of a collection of maps $X_i \to X$ such that for any quasi-compact open subset $U \subset X$, there are finitely many i and quasi-compact open subsets $U_i \subset X_i$ such that the U_i jointly cover U.

8. A *small v-sheaf* is a sheaf Y on the v-topology on (Perf), for which there exists a surjective map of v-sheaves $X \to Y$, for some perfectoid

space[2] X.

9. A *small v-stack* is a sheaf of groupoids Y on the v-topology on (Perf), for which there exists a surjective map of stacks $X \to Y$, for some perfectoid space X, for which $R = X \times_Y X$ is a small v-sheaf.

3.2 The six functor formalism

We will require the "six functor formalism" in the setting of small v-stacks, as described in the introduction to [Sch17].

Let Λ be a ring which is n-torsion for some integer n prime to p.

Definition 3.2.1. 1. For a locally spatial diamond Y we let $D(Y_{\text{ét}}, \Lambda)$ be the derived category of complexes of sheaves of Λ-modules on $Y_{\text{ét}}$.

2. For a small v-stack X, let X_v denote the site of all perfectoid spaces over X, with the v-topology. Let $D_{\text{ét}}(X, \Lambda) \subset D(X_v, \Lambda)$ be the full subcategory whose objects are those $A \in D(X_v, \Lambda)$ such that for all (equivalently, one surjective) $f\colon Y \to X$ from a locally spatial diamond Y, f^*A lies in the left-completion of $D(Y_{\text{ét}}, A)$

Then $D_{\text{ét}}(X, \Lambda)$ admits a derived tensor product, denoted $\otimes$, and a derived internal Hom, denoted $\underline{\mathrm{RHom}}$. When X is a locally spatial diamond $D_{\text{ét}}(X, \Lambda)$ is the left completion of $D(X_{\text{ét}}, \Lambda)$.

Definition 3.2.2. Let $\mathcal{C}$ be the class of morphisms between v-stacks which are compactifiable, representable in locally spatial diamonds, and have finite geometric transcendence degree.

Then $\mathcal{C}$ is closed under composition and fiber products. We need the following constructions:

1. [Sch17, Definition 1.7] For any morphism $f\colon X \to Y$ in $\mathcal{C}$, there is a lower shriek functor[3] $f_!\colon D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$, which agrees with f_* when f is proper, and an upper shriek functor $f^!\colon D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda)$ which is right adjoint to $f_!$.

[2] In [Sch17] there is a "smallness" hypothesis applied to X, which has to do with cutoff cardinals; we will be ignoring such set-theoretic issues.

[3] We have decided to use the notation $f_*, f_!, f^!, \dots$ for functors between derived categories which are often denoted $Rf_*, Rf_!, Rf^!, \dots$.

2. [Sch17, Theorem 1.8(iii)] A projection formula $\mathrm{proj}\colon f_!(\mathcal{F}\otimes f^*\mathcal{G}) \xrightarrow{\cong} f_!\mathcal{F}\otimes\mathcal{G}$. For $\mathcal{F},\mathcal{G}\in D(Y_{\text{ét}},\Lambda)$, the composition

$$f_!(f^!\mathcal{F}\otimes f^*\mathcal{G}) \xrightarrow{\mathrm{proj}} f_!f^!\mathcal{F}\otimes\mathcal{G} \xrightarrow{\mathrm{adj}} \mathcal{F}\otimes\mathcal{G}$$

induces by adjunction a morphism

$$t_{f^!}\colon f^!\mathcal{F}\otimes f^*\mathcal{G}\to f^!(\mathcal{F}\otimes\mathcal{G}).$$

3. [Sch17, Theorem 1.8(iv)] The first local Verdier duality isomorphism:

$$\underline{\mathrm{RHom}}(f_!\mathcal{F},\mathcal{G})\cong f_*\underline{\mathrm{RHom}}(\mathcal{F},f^!\mathcal{G}), \tag{3.2.1}$$

4. [Sch17, Theorem 1.8(v)] The second local Verdier duality isomorphism:

$$f^!\underline{\mathrm{RHom}}(\mathcal{F},\mathcal{G})\cong\underline{\mathrm{RHom}}(f^*\mathcal{F},f^!\mathcal{G}). \tag{3.2.2}$$

5. [Sch17, Theorem 1.9] Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd} \tag{3.2.3}$$

be a cartesian diagram of small v-stacks. Assume g lies in $\mathcal{C}$. There are base change isomorphisms $g^!f_*\mathcal{F}\cong f'_*(g')^!\mathcal{F}$ and $f^*g_!\mathcal{F}\cong g'_!(f')^*\mathcal{F}$. We will denote both of these by BC.

Definition 3.2.3. Let $\pi\colon X\to S$ be a morphism in $\mathcal{C}$. We let $\kappa_{X/S}=\pi^!\Lambda$ be the *relative dualizing complex*. For an object $\mathcal{F}$ of $D(X_{\text{ét}},\Lambda)$, let $\mathbf{D}_S\mathcal{F}=\underline{\mathrm{RHom}}(\mathcal{F},\kappa_{X/S})$ be the relative Verdier dual. (We write these as κ_X and $\mathbf{D}$ in the case that S is a geometric point.) Finally let

$$\mathrm{ev}\colon\mathcal{F}\otimes\mathbf{D}_S\mathcal{F}\to\kappa_{X/S}$$

be the evaluation morphism.

Using (3.2.1) and (3.2.2), an S-morphism $f\colon X\to Y$ in $\mathcal{C}$ interacts with the Verdier dual as follows:

$$\begin{aligned} f_*\mathbf{D}_S &\cong \mathbf{D}_Sf_! && (3.2.4)\\ f^!\mathbf{D}_S &\cong \mathbf{D}_Sf^*. && (3.2.5)\end{aligned}$$

3.3 Locally profinite sets

Let $* = \operatorname{Spa} C$ be a geometric point, let T be a locally profinite topological space, and let $f\colon \underline{T} \to *$ be the corresponding "constant" perfectoid space. $D_{\text{ét}}(*, \Lambda)$ can be identified with the derived category of Λ-modules, and $D_{\text{ét}}(\underline{T}, \Lambda)$ can be identified (via $\mathcal{F} \mapsto R\Gamma(\underline{T}, \mathcal{F})$) with the derived category of modules over the ring $C^\infty(T, \Lambda)$ of locally constant functions $T \to \Lambda$. With respect to these identifications:

1. $f^*M = C^\infty(T, M)$.
2. f_*M =restriction of M along $\Lambda \to C^\infty(T, \Lambda)$.
3. $f_!M \subset M$ is the Λ-submodule of elements whose support in T is compact.
4. $f^!M = \underline{\mathrm{RHom}}_\Lambda(C_c^\infty(T, \Lambda), M)$. We think of $f^!M$ as the module of "derived M-valued distributions on T."

A Λ-valued distribution μ on T is an element of

$$\operatorname{Dist}(T, \Lambda) = \operatorname{Hom}_\Lambda(C_c^\infty(T, \Lambda), \Lambda).$$

It determines a natural transformation $t_\mu\colon f^* \to f^!$, via Hom-$\otimes$-adjunction and the isomorphisms

$$C^\infty(T, M) \otimes_{C^\infty(T,\Lambda)} C_c^\infty(T, \Lambda) \to C_c^\infty(T, M) \leftarrow C_c^\infty(T, \Lambda) \otimes_\Lambda M.$$

If ν is another element of $\operatorname{Dist}(T, \Lambda)$, we say ν is *μ-smooth* if $\nu = (d\nu/d\mu)\mu$ for a function $d\nu/d\mu \in C^\infty(T, \Lambda)$. Then of course $t_\nu = (d\nu/d\mu)t_\mu$.

Lemma 3.3.1. *Let X be a locally spatial diamond. We have an equivalence between $D_{\text{ét}}(X \times \underline{T}, \Lambda)$ and $D_{\text{ét}}(X, C^\infty(T, \Lambda))$.*

Proof. It suffices to show that $D((X \times \underline{T})_{\text{ét}}, \Lambda)$ and $D(X_{\text{ét}}, C^\infty(T, \Lambda))$ are equivalent. If $\mathcal{F}$ is a sheaf of Λ-modules on $(X \times \underline{T})_{\text{ét}}$, we define a sheaf $\mathcal{F}'$ of $C^\infty(T, \Lambda)$-modules on $X_{\text{ét}}$ as follows. Given an étale morphism $Y \to X$, the Λ-module $\mathcal{F}'(Y) = \mathcal{F}(Y \times \underline{T})$ becomes a $C^\infty(T, \Lambda)$ in a natural way: given $T_0 \subset T$ open (and therefore closed) , the restriction map $\mathcal{F}(Y \times T) \to \mathcal{F}(Y \times T_0)$ is split, and one declares that the indicator function 1_{T_0} acts on $\mathcal{F}(Y \times \underline{T})$ by the corresponding idempotent.

Conversely, given a sheaf $\mathcal{F}'$ of $C^\infty(T, \Lambda)$-modules on $X_{\text{ét}}$, the corresponding sheaf $\mathcal{F}$ on $(X \times \underline{T})_{\text{ét}}$ is defined as follows. The étale topology on $X \times \underline{T}$ is generated by objects of the form $U \times \underline{T}_0$, where $U \to X$ is étale and $T_0 \subset T$ is open; then $\mathcal{F}(U \times \underline{T}_0) = 1_{T_0}\mathcal{F}'(U)$. $\square$

Consider the cartesian diagram

$$\begin{tikzcd} X \times \underline{T} \arrow[r, "p'"] \arrow[d, "f'"'] & \underline{T} \arrow[d, "f"] \\ X \arrow[r, "p"'] & *. \end{tikzcd} \tag{3.3.1}$$

Lemma 3.3.2. *Assume that $R\Gamma_c(U, \Lambda)$ is isomorphic to a bounded complex of finitely generated Λ-modules for all quasi-compact $U \in X_{\text{ét}}$. Then $(f')^* p^! \Lambda \cong (p')^! f^* \Lambda$. That is, $(f')^* \kappa_X \cong \kappa_{(X \times \underline{T})/\underline{T}}$.*

Proof. Let $\Lambda' = C^\infty(T, \Lambda)$, and identify $D_{\text{ét}}(\underline{T}, \Lambda)$ and $D_{\text{ét}}(X \times \underline{T}, \Lambda)$ with $D_{\text{ét}}(*, \Lambda')$ and $D_{\text{ét}}(X, \Lambda')$, respectively. Then the lemma is a matter of checking that $p^! \Lambda \otimes_\Lambda \Lambda' \cong (p')^! \Lambda'$. This can be checked on a quasicompact $U \in X_{\text{ét}}$; we have $R\Gamma(U, p^! \Lambda \otimes_\Lambda \Lambda') = R\Gamma(U, p^! \Lambda) \otimes_\Lambda \Lambda' = \mathrm{RHom}_\Lambda(R\Gamma_c(U, \Lambda), \Lambda) \otimes_\Lambda \Lambda'$, whereas $R\Gamma(U, (p')^! f^* \Lambda) = \mathrm{RHom}_{\Lambda'}(R\Gamma_c(U, \Lambda) \otimes_\Lambda \Lambda', \Lambda')$. The natural map between these modules is an isomorphism under the assumption on $R\Gamma_c(U, \Lambda)$. $\square$

3.4 The classifying stack attached to a locally profinite group

Let C be an algebraically closed perfectoid field, and let $* = \operatorname{Spa} C$.

Definition 3.4.1. Let G be a locally profinite group. A *pro-étale $\underline{G}$-torsor* is $\underline{G}$-equivariant morphism $X' \to X$, where X' and X are small v-stacks, such that $X = X'/\underline{G}$, and such that for all compact open subgroups $H \subset G$, $X'/\underline{H} \to X$ is an étale surjection. Let $[*/\underline{G}]$ be the category of pro-étale $\underline{G}$-torsors in perfectoid spaces over C.

Then $[*/\underline{G}]$ is a sheaf of groupoids on (Perf). We have a surjective morphism of stacks $* \to [*/\underline{G}]$, corresponding to the trivial torsor $\underline{G} \to *$. Since $* \times_{[*/\underline{G}]} * \cong \underline{G}$ is a perfectoid space, $[*/\underline{G}]$ is a small v-stack.

Lemma 3.4.2 ([FS])**.** *Let G be a locally profinite group, and let Λ be a ring. Let ΛG-mod be the category of Λ-modules equipped with a smooth action of G. Pullback along $* \to [*/\underline{G}]$ induces an equivalence between $D_{\text{ét}}([*/\underline{G}], \Lambda)$ and the derived category $D(\Lambda G\text{-mod})$. We write $\pi \mapsto \mathcal{L}_\pi$ for the inverse functor. Then $\mathcal{L}_{\pi \otimes \pi'} \cong \mathcal{L}_\pi \otimes \mathcal{L}_{\pi'}$ and $\underline{\mathrm{RHom}}(\mathcal{L}_\pi, \mathcal{L}_{\pi'}) \cong \mathcal{L}_{\underline{\mathrm{RHom}}(\pi, \pi')}$, where $\underline{\mathrm{RHom}}(\pi, \pi')$ is the derived functor of the functor assigning to π and π' the smooth vectors in the ΛG-module of linear maps $\pi \to \pi'$.*

Remark 3.4.3. Let π and π' be admissible, i.e. π^K and π'^K are finite-rank and free for all compact open K. Then the submodule of smooth vectors in the ΛG-module $\mathrm{Hom}_\Lambda(\pi, \pi')$ will in general not be admissible. On the other hand, we can form the $\Lambda(G \times G)$-modules $p_1^*\pi$ and $p_2^*\pi'$ by pulling back along the two projections $p_1, p_2 : G \times G \to G$. The submodule of smooth vectors in the $\Lambda(G \times G)$-module $\mathrm{Hom}_\Lambda(p_1^*\pi, p_2^*\pi')$ is then admissible and in fact isomorphic to $\pi^\vee \boxtimes \pi' = p_1^*\pi \otimes p_2^*\pi'$. Every element of $\mathrm{Hom}_\Lambda(p_1^*\pi, p_2^*\pi)$ has a well-defined trace, which is an element of Λ, and coincides with the contraction of the corresponding element of $\pi^\vee \boxtimes \pi$.

Now suppose G is a locally profinite group and $H \subset G$ is a closed subgroup. The inclusion $H \hookrightarrow G$ induces a morphism of v-stacks

$$i \colon [*/\underline{H}] \to [*/\underline{G}]. \tag{3.4.1}$$

Lemma 3.4.4. *The morphism i lies in $\mathcal{C}$. It is quasi-compact (thus proper) if and only if G/H is compact.*

Proof. Indeed, if S is a perfectoid space and $S \to [*/\underline{G}]$ is a morphism corresponding to a pro-étale $\underline{G}$-torsor $\widetilde{S} \to S$, then $i^*S = \widetilde{S}/\underline{H}$ is a diamond, and $i^*S \to S$ is a pro-étale map with geometric fibers of the form $\underline{G/H}$; all the claims follow from this. $\square$

Let Λ be a ring which is n-torsion for some n prime to p. By Lemma 3.4.4 the functors $i_!$ and $i^!$ are well-defined.

Lemma 3.4.5. *With respect to the equivalence $D_{\text{ét}}([*/\underline{G}], \Lambda) \cong D(\Lambda G\text{-mod})$ (and similarly for H), we have:*

1. *i^* corresponds to restriction.*

2. *i_* corresponds to derived smooth induction.*

3. *$i_!$ corresponds to derived compactly supported smooth induction.*

4. *$i^!$ corresponds to (the derived functor of) the functor*

$$\begin{array}{rcl} \Lambda G\text{-mod} & \to & \Lambda H\text{-mod} \\ M & \mapsto & \varinjlim_{H' \subset H} \mathrm{Hom}_{\Lambda G}(C_c^\infty(G/H', \Lambda), M), \end{array}$$

where in the colimit, H' runs over open subgroups of H.

Proof. Everything is formal except possibly (4), where we must show that the functor described is the right adjoint to compactly supported smooth induction. Indeed, if M is an object of ΛG-mod we have

$$(i^!M)^H = \mathrm{Hom}_{\Lambda H}(\Lambda, i^!H) = \mathrm{Hom}_{\Lambda G}(i_!\Lambda, M) = \mathrm{Hom}_{\Lambda G}(C_c^\infty(G/H, \Lambda), M).$$

Running this argument for the pullback of $i^!M$ along $[*/\underline{H'}] \to [*/\underline{G}]$ for each open subgroup $H' \subset H$ computes $(i^!M)^{H'}$, and finally we use $i^!M = \varinjlim_{H' \subset H}(i^!M)^{H'}$. $\square$

Remark 3.4.6. If $H \subset G$ is an open subgroup of finite index, then i is proper and étale, so that $i_! = i^*$ and $i^! = i^*$. In that case the counit $i^!i_!M = i^*i_*M \to M$ corresponds to the map $F \mapsto \sum_{g \in G/H} g^{-1}f(g)$, where we have identified i^*i_*M with the space of functions $F\colon G \to M$ satisfying $F(hg) = hF(g)$.

Remark 3.4.7. Suppose $H \subset G$ is compact. Given a Λ-valued Haar measure μ on G, we can define a natural transformation $i_!i^* \to 1$ as follows: given an object M of $D(\Lambda G\text{-mod})$, there is a map $i_!i^*M \to M$ given by $F \mapsto \int_G g^{-1}F(g)\, d\mu(g)$. Let $t_\mu\colon i^* \to i^!$ be the adjoint of this transformation.

The following description of $i^!M$ in the case of $H = \{1\}$ can sometimes be useful. Fix a Haar measure on G. For any compact open $K \subset G$ write $e_K \in \mathcal{C}_c^\infty(G, \Lambda)$ for the function $\mathrm{vol}(K)^{-1}\mathbf{1}_K$.

Lemma 3.4.8.

1. *We have the isomorphism* $\mathrm{Hom}_G(\mathcal{C}_c^\infty(G, \Lambda), V) = \varprojlim_K V^K$ *given by* $f \mapsto f(e_K)$*. Here the transition map* $V^{K'} \to V^K$ *for* $K' \subset K$ *is given by* e_K*.*

2. *This isomorphism identifies the submodule of* $G \times G$*-smooth vectors in* $\mathrm{Hom}_G(\mathcal{C}_c^\infty(G, \Lambda), V)$ *with* $V \subset \varprojlim_K V^K$*.*

3. $\mathcal{C}_c^\infty(G, \Lambda)$ *is a projective object in the category of smooth* G*-modules.*

4. $\mathcal{C}_c^\infty(G, \Lambda)$ *is a free* Λ*-module.*

Proof. Fix a countable descending tower K_n of compact open subgroups. This is a co-final sequence in the system of all compact open subgroups, so the projective limit can be taken with respect to this tower.

Given $f \in \mathrm{Hom}_G(\mathcal{C}_c^\infty(G, \Lambda), V)$ let $v_n = f(e_{K_n})$. For $m > n$ we have $e_{K_n}v_m = e_{K_n}f(e_{K_m}) = f(e_{K_n}e_{K_m}) = f(e_{K_n}) = v_n$, thus $(v_n) \in \varprojlim V^{K_n}$.

Conversely given (v_n) we define $f \in \mathrm{Hom}_G(\mathcal{C}_c^\infty(G,\Lambda),V)$ as follows. For $\varphi \in \mathcal{C}_c^\infty(G,\Lambda)$ choose some n s.t. φ is K_n-invariant on the right. Thus

$$\varphi = \sum_{g\in G/K_n} \varphi(g)\mathbf{1}_{gK_n} = \mathrm{vol}(K_n) \sum_{g\in G/K_n} \varphi(g) g e_{K_n}.$$

Set

$$f(\varphi) = \mathrm{vol}(K_n) \sum_{g\in G/K_n} \varphi(g) g v_n.$$

The relationship $e_{K_n} v_m = v_n$ for $m > n$ implies that this is independent of the choice of n. One checks that f is G-equivariant and that the two maps $f \leftrightarrow (v_n)$ are mutual inverses.

For the second point, f is a $G\times G$-smooth vector if and only if the image of f is contained in V^{K_n} for some n. Thus $v_m \in V^{K^n}$ for all $m \geq n$, but then $v_n = e_{K_n} v_m = v_m$, so setting $v = v_m$ for $m > n$ we see $v_n = e_{K_n} v$.

For projectivity of $\mathcal{C}_c^\infty(G,\Lambda)$, consider an exact sequence of ΛG-modules

$$0 \to U \to V \to W \to 0.$$

For every K the sequence

$$0 \to U^K \to V^K \to W^K \to 0$$

remains exact due to the existence of the projector $e_K : V \to V^K$. Then $\varprojlim^1 U^{K_n}$ is zero because the sequence U^{K_n} has surjective transition maps. Thus the inverse limit sequence remains exact.

For the Λ-freeness, write $\mathcal{C}_c^\infty(G,\Lambda) = \varinjlim_n \mathcal{C}_c(G/K_n,\Lambda)$. Each term in the limit is free with basis G/K_n. A simple change of basis shows that $\mathcal{C}_c(G/K_{n+1},\Lambda)$ is the direct sum of $\mathcal{C}_c(G/K_n,\Lambda)$ and a free complement, and the result follows. $\square$

3.5 Exterior tensor products and the Künneth isomorphism

We recall the notion of an exterior tensor product. For morphisms of small v-stacks $X_1, X_2 \to S$, consider the cartesian diagram

$$\begin{tikzcd} & X_1 \times_S X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & S. & \end{tikzcd}$$

Assume that each π_i lies in $\mathcal{C}$. For objects $\mathcal{F}_i \in D_{\text{ét}}(X_i, \Lambda)$ $(i = 1, 2)$ we define

$$\mathcal{F}_1 \boxtimes_S \mathcal{F}_2 = p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2 \in D_{\text{ét}}(X_1 \times X_2, \Lambda).$$

Lemma 3.5.1. *There is a natural isomorphism*

$$(\pi_1 \times \pi_2)_!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \cong \pi_{1!}\mathcal{F}_1 \otimes \pi_{2!}\mathcal{F}_2.$$

Proof. Combining the projection formula and base change isomorphisms gives:

$$\begin{array}{rcl} p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) & = & p_{2!}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2) \\ & \stackrel{\text{proj}}{\to} & p_{2!}p_1^*\mathcal{F}_1 \otimes \mathcal{F}_2 \\ & \stackrel{\text{BC}}{\to} & \pi_2^*\pi_{1!}\mathcal{F}_1 \otimes \mathcal{F}_2, \end{array}$$

so that we have an isomorphism

$$p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \cong \pi_2^*\pi_{1!}\mathcal{F}_1 \otimes \mathcal{F}_2 \tag{3.5.1}$$

Now apply $\pi_{2!}$, and note that $\pi_1 \times \pi_2 = \pi_2 \circ p_2$:

$$\begin{array}{rcl} (\pi_1 \times \pi_2)_!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) & = & \pi_{2!}p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \\ & \cong & \pi_{2!}(\pi_2^*\pi_{1!}\mathcal{F}_1 \otimes \mathcal{F}_2) \\ & \stackrel{\text{proj}}{\to} & \pi_{1!}\mathcal{F}_1 \otimes \pi_{2!}\mathcal{F}_2, \end{array}$$

as claimed. $\square$

In the case $\mathcal{F}_1 = \kappa_{X_1/S} = \pi_1^!\Lambda$, we have a morphism

$$\begin{array}{rcl} p_{2!}(\kappa_{X_1/S} \boxtimes_S \mathcal{F}_2) & \stackrel{3.5.1}{\to} & \pi_2^*\pi_{1!}\kappa_{X_1/S} \otimes \mathcal{F}_2 \\ & = & \pi_2^*\pi_{1!}\pi_1^!\Lambda \otimes \mathcal{F}_2 \\ & \stackrel{\text{adj.}}{\to} & \pi_2^*\Lambda \otimes \mathcal{F}_2 \cong \Lambda \otimes \mathcal{F}_2 \cong \mathcal{F}_2, \end{array}$$

which induces by adjunction a morphism

$$\kappa_{X_1/S} \boxtimes_S \mathcal{F}_2 \to p_2^!\mathcal{F}_2. \tag{3.5.2}$$

If $\mathcal{F}_2 = \kappa_{X_2/S}$, (3.5.2) becomes

$$\kappa_{X_1/S} \boxtimes_S \kappa_{X_2/S} \to \kappa_{X_1 \times_S X_2/S}. \tag{3.5.3}$$

3.6 Reflexive sheaves

Let $X \to S$ be a morphism in $\mathcal{C}$. For an object $\mathcal{F}$ of $D_{\text{ét}}(X, \Lambda)$, the evaluation morphism induces by adjunction a morphism $\mathcal{F} \to \mathbf{D}_S\mathbf{D}_S\mathcal{F}$. Also, for two objects $\mathcal{F}_1, \mathcal{F}_2$ of $D_{\text{ét}}(X, \Lambda)$ we have a morphism

$$(\mathbf{D}_S\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \otimes (\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2) \stackrel{\mathrm{ev}\boxtimes\mathrm{ev}}{\longrightarrow} \kappa_{X/S} \boxtimes_S \kappa_{X/S} \stackrel{(3.5.3)}{\to} \kappa_{X\times_S X/S}$$

which induces by adjuction a morphism

$$(\mathbf{D}_S\mathcal{F}_1) \boxtimes_S \mathcal{F}_2 \to \mathbf{D}_S(\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2). \tag{3.6.1}$$

From now on we will write $\mathbf{D}_S\mathcal{F}_1 \boxtimes_S \mathcal{F}_2$ instead of $(\mathbf{D}_S\mathcal{F}_1) \boxtimes_S \mathcal{F}_2$.

Definition 3.6.1. An object $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$ is *reflexive* (relative to S) if it is isomorphic to a bounded complex, and if $\mathcal{F} \to \mathbf{D}_S\mathbf{D}_S\mathcal{F}$ is an isomorphism. $\mathcal{F}$ is *strongly reflexive* if it is reflexive and if the morphism $\mathbf{D}_S\mathcal{F} \boxtimes_S \mathcal{F} \to \mathbf{D}_S(\mathcal{F} \boxtimes_S \mathbf{D}_S\mathcal{F})$ from (3.6.1) is an isomorphism.

Proposition 3.6.2. *Assume that $\Lambda = \mathcal{O}_E/\ell^n$, where $\ell \neq p$ is prime and $E/\mathbf{Q}_\ell$ is a finitely ramified algebraic extension with ring of integers $\mathcal{O}_E$. Let $* = \operatorname{Spd} C$, so that $D_{\text{ét}}(*, \Lambda)$ is equivalent to $D(\Lambda\text{-mod})$. If $\mathcal{F} \in D_{\text{ét}}(*, \Lambda)$ is reflexive, then each $R^i\Gamma(*, \mathcal{F})$ is finitely generated. If $\mathcal{F}$ is strongly reflexive, then $R\Gamma(*, \mathcal{F})$ is a perfect complex; i.e., it may be represented by a bounded complex of finite free Λ-modules.*

Proof. An object $\mathcal{F}$ of $D_{\text{ét}}(*, \Lambda)$ may be represented by a complex $M^\bullet$ of Λ-modules, with $R^i\Gamma(X, \mathcal{F}) = H^i(M^\bullet)$. Note that Λ is self-injective. The claims follow from Lemma A.4.2. □

Lemma 3.6.3. *Let $X, Y \to S$ be two morphisms of v-stacks which belong to $\mathcal{C}$.*

1. *For a proper S-morphism $f\colon X \to Y$ and a reflexive (respectively, strongly reflexive) $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$, $f_*\mathcal{F}$ is also reflexive (respectively, strongly reflexive). In particular if $X \to *$ is proper and $\Lambda = \mathcal{O}_E/\ell^n$ is as in Proposition 3.6.2, then each $R^i\Gamma(X, \mathcal{F}) = R^if_*\mathcal{F}$ is finitely generated (respectively, $R\Gamma(X, \mathcal{F})$ is a perfect complex).*

2. *For a smooth S-morphism $f\colon X \to Y$ and a reflexive (respectively, strongly reflexive) object $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$, $f^*\mathcal{F}$ is also reflexive (respectively, strongly reflexive).*

Proof. The reflexivity statement (1) follows from (3.2.4) and the relation $f_* = f_!$: we have $\mathbf{D}_S\mathbf{D}_S f_*\mathcal{F} \cong \mathbf{D}_S\mathbf{D}_S f_!\mathcal{F} \cong \mathbf{D}_S f_*\mathbf{D}_S\mathcal{F} \cong \mathbf{D}_S f_!\mathbf{D}_S\mathcal{F} \cong f_*\mathbf{D}_S\mathbf{D}_S\mathcal{F} \cong f_*\mathcal{F}$, so $f_*\mathcal{F}$ is reflexive. For strong reflexivity, we apply $(f \times f)_*$ to the isomorphism $\mathbf{D}_S\mathcal{F} \boxtimes_S \mathcal{F} \cong \mathbf{D}_S(\mathcal{F} \boxtimes_S \mathbf{D}_S\mathcal{F})$, use the Künneth isomorphism (Lemma 3.5.1) and the fact that f_* commutes with $\mathbf{D}_S$.

For (2), suppose $\mathcal{F} \in D_{\text{ét}}(Y, \Lambda)$ is reflexive and $f\colon X \to Y$ is étale. Using the relation (3.2.5) and $f^* = f^!(-d)[-2d]$ we have $\mathbf{D}_S\mathbf{D}_S f^*\mathcal{F} \cong \mathbf{D}_S f^!\mathbf{D}_S\mathcal{F} \cong \mathbf{D}_S f^*(d)[2d]\mathbf{D}_S\mathcal{F} \cong f^!(-d)[-2d]\mathbf{D}_S\mathbf{D}_S\mathcal{F} \cong f^*\mathbf{D}_S\mathbf{D}_S\mathcal{F} \cong f^*\mathcal{F}$, so $\mathcal{F}$ is reflexive. The argument for strong reflexivity is similar, using the fact that f^* commutes with tensor products. $\square$

Example 3.6.4. Let us return to the situation of Lemma 3.3.2. Let $p\colon X \to *$ be in $\mathcal{C}$. Assume that $R\Gamma_c(U, \Lambda)$ is isomorphic to a bounded complex of finitely generated Λ-modules for all quasi-compact $U \in X_{\text{ét}}$. Let T be a locally profinite set, and let $\pi\colon X \times \underline{T} \to X$ be the projection. Then for an object $\mathcal{F}$ of $D_{\text{ét}}(X, \Lambda)$, we have

$$\mathbf{D}_{\underline{T}}\pi^*\mathcal{F} = \underline{\mathrm{RHom}}_{X\times\underline{T}}(\pi^*\mathcal{F}, \kappa_{X\times\underline{T}/\underline{T}}) \cong \underline{\mathrm{RHom}}_{X\times\underline{T}}(\pi^*\mathcal{F}, \pi^*\kappa_X)$$

where in the last step we used Lemma 3.3.2. Applying Lemma 3.3.1, this may be identified with

$$\underline{\mathrm{RHom}}_{D(X,C^\infty(T,\Lambda))}(\mathcal{F} \otimes_\Lambda \Lambda', \kappa_X \otimes \Lambda').$$

If we further assume that $R\Gamma_c(U, \mathcal{F})$ is isomorphic to a bounded complex of finitely generated Λ-modules for all quasi-compact $U \in X_{\text{ét}}$, then the above is isomorphic to $\pi^*\mathbf{D}\mathcal{F}$, so that $\mathbf{D}_{\underline{T}}\pi^*\mathcal{F} \cong \pi^*\mathbf{D}\mathcal{F}$. As a result if $\mathcal{F}$ is assumed reflexive (respectively, strongly reflexive), then $\pi^*\mathcal{F}$ is reflexive (respectively, strongly reflexive) relative to $\underline{T}$.

As we establish further properties of reflexive sheaves, the following "two out of six" lemma will be used repeatedly.

Lemma 3.6.5. *Let $A \to B \to C \to D$ be a sequence of morphisms in any category. If the composites $A \to C$ and $B \to D$ are isomorphisms, then all morphisms in the sequence are isomorphisms.*

Let $X \to S$ be a morphism in $\mathcal{C}$. For any $\mathcal{F}_1, \mathcal{F}_2 \in D_{\text{ét}}(X, \Lambda)$ we have a morphism

$$\underline{\mathrm{RHom}}(\mathcal{F}_1, \mathcal{F}_2) \otimes \mathbf{D}_S\mathcal{F}_2 \to \mathbf{D}_S\mathcal{F}_1,$$

which induces by adjunction a morphism

$$\underline{\mathrm{RHom}}(\mathcal{F}_1, \mathcal{F}_2) \to \underline{\mathrm{RHom}}(\mathbf{D}_S\mathcal{F}_2, \mathbf{D}_S\mathcal{F}_1). \tag{3.6.2}$$

Lemma 3.6.6. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are reflexive then* (3.6.2) *is an isomorphism.*

Proof. Repeatedly applying (3.6.2) gives a sequence of morphisms:

$$\begin{tikzcd}
\underline{\mathrm{RHom}}(\mathcal{F}_1, \mathcal{F}_2) \arrow[r] & \underline{\mathrm{RHom}}(\mathbf{D}_S\mathcal{F}_2, \mathbf{D}_S\mathcal{F}_1) \arrow[d] \\
\underline{\mathrm{RHom}}(\mathbf{D}_S\mathbf{D}_S\mathbf{D}_S\mathcal{F}_1, \mathbf{D}_S\mathbf{D}_S\mathbf{D}_S\mathcal{F}_2) & \underline{\mathrm{RHom}}(\mathbf{D}_S\mathbf{D}_S\mathcal{F}_1, \mathbf{D}\mathbf{D}\mathcal{F}_2) \arrow[l]
\end{tikzcd}$$

The composite of the first and the second morphism is an isomorphism: If we use $\mathbf{D}_S\mathbf{D}_S\mathcal{F}_i \cong \mathcal{F}_i$ to identify $\underline{\mathrm{RHom}}(\mathbf{D}_S\mathbf{D}_S\mathcal{F}_1, \mathbf{D}_S\mathbf{D}_S\mathcal{F}_2)$ with $\underline{\mathrm{RHom}}(\mathcal{F}_1, \mathcal{F}_2)$ then it becomes the identity. Analogously, the composite of the second and third morphisms is an isomorphism. Therefore by Lemma 3.6.5 all morphisms are isomorphisms, and in particular the first one is. □

Lemma 3.6.7. *Let X_1 and X_2 be small v-stacks over S, with each $X_i \to S$ in class $\mathcal{C}$. Let $p_i\colon X_1 \times_S X_2 \to X_i$ be the projections, and let $\mathcal{F}_i \in D_{\acute{e}t}(X_i, \Lambda)$. Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are reflexive, and that the morphism $\mathbf{D}_S\mathcal{F}_1 \boxtimes_S \mathcal{F}_2 \to \mathbf{D}_S(\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2)$ of* (3.6.1) *is an isomorphism. Then we have an isomorphism in $D_{\acute{e}t}(X_1 \times_S X_2, \Lambda)$:*

$$\mathbf{D}_S\mathcal{F}_1 \boxtimes_S \mathcal{F}_2 \xrightarrow{\sim} \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathcal{F}_2)$$

Remark 3.6.8. If $X_1 = X_2 = \operatorname{Spa} C$, then Lemma 3.6.7 reduces to the statement that $V^* \otimes W \to \underline{\mathrm{RHom}}(V, W)$ is an isomorphism whenever V and W are perfect complexes of Λ-modules [Sta17, Tag 0656, Lemma 15.67.14].

Proof. We will describe a sequence of morphisms

$$\mathbf{D}_S\mathcal{F}_1\boxtimes_S\mathcal{F}_2 \to \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathcal{F}_2) \to \mathbf{D}_S(\mathcal{F}_1\boxtimes_S\mathbf{D}_S\mathcal{F}_2) \to \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathcal{F}_2) \tag{3.6.3}$$

with the intent of applying the two-out-of-six lemma once again.

The first morphism in (3.6.3) is adjoint to the composition

$$(\mathbf{D}_S\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \otimes p_1^*\mathcal{F}_1 \cong (\mathbf{D}_S\mathcal{F}_1 \otimes \mathcal{F}_1) \boxtimes_S \mathcal{F}_2 \overset{\mathrm{ev}\otimes 1}{\to} \kappa_{X_1/S} \boxtimes_S \mathcal{F}_2 \overset{3.5.2}{\to} p_2^!\mathcal{F}_2 \tag{3.6.4}$$

The second morphism in (3.6.3) is

$$\begin{aligned}
\underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathcal{F}_2) &\overset{-\otimes 1}{\to} \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}_S\mathcal{F}_2, p_2^!\mathcal{F}_2 \otimes p_2^*\mathbf{D}_S\mathcal{F}_2) \\
&\overset{t_{p_2^!}}{\to} \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}_S\mathcal{F}_2, p_2^!(\mathcal{F}_2 \otimes \mathbf{D}_S\mathcal{F}_2)) \\
&\overset{\mathrm{ev}}{\to} \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}_S\mathcal{F}_2, \kappa_{(X_1\times X_2)/S}) \\
&= \mathbf{D}_S(\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2).
\end{aligned}$$

For the third morphism in (3.6.3), we start with evaluation morphism

$$\mathbf{D}_S(\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2) \otimes p_1^*\mathcal{F}_1 \otimes p_2^*\mathbf{D}_S\mathcal{F}_2 = \mathbf{D}_S(\mathcal{F}_1 \boxtimes \mathbf{D}_S\mathcal{F}_2) \otimes \mathcal{F}_1 \boxtimes \mathbf{D}_S\mathcal{F}_2 \to \kappa_{X_1 \times X_2},$$

which induces by adjuction

$$\mathbf{D}_S(\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2) \otimes p_2^*\mathbf{D}_S\mathcal{F}_2 \to \mathbf{D}_S p_1^*\mathcal{F}_1,$$

which in turn induces by adjunction

$$\begin{aligned} \mathbf{D}_S(\mathcal{F}_1 \boxtimes_S \mathbf{D}_S\mathcal{F}_2) &\to \underline{\mathrm{RHom}}(p_2^*\mathbf{D}_S\mathcal{F}_2, \mathbf{D}_S p_1^*\mathcal{F}_1) \\ &\cong \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, \mathbf{D}_S p_2^*\mathbf{D}_S\mathcal{F}_2) \\ &\cong \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathbf{D}_S\mathbf{D}_S\mathcal{F}_2) \\ &\cong \underline{\mathrm{RHom}}(p_1^*\mathcal{F}_1, p_2^!\mathcal{F}_2). \end{aligned}$$

In (3.6.3), the composition of the first and second morphisms is an isomorphism by hypothesis, and the composition of the second and third morphisms is the identity. Thus we conclude by the two-out-of-six lemma. □

Definition 3.6.9. Let $X_1, X_2 \to S$ be morphisms of small v-stacks which are compactifiable, representable in diamonds, and of finite geometric transcedence degree. A *correspondence* from X_1 to X_2 is an S-morphism $c = (c_1, c_2)\colon Y \to X_1 \times_S X_2$, where each $c_i\colon Y \to X_i$ is compactifiable, representable in diamonds, and of finite geometric transcedence degree.

Combining (3.2.2) with Lemma 3.6.7 gives the following corollary.

Corollary 3.6.10. *Let $c\colon Y \to X \times_S X$ be a correspondence. Let $\mathcal{F} \in D(X_{\text{ét}}, \Lambda)$ be strongly reflexive. Then we have an isomorphism*

$$\underline{\mathrm{RHom}}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \xrightarrow{\sim} c^!(\mathbf{D}_S\mathcal{F} \boxtimes_S \mathcal{F}).$$

3.7 Cohomological correspondences

Definition 3.7.1. Let $c\colon Y \to X_1 \times_S X_2$ be a correspondence as in Definition 3.6.9, and let $\mathcal{F}_i$ be an object of $D_{\text{ét}}(X_i, \Lambda)$ for $i = 1, 2$. A *cohomological correspondance from $\mathcal{F}_1$ to $\mathcal{F}_2$ lying over c* is a morphism $u\colon c_1^*\mathcal{F}_1 \to c_2^!\mathcal{F}_2$.

Let $b\colon Y' \to X_1' \times_S X_2'$ be another correspondence. There is an evident notion of a morphism of correspondences $f\colon c \to b$; it is a commutative diagram of the form

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & Y \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "f^\natural"'] & X_2 \arrow[d, "f_2"'] \\ X_1' & Y' \arrow[l, "b_1"] \arrow[r, "b_2"'] & X_2'. \end{tikzcd} \tag{3.7.1}$$

We say that f is proper if each of its components $f_1, f_2, f^\natural$ are.

We have the following pushforward construction for cohomological correspondences [Var07, 1.1.6]. Assume at least one of the following:

1. The left inner square of (3.7.1) is cartesian.

2. The morphisms f_1 and $f^\natural$ are proper.

3. The morphisms c_1 and b_1 are proper.

Given $u\colon f_1^*\mathcal{F}_1 \to f_2^!\mathcal{F}_2$, the composition

$$b_{2!}b_1^*(f_{1!}\mathcal{F}_1) \overset{\mathrm{BC}}{\to} b_{2!}f^\natural_! c_1^*\mathcal{F}_1 = f_{2!}c_{2!}c_1^*\mathcal{F}_1 \overset{f_{2!}(u)}{\to} f_{2!}\mathcal{F}_2$$

induces by adjunction a cohomological correspondence $f_!u$ from $f_{1!}\mathcal{F}_1$ to $f_{2!}\mathcal{F}_2$ lying over b.

Remark 3.7.2. In the special case that $S = *$ and $b\colon S \to S \times_S S = S$ is the identity, the above construction shows that cohomological correspondence u from $\mathcal{F}_1$ to $\mathcal{F}_2$ determines a morphism $R\Gamma_c(u) = f_!u\colon R\Gamma_c(X_1, \mathcal{F}_1) \to R\Gamma_c(X_2, \mathcal{F}_2)$.

3.8 Self-correspondences and the trace morphism

Let $X \to S$ be a proper morphism of small v-stacks which is in $\mathcal{C}$. Suppose we are given the following data:

1. A correspondence $c\colon Y \to X \times_S X$:

$$\begin{tikzcd} & Y \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X & & X \end{tikzcd},$$

   where we assume c_2 is proper. In particular, Y is proper over S.

2. An object $\mathcal{F}$ in $D_{\text{ét}}(X, \Lambda)$ which is strongly reflexive relative to S.

Let $\Delta\colon X \to X \times_S X$ be the diagonal map, and let $\mathrm{Fix}(c)$ be the fixed point locus of c, defined as the fiber product

$$\begin{tikzcd} \mathrm{Fix}(c) \arrow[r, "c'"] \arrow[d, "\Delta'"'] & X \arrow[d, "\Delta"] \\ Y \arrow[r, "c"'] & X \times_S X. \end{tikzcd} \tag{3.8.1}$$

The composition $\Delta^*(\mathbf{D}_S\mathcal{F}\boxtimes_S\mathcal{F}) = \mathbf{D}_S\mathcal{F}\otimes\mathcal{F} \xrightarrow{\text{ev}} \kappa_{X/S}$ induces by adjunction a morphism

$$\mathbf{D}_S\mathcal{F}\boxtimes_S\mathcal{F} \to \Delta_*\kappa_{X/S}. \tag{3.8.2}$$

We have a series of morphisms

$$\begin{aligned}
\underline{\mathrm{RHom}}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) &\overset{\text{Cor. 3.6.10}}{\to} c^!(\mathbf{D}_S\mathcal{F}\boxtimes_S\mathcal{F}) \\
&\overset{(3.8.2)}{\to} c^!\Delta_*\kappa_{X/S} \\
&\overset{\text{BC}}{\to} \Delta'_*c'^!\kappa_{X/S} \\
&= \Delta'_*\kappa_{\mathrm{Fix}(c)/S}
\end{aligned}$$

whose composition we will call tr_c.

Example 3.8.1. Let $c\colon * \times * \to *$ be the trivial correspondence, and let $\mathcal{F}$ be a strongly reflexive sheaf on $*$ corresponding to a perfect complex $N^\bullet$. Then $\mathrm{Hom}(c_1^*\mathcal{F}, c_2^!\mathcal{F})$ can be identified with $\mathrm{End}\, N^\bullet$, the Λ-module of homotopy classes of chain maps $f^\bullet\colon N^\bullet \to N^\bullet$. Under these circumstances, $\mathrm{tr}_c(f^\bullet) = \sum_{i\in\mathbf{Z}}(-1)^i \operatorname{tr} f^i$.

3.9 Proper pushforward of cohomological correspondences

Proposition 3.9.1. *Let $[f] = (f, f^\natural, f)\colon c \to b$ be a proper morphism of correpondences over S, where $c\colon Y \to X\times_S X$. The map $f'\colon \mathrm{Fix}(c) \to \mathrm{Fix}(b)$ is proper as well, and for a strongly reflexive object $\mathcal{F}\in D_{\text{ét}}(X,\Lambda)$ the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \arrow[r, "\mathrm{tr}_c"] \arrow[d, "{[f]_!}"'] & H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S}) \arrow[d, "f'_!"] \\
\mathrm{Hom}(b_1^*f_!\mathcal{F}, b_2^!f_!\mathcal{F}) \arrow[r, "\mathrm{tr}_b"'] & H^0(\mathrm{Fix}(b), \kappa_{\mathrm{Fix}(b)/S})
\end{tikzcd} \tag{3.9.1}$$

Proof. This is the analogue of [Var07, Proposition 1.2.5]. The proof of the latter ((4.3.4) in [Var07]) relies only upon the tools we've developed so far (especially the base change theorems), which are available in the situation of small v-stacks. □

In the context of Proposition 3.9.1, suppose $\beta \subset \mathrm{Fix}(c)$ is an open and closed subset, and let $p\colon \beta \to S$ be the structure map. For a cohomological correspondence u lying over c, we define the *local term* $\mathrm{loc}_\beta(u)$ as the image of $\mathrm{tr}_c(u)$ under the composition

$$H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S}) \to H^0(\beta, p^!\Lambda) = H^0(S, p_!p^!\Lambda) \to H^0(S,\Lambda).$$

Example 3.9.2. Consider the case that $S = *$ and b is the trivial correspondence on S, so that $f: c \to b$ is the structure morphism. Suppose $\mathrm{Fix}(c)$ is the disjoint union of open and closed subsets $\beta_1, \dots, \beta_n$. Then applying Proposition 3.9.1 to the structure morphism $f: c \to b$ gives the following familiar form of the Lefschetz fixed-point formula:

$$\mathrm{tr}\, f_! u | R\Gamma(X, \mathcal{F}) = \sum_{i=1}^{n} \mathrm{loc}_{\beta_i}(u).$$

3.10 Reflexivity over the classifying stack of a locally profinite group

Suppose we are given a locally pro-p group G and a morphism $p: X \to *$ in $\mathcal{C}$, which admits an action $\alpha: X \times \underline{G} \to X$. Then we have a cartesian diagram of morphisms in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\pi'"'] & * \arrow[d, "\pi"] \\ {[X/\underline{G}]} \arrow[r, "p'"'] & {[*/\underline{G}].} \end{tikzcd}$$

Lemma 3.10.1. *There is an equivalence between $D_{\text{ét}}([X/\underline{G}], \Lambda)$ and the category of objects $\mathcal{F}$ in $D_{\text{ét}}(X, \Lambda)$ which come equipped with an $\underline{G}$-equivariance $pr_X^* \mathcal{F} \xrightarrow{\sim} \alpha^* \mathcal{F}$, where $pr_X: X \times \underline{G} \to X$ is the projection.*

(We are calling a morphism $e: \mathrm{pr}_X^* \mathcal{F} \xrightarrow{\sim} \alpha^* \mathcal{F}$ a $\underline{G}$-equivariance if it satisfies the appropriate cocycle condition.)

Proof. Let $\mathcal{F}_G$ be an object of $D_{\text{ét}}([X/\underline{G}], \Lambda)$, and let $\mathcal{F} = (\pi')^* \mathcal{F}_G$. Then the descent datum of $\mathcal{F}$ relative to π' is exactly the equivariance $\mathrm{pr}_X^* \mathcal{F} \xrightarrow{\sim} \alpha^* \mathcal{F}$.

Going in the other direction, let $\mathcal{F}$ be a $\underline{G}$-equivariant object of $D(X_{\text{ét}}, \Lambda)$. Since $X \to [X/\underline{G}]$ is a v-cover, $\mathcal{F}$ descends to an object $\mathcal{F}_G$ of $D(X_v, \Lambda)$. Suppose $f: Y \to [X/\underline{G}]$ is a perfectoid space, so that we have a cartesian diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "f'"] \arrow[d, "\pi''"'] & X \arrow[d, "\pi'"] \\ Y \arrow[r, "f"'] & {[X/\underline{G}]} \end{tikzcd}$$

Then $\widetilde{Y}$ is a perfectoid space [Sch17, Lemma 10.13], and $(f')^* \mathcal{F} = (\pi'')^* f^* \mathcal{F}_G$ is a $\underline{G}$-equivariant object of $D(\widetilde{Y}_{\text{ét}}, \Lambda)$. Descent data for étale sheaves on the pro-étale $\underline{G}$-torsor $\widetilde{Y} \to Y$ is effective, ultimately because of [Sch12, Lemma 7.5(i)]: any quasi-compact étale $U \in \widetilde{Y}_{\text{ét}}$ descends to $\widetilde{Y}/\underline{H}$ for some open compact subgroup $H \subset G$. Thus the descent $f^*\mathcal{F}_G$ lies in $D(Y_{\text{ét}}, \Lambda)$, which proves that $\mathcal{F}_G$ lies in $D_{\text{ét}}([X/\underline{G}], \Lambda)$.

We have thus defined a functor $\mathcal{F} \mapsto \mathcal{F}_G$ from G-equivariant objects of $D(X_{\text{ét}}, \Lambda)$ to $D_{\text{ét}}([X/\underline{G}], \Lambda)$, which (after passing to the left-completion $D_{\text{ét}}(X, \Lambda)$) gives the required inverse functor. $\square$

Lemma 3.10.2. *Assume that for all quasi-compact $U \in X_{\text{ét}}$, $R\Gamma_c(U, \Lambda)$ is isomorphic to a bounded complex of finitely generated Λ-modules. Then $(\pi')^*\kappa_{[X/\underline{G}]/[*/\underline{G}]} \cong \kappa_X$.*

Proof. The composition

$$(p')^! \to (p')^!\pi_*\pi^* \overset{\text{BC}}{\tilde{\to}} \pi'_* p^! \pi^*$$

induces by injuction a morphism $(\pi')^*(p')^! \to p^!\pi^*$, which we claim is an isomorphism when applied to Λ. Consider the diagram

$$\begin{tikzcd} X \times \underline{G} \arrow[r, "q'"] \arrow[d, "p''"'] & X \arrow[d, "p"] \\ \underline{G} \arrow[r, "q"'] & *. \end{tikzcd}$$

By Lemma 3.3.2 we have $(q')^*\kappa_X \cong \kappa_{X \times \underline{G}/\underline{G}}$. The action map $\alpha\colon X \times \underline{G} \to X$ factors as $\alpha = q' \circ \alpha'$, where α' is an automorphism of $X \times \underline{G}$; therefore $\alpha^*\kappa_X \cong (\alpha')^*(q')^*\kappa_X \cong (\alpha')^*(p'')^!\Lambda = (\alpha')^!(p'')^!\Lambda = (p'')^!\Lambda = \kappa_{X \times \underline{G}/X}$. Therefore we have an isomorphism $(q')^*\kappa_X \cong \alpha^*\kappa_X$, which constitutes a $\underline{G}$-equivariace. It follows that $\kappa_X = (\pi')^*K$ for an object $K \in D_{\text{ét}}([X/\underline{G}], \Lambda)$.

For an object $\mathcal{F}_G$ in $D_{\text{ét}}([X/\underline{G}], \Lambda)$, let $\mathcal{F} = (\pi')^*\mathcal{F}_G$. In the following chain of bijections, the subscript G indicates those morphisms which commute with the G-equivariance:

$$\begin{aligned} \operatorname{Hom}(\mathcal{F}_G, K) &\cong \operatorname{Hom}_G(\mathcal{F}, (\pi')^*K)) \\ &\cong \operatorname{Hom}_G(\mathcal{F}, \kappa_X) \\ &\cong \operatorname{Hom}_G(p_!\mathcal{F}, \Lambda) \\ &\cong \operatorname{Hom}_G(\pi^*(p')_!\mathcal{F}_G, \Lambda) \\ &\cong \operatorname{Hom}((p')_!\mathcal{F}_G, \Lambda) \\ &\cong \operatorname{Hom}(\mathcal{F}_G, (p')^!\Lambda). \end{aligned}$$

Thus $K \cong (p')^!\Lambda = \kappa_{[X/\underline{G}]/[*/\underline{G}]}$. $\square$

The morphism $[X/\underline{G}] \to *$ is not necessarily in class $\mathcal{C}$, but nonetheless we may define an absolute dualizing complex by

$$\kappa_{[X/\underline{G}]} = \kappa_{[X/\underline{G}]/[*/\underline{G}]}.$$

This definition only depends on the v-stack $[X/\underline{G}]$: if $[X/\underline{G}] \cong [X'/\underline{G}']$, then there exists a class $\mathcal{C}$ morphism $Y \to *$ and an action of $\underline{G \times G'}$ on Y such that $X = Y/\underline{G}'$ and $X' = Y/\underline{G}$. Then by repeated applying Lemma 3.10.2, $\kappa_{[X/\underline{G}/[*/\underline{G}]}$ is the descent of the $\underline{G}$-equivariant object κ_X, which is in turn the descent of the $\underline{G \times G'}$-equivariant object κ_Y; this is naturally isomorphic to $\kappa_{[X'/\underline{G}']/[*/\underline{G}']}$.

Therefore it is possible to talk about the (absolute) Verdier dual of an object of $D_{\text{ét}}([X/\underline{G}], \Lambda)$, as well as the property of being reflexive or strongly reflexive (relative to $*$).

Lemma 3.10.3. *Let G be a locally pro-p group, and assume that $\Lambda = \mathcal{O}_E/\ell^n$ for a finite extension $E/\mathbf{Q}_p$, $\ell \neq p$. For a derived ΛG-module π, the following are equivalent:*

1. *$\mathcal{L}_\pi$ is strongly reflexive (relative to $*$).*
2. *π^K is a perfect complex for all open pro-p subgroups $K \subset G$.*

We will call such a π an *admissible derived ΛG-module.*

Proof. By Lemma 3.4.2, $\mathbf{D}\mathcal{L}_\pi \cong \mathcal{L}_{\check{\pi}}$, where $\check{\pi}$ is the smooth dual. For a pro-p open subgroup $K \subset G$, we have an idempotent operator $e_K\colon \pi \to \pi^K$, which identifies $(\check{\pi})^K$ with $(\pi^K)^\vee$. Therefore if $\mathcal{L}_\pi$ is reflexive, then $(\pi^K)^{\vee\vee} \cong \pi^K$. If $\mathcal{L}_\pi$ is strongly reflexive, then $\check{\pi} \boxtimes \pi \cong (\pi \boxtimes \check{\pi})^\vee$ as $\Lambda(G \times G)$-modules; taking $(K \times K)$-invariants shows that $(\pi^K)^\vee \otimes \pi^K \to (\pi^K \otimes (\pi^K)^\vee)^\vee$ is an isomorphism. By Lemma A.4.2, that π^K is a perfect complex.

Conversely, if π^K is perfect then $\pi^K \to (\pi^K)^{\vee\vee}$ and $(\pi^K)^\vee \otimes \pi^K \to (\pi^K \otimes (\pi^K)^\vee)^\vee$ are isomorphisms; applying $\varinjlim_K$ and noticing that groups of the form $K \times K$ form a co-final sequence in the set of all compact open subgroups of $G \times G$ we see that $\pi^{\vee\vee} \cong \pi$ as G-representations and $(\pi^\vee \boxtimes \pi)^\vee \cong \pi \boxtimes \pi^\vee$ as $G \times G$-representations. $\square$

Example 3.10.4. Let G be a locally pro-p group, and consider the correspondence $c\colon * \to [*/\underline{G}] \times [*/\underline{G}]$. Then $\mathrm{Fix}(c)$ classifies $\underline{G}$-torsors $\widetilde{S} \to S$ equipped with two trivializations $\widetilde{S} \cong S \times \underline{G}$; from this we see that $\mathrm{Fix}(c) \cong \underline{G}$.

Using that $[*/G] \times [*/G] = [*/G \times G]$, the morphisms in diagram (3.8.1) can be described as follows: Δ is a special case of (3.4.1) for the diagonal inclusion $G \to G \times G$, c is also a special case of (3.4.1) for $\{1\} \to G \times G$, c' factors as the composition of the structure map $\underline{G} \to *$ and $* \to [*/G]$, and Δ' is the structure map.

Let π be an admissible derived ΛG-module, corresponding to the strongly reflexive object $\mathcal{L}_\pi$ in $D_{\text{ét}}([*/\underline{G}], \Lambda)$. We spell out the trace morphism tr_c in this case. In the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(c_1^*\mathcal{L}_\pi, c_2^!\mathcal{L}_\pi) \arrow[r, "\mathrm{tr}_c"] \arrow[d, "\sim"'] & H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)}) \arrow[d, "\sim"'] \\
\mathrm{Hom}_\Lambda(\pi, \mathrm{Hom}_{\Lambda G}(C_c^\infty(G, \Lambda), \pi)) \arrow[r] & \mathrm{Dist}(G, \Lambda),
\end{tikzcd}$$

the lower horizontal arrow sends $F\colon \pi \to \mathrm{Hom}_{\Lambda G}(C_c^\infty(\pi, \Lambda), \pi)$ to the distribution which assigns to $f \in C_c^\infty(G, \Lambda)$ the trace of the endormorphism $m \mapsto F(m)(f)$ of π. (Note that since f is left K-invariant for some compact open $K \subset G$, this endomorphism factors through the perfect complex π^K; thus this trace is well-defined.)

As in Remark 3.4.7, a Λ-valued Haar measure μ on G determines a natural transformation $t_\mu\colon i^* \to i^!$. Note the commutativity of

$$\begin{tikzcd}
\mathrm{Hom}(c_1^*\mathcal{L}_\pi, c_2^*\mathcal{L}_\pi) \arrow[r, "t_\mu"] \arrow[d, "\sim"'] & \mathrm{Hom}(c_1^*\mathcal{L}_\pi, c_2^!\mathcal{L}_\pi) \arrow[d] \\
\mathrm{End}_\Lambda \pi \arrow[r] & \mathrm{Dist}(G, \Lambda),
\end{tikzcd}$$

where the lower horizontal arrow sends $F\colon \pi \to \pi$ to the distribution which sends $f \in C_c^\infty(G, \Lambda)$ to the trace of $v \mapsto \int_G f(g)gF(v)\, d\mu(g)$.

3.11 A Lefschetz fixed-point formula for diamonds with an action of a pro-p group

Suppose we are given a proper morphism $p\colon X \to *$ in $\mathcal{C}$, an action of a locally pro-p group G on X, and a strongly reflexive object $\mathcal{F}_G$ of $D_{\text{ét}}([X/\underline{G}], \Lambda)$ (relative to $*$). Then $R\Gamma(X, \mathcal{F}) = p_!\mathcal{F}$ is an admissible derived representation of G. If μ is a Λ-valued Haar measure on G, then we can talk about the trace distribution $\chi_{\mathcal{F}} \in \mathrm{Dist}(G, \Lambda)$ of $R\Gamma(X, \mathcal{F})$ relative to μ. That is, $\chi_{\mathcal{F}}(f)$ is the trace of the operator $v \mapsto \int_G f(g)gv\, d\mu(g)$ on $R\Gamma(X, \mathcal{F})$. The goal of this discussion is to present a version of the Lefschetz fixed-point formula which computes $\chi_{\mathcal{F}}$ in terms of local terms at fixed points.

For a closed subgroup $K \subset G$, consider the following correspondence c_K on X:

$$\begin{tikzcd} & (X \times \underline{G})/\underline{K} \arrow[dl, "c_{1K}"'] \arrow[dr, "c_{2K}"] & \\ {[X/\underline{K}]} & & {[X/\underline{K}]} \end{tikzcd}$$

where c_{1K} is projection onto the first factor, and c_{2K} is the action map. Let $\mathcal{F}_K$ be the pullback of $\mathcal{F}_J$ to $[X/\underline{K}]$. The descent datum of $\mathcal{F}_K$ back to $[X/\underline{G}]$ is a morphism $v_K = c_{1K}^*\mathcal{F} \to c_{2K}^*\mathcal{F}$. The measure μ determines a functor $t_\mu\colon c_{2K}^* \to c_{2K}^!$ (see Remark 3.4.7), so we get a cohomological correspondence $u_K = t_\mu \circ v_K\colon c_{1K}^*\mathcal{F} \to c_{2K}^!\mathcal{F}$ lying over c_K. When $K \subset G$ is open, $\mathcal{F}_K$ is strongly reflexive (since it is the pullback of $\mathcal{F}_G$ through an étale morphism), and so its trace $\mathrm{tr}_c(u_K) \in H^0(\mathrm{Fix}(c_K), \kappa_{[\mathrm{Fix}(c_K)/\underline{G}]})$ is defined. Finally, let $c = c_{\{1\}}$ and $u = u_{\{1\}}$. By Proposition 3.9.1, the family of $\mathrm{tr}_c(u_K)$ for $K \subset G$ open determine a well-defined element

$$\mathrm{tr}_c(u) \in \varprojlim H^0(\mathrm{Fix}(c_K), \kappa_{\mathrm{Fix}(c_K)/[*/\underline{K}]}) = H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)}).$$

We have that $\mathrm{Fix}(c)$ is the locus of points $(x,g) \in X \times \underline{G}$ with $x^g = x$; let $q\colon \mathrm{Fix}(c) \to \underline{G}$ be the projection. Then $q_!\,\mathrm{tr}_c(u) \in H^0(\underline{G}, \kappa_{\underline{G}}) \cong \mathrm{Dist}(G,\Lambda)$ is a Λ-valued distribution on G.

Proposition 3.11.1. *Assume that X is proper and $\mathcal{F}_G$ is strongly reflexive. Let $\chi_{\mathcal{F}} \in \mathrm{Dist}(G,\Lambda)$ be the trace distribution of the admissible derived ΛG-module $R\Gamma(X,\mathcal{F})$. Then $\chi_{\mathcal{F}} = q_!\,\mathrm{tr}_c(u)$.*

Proof. Let $f \in C_c^\infty(G,\Lambda)$. Since f is smooth, it must be left K-invariant for a compact open subgroup $K \subset G$. Statement (1) is then a repackaging of Proposition 3.9.1 applied to the proper morphism $c_K \to b_K$, where b_K is the correspondence $\underline{G/K} \to [*/\underline{K}] \times [*/\underline{K}]$. ☐

If $\mathcal{F}$ is itself strongly reflexive, then for each $g \in G$ we have a well-defined local term. To wit: let $c_g\colon X \to X \times X$ and $u_g\colon c_{g1}^*\mathcal{F} \to c_{g2}^*\mathcal{F} = c_{g2}^!\mathcal{F}$ be the specializations of c and u at g. Let $X^g = \mathrm{Fix}(c_g)$; then X^g is proper and we have the local term $\mathrm{loc}_{X^g}(u_g) \in \Lambda$.

Proposition 3.11.2. *Assume that $\mathcal{F}$ is strongly reflexive, and also assume that $R\Gamma(U,\mathcal{F})$ and $R\Gamma(U,\Lambda)$ are isomorphic to a bounded complex of finitely generated Λ-modules. In addition, let $G^e \subset G$ be an open subset consisting of elements g for which X^g is finite, so that $\mathrm{Fix}(c)|_{G^e} = \underline{F}$ for a locally* *profinite set F. Then $\mathrm{tr}_c(u)|_{G^e}$, considered as an element of* $\mathrm{Dist}(F,\Lambda)$*, is smooth with respect to $\mu' = q^*\mu$, and for $g \in G^e$ we have*

$$\frac{d\,\mathrm{tr}_c(u)}{d\mu'}(g) = \mathrm{loc}_{X^g}(u_g).$$

Proof. Let $\widetilde{c} = \widetilde{c}_1 \times_{\underline{G}} \widetilde{c}_2 \colon X \times \underline{G} \to (X \times \underline{G}) \times_{\underline{G}} (X \times \underline{G}) = X \times X \times \underline{G}$, where $\widetilde{c}_1$ is the identity and $\widetilde{c}_2(x,g) = (x^g, g)$. Then $\widetilde{c}$ is a correspondence over the base $\underline{G}$, and $\mathrm{Fix}(\widetilde{c}) = \mathrm{Fix}(\widetilde{c})$. Let $q \colon X \times \underline{G} \to X$ be the projection, and let $\widetilde{\mathcal{F}} = q^*\mathcal{F}$.

The object $\mathcal{F}$ is strongly reflexive. Then $\widetilde{\mathcal{F}}$ is strongly reflexive with respect to $\underline{G}$ as in Example 3.6.4.

We have a diagram

$$\begin{tikzcd}
\mathrm{Hom}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \arrow[r, "\mathrm{tr}_c"] & H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)}) \\
\mathrm{Hom}(\widetilde{c}_1^*\widetilde{\mathcal{F}}, \widetilde{c}_2^!\widetilde{\mathcal{F}}) \arrow[u, "(1)"] \arrow[d, "(2)"'] \arrow[r, "\mathrm{tr}_{\widetilde{c}}"] & H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/\underline{G}}) \arrow[u] \arrow[d] \\
\mathrm{Hom}(c_{1g}^*\mathcal{F}, c_{2g}^*\mathcal{F}) \arrow[r, "\mathrm{tr}_{c_g}"'] & H^0(X^g, \kappa_{X^g})
\end{tikzcd} \tag{3.11.1}$$

The morphism (1) is induced from $\widetilde{c}_1^*\widetilde{\mathcal{F}} = \widetilde{c}_1^*q^*\mathcal{F} = c_1^*\mathcal{F}$ and $\widetilde{c}_2^!\widetilde{\mathcal{F}} = \widetilde{c}_2^!q^*\mathcal{F} \to c_2^!\widetilde{\mathcal{F}}$, where the last arrow is adjoint to $c_{2!} \to \widetilde{c}_2^!q^* = q_!\widetilde{c}_{2!}\widetilde{c}_2^!q^* \to q_!q^* \xrightarrow{t_\mu} q_!q^! \to \mathrm{id}$. Let us comment on the commutativity of the above left square; it follows from the commutativity of

$$\begin{tikzcd}
\underline{\mathrm{RHom}}(\widetilde{c}_1^*\widetilde{\mathcal{F}}, \widetilde{c}_2^!\widetilde{\mathcal{F}}) \arrow[r, "\sim"] \arrow[d] & \widetilde{c}^!(\mathbf{D}_{\underline{G}}\widetilde{\mathcal{F}} \boxtimes_{\underline{G}} \widetilde{\mathcal{F}}) \arrow[d] \\
\underline{\mathrm{RHom}}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \arrow[r, "\sim"'] & c^!(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F})
\end{tikzcd}$$

The right vertical arrow uses the fact that $\mathbf{D}_{\underline{G}}\widetilde{\mathcal{F}} = q^*\mathbf{D}\mathcal{F}$ (Example 3.6.4), so that $\widetilde{c}^!(\mathbf{D}_{\underline{G}}\widetilde{\mathcal{F}} \boxtimes_{\underline{G}} \widetilde{\mathcal{F}}) \cong \widetilde{c}^!(q^* \times q^*)(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F})$, and then we use the natural transformation $\widetilde{c}^!(q^* \times q^*) \to c^!$, which is constructed analogously.

Now let $i_g \colon X \to X \times \underline{G}$ be the inclusion along $g \in G$. The morphism (2) comes from

$$\begin{aligned}
\mathrm{Hom}(\widetilde{c}_1\widetilde{\mathcal{F}}, \widetilde{c}_2^!\widetilde{\mathcal{F}}) &\to \mathrm{Hom}(i_g^*\widetilde{c}_1^*q^*\mathcal{F}, i_g^*\widetilde{c}_2^*q^*\mathcal{F}) \\
&\cong \mathrm{Hom}(c_{1g}^*\mathcal{F}, c_{2g}^!\mathcal{F})
\end{aligned}$$

where we have used the fact that $\widetilde{c}_2^! = \widetilde{c}_2^*$ and $\widetilde{c}_{2g}^! = \widetilde{c}_{2g}^*$ since these morphisms are étale. The commutativity of the lower square is routine.

Now let us restrict the objects in (3.11.1) to G^e to get a diagram

$$\begin{tikzcd}
\mathrm{Hom}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \arrow[r, "\mathrm{tr}_c"] & \mathrm{Dist}(F,\Lambda) \\
\mathrm{Hom}(\widetilde{c}_1^*\widetilde{\mathcal{F}}, \widetilde{c}_2^!\widetilde{\mathcal{F}}) \arrow[r, "\mathrm{tr}_{\widetilde{c}}"] \arrow[u, "(1)"] \arrow[d, "(2)"'] & C^\infty(F,\Lambda) \arrow[u, "\mu"'] \arrow[d, "\mathrm{res.}"] \\
\mathrm{Hom}(c_{1g}^*\mathcal{F}, c_{2g}^*\mathcal{F}) \arrow[r, "\mathrm{tr}_{c_g}"'] & C^\infty(X^g,\Lambda)
\end{tikzcd}$$

whose commutativity is the content of the proposition.

□

3.12 Étale pullback of cohomological correspondences

We need one more result about the behavior of correspondences and trace morphisms under étale pullback; the proof is a tedious diagram chase.

Proposition 3.12.1. *Let $[f] = (f, f^\natural, f)\colon c \to b$ be an étale morphism of self-correspondences over S, where $b\colon Y \to X \times_S X$. The map $f'\colon \mathrm{Fix}(c) \to \mathrm{Fix}(b)$ is étale as well, and for a strongly reflexive object $\mathcal{F} \in D_{\text{ét}}(X,\Lambda)$ there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}(b_1^*\mathcal{F}, b_2^!\mathcal{F}) \arrow[r, "\mathrm{tr}_c"] \arrow[d, "{[f]^*}"'] & H^0(\mathrm{Fix}(b), \kappa_{\mathrm{Fix}(b)/S}) \arrow[d] \\
\mathrm{Hom}(c_1^* f^*\mathcal{F}, c_2^! f^*\mathcal{F}) \arrow[r, "\mathrm{tr}_b"'] & H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S})
\end{tikzcd}$$

Here the left vertical arrow is the composition

$$\begin{aligned}
\mathrm{Hom}(b_1^*\mathcal{F}, b_2^!\mathcal{F}) &\to \mathrm{Hom}(f^{\natural *} b_1^*\mathcal{F}, f^{\natural *} b_2^!\mathcal{F}) \\
&= \mathrm{Hom}(f^{\natural *} b_1^*\mathcal{F}, f^{\natural !} b_2^!\mathcal{F}) \\
&= \mathrm{Hom}(c_1^* f^*\mathcal{F}, c_2^! f^!\mathcal{F}) \\
&= \mathrm{Hom}(c_1^* f^*\mathcal{F}, c_2^! f^*\mathcal{F}),
\end{aligned}$$

and the right vertical arrow is

$$\begin{aligned} H^0(\mathrm{Fix}(b), \kappa_{\mathrm{Fix}(b)/S}) &\to H^0(\mathrm{Fix}(b), f'_* f'^* \kappa_{\mathrm{Fix}(b)/S}) \\ &= H^0(\mathrm{Fix}(c), f'^* \kappa_{\mathrm{Fix}(b)/S}) \\ &= H^0(\mathrm{Fix}(c), f'^! \kappa_{\mathrm{Fix}(b)/S}) \\ &= H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S}). \end{aligned}$$

3.13 Continuity of the trace morphism

Suppose $p\colon X \to S$ is a morphism in $\mathcal{C}$. We have

$$H^0(X, \kappa_{X/S}) = \mathrm{Hom}_X(\Lambda, p^!\Lambda) = \mathrm{Hom}_S(p_!\Lambda, \Lambda).$$

For an element $\mu \in H^0(X, \kappa_{X/S})$ and a morphism $N \to p_!\Lambda$ in $D_{\text{ét}}(S, \Lambda)$, let $\mu|_N$ be the image of μ in $\mathrm{Hom}_S(N, \Lambda)$.

The module $H^0(X, \kappa_{X/S})$ has a natural "weak" topology, defined as follows: $\mu_i \to \mu$ if, for every compact object N in $D_{\text{ét}}(S, \Lambda)$ and every morphism $N \to p_!\Lambda$, $(\mu_i)|_N = \mu|_N$ for $i \gg 0$.

Example 3.13.1. Let $p\colon \underline{X} \to *$, where X is a locally profinite set. Then $H^0(\underline{X}, \kappa_X)$ is the space of Λ-valued distributions on X. Under this identification, $\mu_i \to \mu$ if for every $f \in C_c^\infty(X, \Lambda)$, $\int_X f\, d\mu_i = \int_X f\, d\mu$ for $i \gg 0$.

Now suppose $c\colon Y \to X \times X$ is a correspondence over S, and $\mathcal{F}_i$ is a directed system of strongly reflexive (relative to S) objects of $D_{\text{ét}}(X, \Lambda)$, such that $\mathcal{F} = \varinjlim \mathcal{F}_i$ exists and is strongly reflexive. Then for a compatible system of cohomological correspondence $u_i\colon c_1^*\mathcal{F}_i \to c_2^!\mathcal{F}_i$, we can form the colomit u as the composition.

$$u\colon c_1^*\mathcal{F} = \varinjlim c_1^*\mathcal{F}_i \stackrel{u_i}{\to} \varinjlim c_2^!\mathcal{F}_i \to c_2^!\mathcal{F}.$$

Proposition 3.13.2. *The* $\mathrm{tr}_c(u_i)$ *converge to* $\mathrm{tr}_c(u)$ *in* $H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S})$.

Proof. Let $p\colon \mathrm{Fix}(c) \to S$ be the structure map; this factors as $\mathrm{Fix}(c) \stackrel{c'}{\to} X \stackrel{q}{\to} S$. The trace morphism $\mathrm{tr}_c\colon \mathrm{Hom}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \to H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S})$ factors through

$$H^0(\mathrm{Fix}(c), (c')^!(\mathbf{D}_S\mathcal{F} \otimes \mathcal{F})) = \mathrm{Hom}_X(c'_!\Lambda, \mathbf{D}_S\mathcal{F} \otimes \mathcal{F}) \to \mathrm{Hom}_S(p_!\Lambda, q_!(\mathbf{D}_S\mathcal{F} \otimes \mathcal{F})).$$

Now suppose $N \in D_{\text{ét}}(S, \Lambda)$ is compact, and let $N \to p_!\Lambda$ be given; then $u \mapsto \mathrm{tr}_c(u)|_N$ factors through $\mathrm{Hom}_S(N, q_!(\mathbf{D}_S\mathcal{F} \otimes \mathcal{F}) \stackrel{\mathrm{ev}}{\to} \mathrm{Hom}_S(N, q_!q^!\Lambda) \to \mathrm{Hom}_S(N, \Lambda)$. But since $\mathcal{F} = \varinjlim \mathcal{F}_i$ and N is compact, we have

$$\mathrm{Hom}_S(N, p_{X!}(\mathbf{D}_S\mathcal{F} \otimes \mathcal{F})) \cong \varinjlim_i \mathrm{Hom}_S(N, p_{X!}(\mathbf{D}_S\mathcal{F} \otimes \mathcal{F}_i)),$$

which shows that $\mathrm{tr}_c(u)|_N = \mathrm{tr}_c(u_i)|_N$ for $i \gg 0$. $\square$

Example 3.13.3. Let G be a locally pro-p group, and π be an admissible ΛG-module, with associated strongly reflexive object $\mathcal{L}_\pi$ in $D_{\text{ét}}([*/\underline{G}], \Lambda)$. We have $\pi = \varinjlim_K \pi^K$, where $K \subset G$ runs through normal open subgroups, and thus $\mathcal{L}_\pi = \varinjlim_K \mathcal{L}_{\pi^K}$. Consider the correspondence $c = (i, i)\colon * \to [*/\underline{G}] \times [*/\underline{G}]$. Then $t_\mu\colon i^*\mathcal{L}_\pi \to i^!\mathcal{L}_\pi$ is a cohomological correspondence lying over c. In this case Proposition 3.13.2 reduces to the fact that the trace distribution of π^K converges weakly to the trace distribution of π.

3.14 Twisting by étale local systems

Let $c\colon Y \to X \times X$ be a correspondence over a connected base S. Suppose V is a free Λ-module. We can consider V as a constant object in $D_{\text{ét}}(X, \Lambda)$. Assume we are given a c-equivariant structure on V, which is to say a morphism $v\colon c_1^*V \to c_2^*V$. For a point $y \in \mathrm{Fix}(c)$ with image $x \in X$, the fiber of v over y is an endomorphism of V. Thus we see that v induces a morphism $\mathrm{Fix}(c) \to \underline{\mathrm{End}\, V}$. Define $\theta_v \in H^0(\mathrm{Fix}(c), \Lambda)$ as the composition $\mathrm{Fix}(c) \to \underline{\mathrm{End}\, V} \xrightarrow{\mathrm{tr}} \underline{H^0(S, \Lambda)} = \underline{\Lambda}$.

Now suppose $\mathcal{F}$ is a strongly reflexive object in $D_{\text{ét}}(X, \Lambda)$ (relative to S), and let $u\colon c_1^*\mathcal{F} \to c_2^!\mathcal{F}$ be a cohomological correspondence lying over c. We get a new cohomological correspondence $u \otimes v\colon c_1^*(\mathcal{F} \otimes V) \to c_2^!(\mathcal{F} \otimes V)$. (Note that $c_2^!(\mathcal{F} \otimes V) \cong c_2^!\mathcal{F} \otimes c_2^*V$ since V is free.)

Lemma 3.14.1. *In $H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)/S})$ we have* $\mathrm{tr}_c(u \otimes v) = \mathrm{tr}_c(u)\theta_v$

Proof. Let $V^* = \mathrm{Hom}(V, \Lambda)$, considered as an object of $D_{\text{ét}}(X, \Lambda)$. Using the notation of (3.8.1), we observe that the composition

$$\underline{\mathrm{RHom}}(c_1^*V, c_2^*V) \to c^*(V^* \otimes V) \to c^*\Delta_*\Lambda = \Delta'_*\Lambda$$

carries a morphism $w \in \mathrm{Hom}(c_1^*V, c_2^*V)$ to $\theta_w \in H^0(X, \Delta'_*\Lambda) = H^0(\mathrm{Fix}(c), \Lambda)$. The lemma is then reduced to checking the commutativity of the following diagram

$$\begin{tikzcd}
\underline{\mathrm{RHom}}(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \otimes \underline{\mathrm{RHom}}(c_1^*V, c_2^*V) \arrow[r] \arrow[d] & \underline{\mathrm{RHom}}(c_1^*(\mathcal{F}\otimes V), c_2^!(\mathcal{F}\otimes V)) \arrow[d] \\
c^!(\mathbf{D}_S\mathcal{F} \boxtimes \mathcal{F}) \otimes c^*(V^* \boxtimes V) \arrow[r] \arrow[d] & c^!(\mathbf{D}_S(\mathcal{F}\otimes V) \boxtimes (\mathcal{F}\otimes V)) \arrow[d] \\
c^!\Delta_*\kappa_X \otimes c^*\Delta_*\Lambda \arrow[r] \arrow[d] & c^!\Delta_*\kappa_X \arrow[d] \\
\Delta'_*\kappa_{\mathrm{Fix}(c)} \otimes \Delta'_*\Lambda \arrow[r] & \Delta'_*\kappa_{\mathrm{Fix}(c)}
\end{tikzcd}$$

□

As a slight variation on this idea, suppose G is a finite group and suppose that $[f] = (f, f^\natural, f)\colon c \to b$ is a morphism from c to the trivial correspondence on $[*/\underline{G}]$. Let us write $\widetilde{c}\colon \widetilde{Y} \to \widetilde{X} \times \widetilde{X}$ for the pullback of this correspondence along $* \to [*/\underline{G}]$.

In this situation we can give a morphism $i\colon \mathrm{Fix}(c) \to \underline{\langle G\rangle}$, where $\langle G\rangle$ is the set of conjugacy classes of G. It is defined as follows: For a geometric point $y \to \mathrm{Fix}(c)$, let $\widetilde{y} \to \widetilde{Y}$ be a geometric point in its preimage; then there exists a unique $g \in G$ such that $c_1(\widetilde{y})^g = c_2(\widetilde{y})$, and then $i(y) = \langle g\rangle$ is well-defined.

Let π be an object of $D(\Lambda G\text{-mod})$ whose underlying Λ-module is perfect. We have a morphism $v\colon c_1^* f^*\mathcal{L}_\pi \to c_2^* f^*\mathcal{L}_\pi$, namely the identity on $f^{\natural *}\mathcal{L}_\pi$. Let $\theta_v \in H^0(\mathrm{Fix}(c), \Lambda)$ be the map $y \mapsto \mathrm{tr}\,\pi(i(y))$. For a cohomological correspondence u over c we have the twist $u \otimes v$.

Lemma 3.14.2. *In $H^0(\mathrm{Fix}(c), \kappa_{\mathrm{Fix}(c)})$ we have $\mathrm{tr}_c(u\otimes v) = \mathrm{tr}_c(u)\theta_v$.*

Proof. $\mathrm{tr}_c(u)$ can be computed étale locally on X. After passing to the étale cover $\widetilde{X} \to X$, $f^*\mathcal{L}_\pi$ becomes constant, and then we can apply Lemma 3.14.1 □

3.15 Twisting by pro-étale local systems

Let G be a locally pro-p group, let $f\colon X \to [*/\underline{G}]$ be a morphism of class $\mathcal{C}$. For an admissible ΛG-module π we have the object $f^*\mathcal{L}_\pi$, a pro-étale local system on X.

Example 3.15.1. We give an example of a reflexive pro-étale local system on a diamond which has infinite fibers. Let D/C be the rigid unit disk around 1, considered as a group under multiplication, and let $\widetilde{D} = \varprojlim_p D$. Then $\widetilde{D}^* = \widetilde{D}\backslash\{0\}$ is a perfectoid space admitting a continuous action of $\mathbf{Q}_p^\times$ without geometric fixed points, so that $X = \widetilde{D}^*/\underline{\mathbf{Q}}_p^\times$ is a diamond. (This diamond appears in [Wei16]). Let $\Lambda = \overline{\mathbf{F}}_\ell$, and let π be an infinite-dimensional admissible representation of $\mathbf{Z}_p^\times$ on a Λ-vector space (namely, the direct sum of a sequence of characters of increasing conductor). Let $\mathcal{L}_\pi = f^*\pi$, where $f\colon X \to [*/\underline{\mathbf{Z}}_p^\times]$ corresponds to the pro-étale $\underline{\mathbf{Z}}_p$-torsor $\widetilde{D}^*/p^{\mathbf{Z}} \to X$. We claim that $\mathcal{L}_\pi$ is reflexive.

For $U \in (\widetilde{D}^*/\mathbf{Q}_p^\times)_{\text{ét}}$ qcqs, let $U' \in (\widetilde{D}^*/p^{\mathbf{Z}})_{\text{ét}}$ be its preimage, a qcqs perfectoid space with a continuous $\mathbf{Z}_p^\times$-action. Then U' is the completed perfection of a quasi-compact rigid space. By [Hub96, Proposition 6.1.1], each $H^i(U'_{\text{ét}}, \Lambda)$ is finite, and so there exists an open subgroup $H \subset \mathbf{Z}_p^\times$ which acts trivially on each one. From this one can deduce that $\mathcal{L}_\pi \cong \varprojlim_{H\subset\mathbf{Z}_p^\times} \mathcal{L}_{\pi^H}$; where the transition maps are the projections. Therefore $\mathbf{D}\mathcal{L}_\pi \cong \varinjlim_H \mathbf{D}\mathcal{L}_{\pi^H} \cong \varinjlim_H \mathcal{L}_{(\pi^\vee)^H}[2](1) \cong \mathcal{L}_{\pi^\vee}[2](1)$, and so $\mathbf{D}\mathbf{D}\mathcal{L}_\pi \cong \mathcal{L}_\pi$.

Returning to the more general scenario, suppose that G is a pro-p group. Let μ be a Λ-valued invariant measure on G. Suppose that $[f] = (f, f^\natural, f)\colon c \to b$ is a class $\mathcal{C}$ morphism from a correspondence $c\colon Y \to X \times X$ to the trivial correspondence b on $[*/\underline{G}]$. As in the previous section, we have a map $\iota\colon \operatorname{Fix}(c) \to \underline{\langle G\rangle}$, where $\langle G\rangle$ is the set of conjugacy classes of G. Let π be a derived admissible Λ-module, so that for every open normal subgroup $K \subset G$, π^K is a perfect complex. Let $\theta_{\pi^K} \in H^0(\operatorname{Fix}(c), \Lambda)$ be the composition

$$\theta_{\pi^K}\colon \operatorname{Fix}(c) \xrightarrow{\iota} \underline{\langle G\rangle} \xrightarrow{g\mapsto \operatorname{tr}\pi(1_{Kg})} \underline{\Lambda}.$$

Let $\mathcal{F}$ be a strongly reflexive object in $D_{\text{ét}}(X,\Lambda)$, and let $u\colon c_1^*\mathcal{F} \to c_2^!\mathcal{F}$ be a cohomological correspondence. We can "twist u by π" to obtain a cohomological correspondence $u_\pi\colon c_1^*(\mathcal{F}\otimes f^*\mathcal{L}_\pi) \to c_2^!(\mathcal{F}\otimes f^*\mathcal{L}_\pi)$. Explicitly, u_π is the composition

$$\begin{aligned}
c_1^*(\mathcal{F} \otimes f^*\mathcal{L}_\pi) &= \varinjlim_K c_1^*(\mathcal{F} \otimes f^*\mathcal{L}_{\pi^K}) \\
&\cong \varinjlim_K c_1^*\mathcal{F} \otimes f^{\natural *}\mathcal{L}_{\pi^K} \\
&\overset{u\otimes f^{\natural *}(1)}{\to} \varinjlim_K c_2^!\mathcal{F} \otimes f^{\natural *}\mathcal{L}_{\pi^K} \\
&\cong \varinjlim_K c_2^!(\mathcal{F} \otimes f^*\mathcal{L}_{\pi^K}) \\
&\to c_2^!(\mathcal{F} \otimes f^*\mathcal{L}_\pi).
\end{aligned}$$

Proposition 3.15.2. *Assume that $\mathcal{F} \otimes f^*\mathcal{L}_\pi$ is strongly reflexive. Then $\mathrm{tr}_c(u)\theta_{\pi^K}$ weakly converges to $\mathrm{tr}_c(u_\pi)$, where K runs over open normal subgroups of G.*

Proof. In $D(\Lambda G)$ we can write $\pi = \varinjlim_K \pi^K$, where K runs over open normal subgroups of G. Then $\mathrm{tr}_c(u_{\pi^K}) \to \mathrm{tr}_c(\pi)$ by Proposition 3.13.2; by Lemma 3.14.2 $\mathrm{tr}_c(u_{\pi^K}) = \mathrm{tr}_c(u)\theta_{\pi^K}$. $\square$

3.16 An application

We close the section with a corollary involving the following set-up, which will be used in 5.5:

- G and J, two locally pro-p groups endowed with invariant Λ-valued measures μ and ν, respectively.
- X, a spatial diamond proper over $\mathrm{Spa}\, C$ admitting an action of $\underline{J}$. Let $c_1\colon X \times \underline{J} \to X$ be the projection, and $c_2\colon X \times \underline{J} \to X$ be the action map.
- $\mathcal{F}$, a strongly reflexive object of $D_{\text{ét}}(X, \Lambda)$ which is $\underline{J}$-equivariant. That is, we have an isomorphism $v\colon c_1^*\mathcal{F} \to c_2^*\mathcal{F}$.
- $j\colon X^a \hookrightarrow X$ an open subdiamond, and $\widetilde{X} \to X^a$ a pro-étale $\underline{G}$-torsor which is also $\underline{J}$-equivariant; this corresponds to a morphism $p\colon X^a \to [*/\underline{G}]$.
- π, an admissible derived ΛG-module.
- $\mathcal{F}_\pi = \mathcal{F} \otimes j_! p^*\mathcal{L}_\pi$, an object which admits a $\underline{J}$-equivariance $v_\pi = v \otimes p^*(1)\colon c_1^*\mathcal{F}_\pi \to c_2^*\mathcal{F}_\pi$. The descent of $\mathcal{F}_\pi$ to $[X/\underline{J}]$ is assumed to be strongly reflexive (relative to $*$).

- $J^e \subset J$, an open subset having the property that for all $\gamma \in J^e$, X^γ is a finite subset of X^a.
- For $\gamma \in J^e$ and $x \in X^\gamma$ let $i_x(\gamma) \in \langle G \rangle$ be the conjugacy class by which γ acts on the fiber $\widetilde{X}_x$.
- Assume that the Harish-Chandra character Θ_π of π is well-defined at each $i_x(\gamma)$. That is, if $f_i \in C(G, \Lambda)$ is a sequence of functions for which $f_i \mu$ weakly converges to the Dirac distribution at $i_x(\gamma)$, then the sequence $\operatorname{tr}(f_i|\pi)$ converges to $\Theta_\pi(i_x(\gamma))$.

Under these circumstances $R\Gamma(X, \mathcal{F}_\pi)$ is a derived admissible ΛG-module.

Theorem 3.16.1. *Let χ be the trace distribution of the admissible derived ΛJ-module $R\Gamma(X, \mathcal{F}_\pi)$. Then $\chi|_{J^e}$ is ν-smooth, and for $\gamma \in J^e$ we have*

$$\frac{d}{d\nu}\chi(\gamma) = \sum_{x \in X^\gamma} \operatorname{loc}_x(u_\gamma)\Theta_\pi(i_x(\gamma)).$$

Proof. The distribution ν induces a natural transformation $t_\nu \colon c_2^* \to c_2^!$. Let $u = t_\nu \circ v$ and $u_\pi = t_\nu \circ v_\pi$, so that u and u_π are cohomological correspondences over c.

By Proposition 3.11.1, the trace distribution of $R\Gamma(X, \mathcal{F}_\pi)$ equals $p_! \operatorname{tr}_c(u_\pi)$, where $p\colon \operatorname{Fix}(c) \to \underline{J}$ is the projection. Let $p^e\colon \operatorname{Fix}(c)^e \to \underline{J}^e$ be the base change of p to $\underline{J}^e$. By hypothesis, $\operatorname{Fix}(c)^e \subset X^a \times \underline{J}^e$, and p^e is étale, so that

$$\operatorname{Fix}(c)^e = \left\{ (x, \gamma) \in X^a \times J^e \;\middle|\; x^\gamma = x \right\}$$

is a locally profinite set. Our goal is to use the results of 3.9 and 3.10 to calculate the restriction of $\operatorname{tr}_c(u_\pi)$ to $\operatorname{Fix}(c)^e$. Let $\nu' = (p^e)^*\nu$.

Let $(x, \gamma) \in \operatorname{Fix}(c)^e$, and let $\widetilde{x} \in \widetilde{X}_x$. Let $g \in G$ be such that $\widetilde{x}^g = \widetilde{x}^\gamma$, so that $i_x(\gamma)$ is the conjugacy class of g. Let $H, H' \subset G$ be open pro-p subgroups such that $H', g^{-1}H'g \subset H$. Let $X_H = \widetilde{X}^a/H$, $X_{H'} = \widetilde{X}^a/H'$. We have an étale morphism of correspondences $f = (f, f^\natural, f)\colon c_H \to c$, where c_H is the correspondence

$$\begin{tikzcd}[column sep=small] & X_{H'} \times \underline{J}^e \arrow[dl, "c_{H1}"'] \arrow[dr, "c_{H2}"] & \\ X_H & & X_H, \end{tikzcd}$$

c_{H1} is the composition of projections $X_{H'} \times \underline{J}^e \to X_{H'} \to X_H$, and c_{H2} is the composition of the action map $X_{H'} \times \underline{J}^e \to X_{H'}$ with the map $X_{H'} \xrightarrow{g} X_{g^{-1}H'g} \to X_H$. Let $\mathrm{Fix}(c_H)^e$ be the subset of $\mathrm{Fix}(c_H)$ lying over $\underline{J}^e$; then $\mathrm{Fix}(c_H)^e$ is a locally profinite set and $\mathrm{Fix}(c_H)^e \to \mathrm{Fix}(c)^e$ is étale. Let x_H be the image of $\widetilde{x}$ in X_H, so that $(x_H, \gamma) \in \mathrm{Fix}(c)^e$.

Let $p_H \colon X_H \to [*/\underline{H}]$ correspond to the pro-étale $\underline{H}$-torsor $\widetilde{X} \to X_H$. We have $f^*\mathcal{L}_\pi = \varinjlim_{K \subset H} p_H^* \mathcal{L}_{\pi^K}$, where K runs over open normal subgroups of H. The cohomological correspondences u and u_π over c lift to cohomological correspondences f^*u and f^*u_π over c_H as in Proposition 3.12.1. For $(y, \delta) \in \mathrm{Fix}(c_H)$, we have a well-defined class $i_y(\delta) \in \langle H \rangle$ giving the action of δ on the fiber $\widetilde{X}_y$. By Proposition 3.15.2, $\mathrm{tr}_{c_H}(f^*u)\theta_{\pi^K} \to \mathrm{tr}_{c_H}(f^*u_\pi)$ in $H^0(\mathrm{Fix}(c_H), \kappa_{\mathrm{Fix}(c_H)})$, where $\theta_{\pi^K}(y, \delta) = \mathrm{tr}\, \pi^K(i_y(\delta))$.

On the other hand, the existence of the Harish-Chandra character shows that over $\mathrm{Fix}(c_H)^e$ we have $\theta_{\pi^K} \to \Theta_\pi \circ i$. Therefore

$$\mathrm{tr}_c(u_\pi) = \mathrm{tr}_{c_H}(f^*u)(\Theta_\pi \circ i) = \mathrm{tr}_c(u)(\Theta_\pi \circ i).$$

Finally we apply Proposition 3.11.2: Since $\mathcal{F}$ is strongly reflexive on X, $\mathrm{tr}_c(u)|_{\mathrm{Fix}(c)^e}$ is ν'-smooth, and $(d/d\nu')\, \mathrm{tr}_c(u)(x, \gamma) = \mathrm{loc}_x(u_\gamma)\Theta_\pi(g)$. Therefore $\mathrm{tr}_c(u_\pi)|_{\mathrm{Fix}(c)^e}$ is also ν'-smooth, and

$$\mathrm{tr}_c(u_\pi)(x, \gamma) = \mathrm{loc}_x(u_\gamma)\Theta_\pi(g).$$

Pushing this identity forward along $p_!$ gives the result. $\square$

4 Geometric Langlands for a p-adic field

4.1 The B_{dR}^+-affine Grassmannian

In this section we recall the B_{dR}^+-affine Grassmannian from [SW14]. Recall that the usual affine Grassmannian for a reductive group G over a field k is the ind-scheme representing the functor $R \mapsto G(R(\!(t)\!))/G(R[\![t]\!])$ for a k-algebra R. In the B_{dR}^+-version, the power series ring $R(\!(t)\!)$ is replaced with the de Rham period ring $B_{\mathrm{dR}}(R)$, which we review below.

Recall that F is our fixed p-adic field. Write $\mathcal{O}_F$ for its valuation ring and $\mathbf{F}_q$ for its residue field. For a perfect $\mathbf{F}_q$-algebra R, we have the $\mathcal{O}_F$-Witt vectors $W_{\mathcal{O}_F}(R) = W(R) \otimes_{W(\mathbf{F}_q)} \mathcal{O}_F$.

Definition 4.1.1. Let R be a perfectoid F-algebra, and let $\theta \colon W_{\mathcal{O}_F}(R^\flat)[1/p] \to R$ be the usual map; let $\xi \in W_{\mathcal{O}_F}(R^\flat)$ generate the kernel. Let $B_{\mathrm{dR}}^+(R)$ be the ξ-adic completion of $W_{\mathcal{O}_F}(R^\flat)[1/p]$, and let $B_{\mathrm{dR}}(R) = B_{\mathrm{dR}}^+(R)[1/\xi]$.

In the case that $R = C$ is a perfectoid field, $B^+_{\mathrm{dR}}(C)$ is a discrete valuation ring with uniformizer ξ, which contains an algebraic closure of F. Therefore we have the Cartan decomposition

$$G(B^+_{\mathrm{dR}}(C))\backslash G(B_{\mathrm{dR}}(C))/G(B^+_{\mathrm{dR}}(C)) \xrightarrow{\sim} X_*(T)/W,$$

where T is any maximal torus of G defined over $\bar{F}$ and W is its Weyl group. Since any two maximal tori of G defined over $\bar{F}$ are conjugate in $G(\bar{F}) \subset G(B^+_{\mathrm{dR}})$, this decomposition is independent of the choice of maximal torus T. We can thus take T to be the universal maximal torus, which is part of the universal Borel pair (T, B) (i.e. the limit over all Borel pairs of G).

Definition 4.1.2. Let C be an algebraically closed perfectoid field containing F and let $\mu \in X_*(T)/W$. Define sheaves Gr_G and $\mathrm{Gr}_{G,\mu}$ on Perf_C as follows.

1. Let Gr_G be the sheaf on Perf_C associated to the presheaf

$$\mathrm{Spa}(R, R^+) \mapsto G(B_{\mathrm{dR}}(R))/G(B^+_{\mathrm{dR}}(R)).$$

2. Let $\mathrm{Gr}_{G,\leq\mu}$ be the subsheaf of Gr_G whose points over (R, R^+) consist of those $L \in \mathrm{Gr}_G(R, R^+)$ such that, for every geometric point $x\colon \mathrm{Spa}(C', (C')^\circ) \to \mathrm{Spa}(R, R^+)$, the pullback $x^*L \in \mathrm{Gr}_G(C', (C')^\circ)$ corresponds under the Cartain decomposition to $\lambda \in X_*(T)/W$ with $\lambda \leq \mu$.

It is possible to define Gr_G over $\mathrm{Spd}\, F$ and $\mathrm{Gr}_{G,\leq\mu}$ over $\mathrm{Spd}\, E$ (where E is a reflex field for μ), but since we are not considering the action of the Weil group of F, we are content with working over $\mathrm{Spd}\, C$.

Theorem 4.1.3 ([SW14, Theorem 21.3.6])**.** *$\mathrm{Gr}_{G,\leq\mu}$ is a proper spatial diamond.*

Example 4.1.4. If $G = T$ is a torus, then there is an isomorphism of diamonds over $\mathrm{Spd}\, C$:

$$\begin{array}{rcl} \underline{X_*(T)} & \to & \mathrm{Gr}_T \\ \nu & \mapsto & L_\nu := \nu(\xi)T(B^+_{\mathrm{dR}}). \end{array}$$

4.2 Geometric Satake equivalence

In this section we explain Assumption 2 of Theorem 1.0.4. Essentially, it is that the geometric Satake equivalence of [MV07] holds for the B_{dR}^+-Grassmannian Gr. Note that [Zhu17] introduces a different sort of mixed-characteristic affine Grassmannian and proves a geometric Satake equivalence for it, but unfortunately it does not seem obvious to us how to derive what we need directly from this.

Let $\ell \neq p$, and let $\overline{\mathbf{Q}}_\ell$-Vect be the category of vector spaces over $\overline{\mathbf{Q}}_\ell$. Let $\mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G)$ denote the category of perverse $\overline{\mathbf{Q}}_\ell$-sheaves on $\mathrm{Gr}_{G,\text{ét}}$ which are $G(B_{\mathrm{dR}}^+)$-equivariant, and let

$$\begin{aligned} H\colon \mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G) &\to \overline{\mathbf{Q}}_\ell\text{-Vect} \\ \mathcal{F} &\to \bigoplus_{i\in\mathbf{Z}} H^i(\mathrm{Gr}_{G,\text{ét}}, \mathcal{F}) \end{aligned}$$

be the global cohomology functor.

Assumption 2 is the following:

1. $\mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G)$ is preserved under the convolution product. The convolution and fusion products are equivalent, so that $\mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G)$ is even a commutative tensor category.

2. H is an exact faithful tensor functor. By Tannakian duality, $\widehat{G} = \mathrm{Aut}^{\otimes} H$ is a pro-algebraic group over $\overline{\mathbf{Q}}_\ell$, for which there is an exact tensor equivalence $\mathrm{Rep}_{\widehat{G}} \cong \mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G)$ which pulls back H to the canonical fiber functor on $\mathrm{Rep}_{\widehat{G}}$.

3. Given $\mu \in X_*(T)$, let r_μ be the algebraic representation of $\mathrm{Rep}_{\widehat{G}}$ with highest weight μ, let L_μ be the image of μ under $X_*(T) \to \mathrm{Gr}_T \to \mathrm{Gr}_G$, and let IC_μ be the intersection complex associated to the $G(B_{\mathrm{dR}}^+)$-orbit of L_μ. The above tensor equivalence carries IC_μ onto r_μ.

4. (The weight decomposition, [MV07, Theorems 3.5 and 3.6].) Fix a Borel pair (T, B) of G_C and write $B = TN$. For $\nu \in X_*(T)$, let $S_\nu \subset \mathrm{Gr}_G$ denote the $N(B_{\mathrm{dR}})$-orbit of L_ν. This is an ind-diamond; its intersection with any $\mathrm{Gr}_{\leq\mu}$ is a diamond. We have

$$\overline{S}_\nu = \bigcup_{\nu' \leq \nu} S_{\nu'}$$

[MV07, Proposition 3.1(a)].

For $\mathcal{F} \in \mathcal{P}_{G(B_{\mathrm{dR}}^+)}$, the cohomology $H_c^i(S_\nu, \mathcal{F})$ is zero unless $i = 2\rho(\nu)$. We have a natural equivalence of functors $\mathcal{P}_{G(B_{\mathrm{dR}}^+)} \to \overline{\mathbf{Q}}_\ell$-Vect:

$$H \cong \bigoplus_{\nu \in X_*(T)} H_c^{2\rho(\nu)}(S_\nu, -) \colon \mathcal{P}_{G(B_{\mathrm{dR}}^+)}(\mathrm{Gr}_G) \to \overline{\mathbf{Q}}_\ell\text{-Vect}$$

The ν-weight space of H is $H_c^{2\rho(\nu)}(S_\nu, -)$.

5. (The Demazure resolution, [NP01, Proposition 9.4].) $\mathrm{Gr}_{G,\leq\mu}$ is not smooth over $\operatorname{Spa} C$ for general non-minuscule μ, but there exists a morphism $r\colon \mathrm{Gr}'_{G,\leq\mu} \to \mathrm{Gr}_{G,\leq\mu}$ from a smooth proper diamond, such that r is an isomorphism over the open cell $\mathrm{Gr}_{G,\mu}$. Let Λ be a prime-to-p torsion ring, and let $\mathrm{IC}_{\mu,\Lambda}$ be the intersection complex with coefficients in Λ. Then $\mathrm{IC}_{\mu,\Lambda}$ is a direct summand of $r_*\Lambda$.

We need the last item for the following lemma.

Lemma 4.2.1. *Let $U \to \mathrm{Gr}_{G,\leq\mu}$ be an étale morphism from a quasi-compact diamond. Then each $R\Gamma_c(U_{\text{ét}}, \mathrm{IC}_{\mu,\Lambda})$ and $R\Gamma_c(U_{\text{ét}}, \Lambda)$ are isomorphic to bounded complexes of finitely generated Λ-modules.*

Proof. Considering item (5) above, $R\Gamma_c(U_{\text{ét}}, \mathrm{IC}_{\mu,\Lambda})$ is a summand of $R\Gamma_c(U', \Lambda)$, where U' is the pullback of U to $\mathrm{Gr}'_{G,\leq\mu}$; we can then apply the general fact f is any smooth morphism of spatial diamonds, then $f_!$ preserves constructible sheaves; this follows from [Sch17, Proposition 18.9.ii].

Concerning the statement about $R\Gamma_c(U_{\text{ét}}, \Lambda)$: the statement is true for minuscule μ by smoothness, and the general case proceeds by induction on μ. Let $V = U \cap \mathrm{Gr}_{G,\mu}$ and $Z = U \backslash V$, so that there is an exact triangle $R\Gamma_c(V, \Lambda) \to R\Gamma_c(U, \Lambda) \to R\Gamma_c(Z, \Lambda)$. $R\Gamma_c(V, \Lambda)$ is finitely generated by smoothness, and since $Z = \cup_{\lambda<\mu} Z_\lambda$, where Z_λ is quasicompact in $\mathrm{Gr}_{G,\lambda,\text{ét}}$, we can apply the induction hypothesis to show the same for $R\Gamma_c(Z, \Lambda)$. $\square$

4.3 The Fargues-Fontaine curve

The following is a review of [Far, §1]. All of the following constructions are relative to our local field F (which is E in [Far]); often we will suppress the F from the notation. We remark that all of the constructions have analogues in the case that F has equal characteristic.

Let $F_0 \subset F$ denote the maximal subfield which is unramified over $\mathbf{Q}_p$. Choose a uniformizer $\pi_F \in \mathcal{O}_F$.

For a perfectoid $\mathbf{F}_q$-algebra R with pseudo-uniformizer ϖ, we define

$$\begin{aligned}\mathcal{Y}_R &= \operatorname{Spa} W_{\mathcal{O}_F}(R^\circ)\backslash\{\pi_F[\varpi]=0\}\\ \mathcal{X}_R &= Y_R/\phi_R^{\mathbf{Z}},\end{aligned}$$

where $\phi_R\colon R\to R$ is the qth power Frobenius automorphism. $\mathcal{X}_R$ is the *adic Fargues-Fontaine curve.* We also need the schematic Fargues-Fontaine curve X_R, defined as

$$X_R=\operatorname{Proj}\bigoplus_{d=0}^{\infty} H^0(\mathcal{Y}_R,\mathcal{O}_{\mathcal{Y}_R})^{\phi_R=\pi_F^d}.$$

When R is a perfectoid field, X_R is a noetherian scheme of degree 1.

Now suppose $R^\sharp$ is a perfectoid F-algebra equipped with an isomorphism ι of $\mathbf{F}_q$-algebras $R^{\sharp\flat}\cong R$. The pair $(R^\sharp,\iota)$ is called an *untilt* of R. The kernel of the homomorphism $\theta\colon W(R^\circ)[1/p]\to R^\sharp$ is a primitive element ξ of degree 1, which determines a Cartier divisor $D_{R^\sharp}\hookrightarrow X_R$. The completion of X_R along $D_{R^\sharp}$ is $\operatorname{Spf} B_{dR}^+(R^\sharp)$ [Far, Proposition 1.33].

4.4 Kottwitz's theory of σ-conjugacy classes

We recall here some facts about Kottwitz's set $B(G)$ of σ-conjugacy classes in $G(\breve{F})$ [Kot85]. Associated to such a class $[b]\in B(G)$ there are two invariants: $\kappa([b])\in\pi_1(G)_\Gamma$ and $\nu_{[b]}\in\bar{C}_{\mathbf{Q}}$. Let us explain the notation.

First, $\pi_1(G)$ is the algebraic fundamental group introduced by Borovoi [Bor98]. It can be described as follows. Given a maximal torus $S\subset G$ defined over F let S_{sc} be the preimage of S in the simply connected cover of the derived subgroup of G. The natural map $S_{\mathrm{sc}}\to S$ induces an injective group homomorphism $X_*(S_{\mathrm{sc}})\to X_*(S)$. If S' is another maximal torus of G defined over F, then the isomorphism $X_*(S)/X_*(S_{\mathrm{sc}})\to X_*(S')/X_*(S'_{\mathrm{sc}})$ induced by conjugation by any $g\in G(\bar{F})$ with $gSg^{-1}=S'$ is independent of the choice of g, and in particular Γ-equivariant. Taking the limit over all possible maximal tori S of the quotient $X_*(S)/X_*(S_{\mathrm{sc}})$ thus gives a finitely generated abelian group with Γ-action, and this is the definition of $\pi_1(G)$. The assignment of $\kappa([b])\in\pi_1(G)_\Gamma$ to $[b]$ is explained in [Kot97, §4.9,§7.5]. It produces a map

$$\kappa: B(G)\to\pi_1(G)_\Gamma$$

functorial with respect to all homomorphisms of reductive groups. Note that if $\widehat{S}$ is the torus dual to S we have $X_*(S)/X_*(S_{\mathrm{sc}})=X^*(\widehat{S})/X^*([\widehat{S}]_{\mathrm{ad}})=X^*(Z(\widehat{G}))$ and in this way one obtains $\pi_1(G)_\Gamma=X^*(Z(\widehat{G})^\Gamma)$. The invariant

$\kappa([b])$ was initially taken to lie in $X^*(Z(\widehat{G})^\Gamma)$, but following [RR96] we use $\pi_1(G)_\Gamma$ instead because it is obviously functorial in G, while the functoriality of $X^*(Z(\widehat{G}))$ is less obvious, due to the fact that $G \mapsto \widehat{G}$ is functorial only with respect to homomorphisms with normal image.

Next we turn to the *Newton point* $\nu_{[b]} \in \bar{C}_{\mathbf{Q}}$. Let $\mathbb{D}$ be the pro-torus determined by $X^*(\mathbb{D}) = \mathbf{Q}$. Then associated to $[b]$ is a σ-stable $G(\breve{F})$-conjugacy class of homomorphisms $\mathbb{D} \to G$ defined over $\breve{F}$, called *slope morphisms*. There are different ways to think about them. One way, following [Kot85, §4.2] is to obtain from a representative b of $[b]$ and a finite-dimensional rational representation $\rho : G \to \mathrm{GL}(V)$ the structure of a σ-$\breve{F}$-space on V (an isocrystal in the classical sense when $F = \mathbf{Q}_p$) and use its slope decomposition to obtain a homomorphism from $\mathbb{D}$ into $\mathrm{GL}(V)$ defined over $\breve{F}$. For varying ρ these homomorphisms splice together to a homomorphism $\mathbb{D} \to G$ defined over $\breve{F}$. Varying the representative b replaces this homomorphism by a $G(\breve{F})$-conjugate.

A second way to think about the slope morphism is via the reinterpretation of $B(G)$ as the set of cohomology classes of algebraic 1-cocycles of the Galois gerbe $1 \to \mathbb{D} \to \mathcal{E} \to \Gamma$ with values in $G(\bar{F})$, alluded to in §2.3. The restriction of such a 1-cocycle to $\mathbb{D}$ is by definition an algebraic homomorphism $\mathbb{D} \to G$ defined over $\bar{F}$.

In either interpretation, we may compose the slope morphism with $\psi^{-1} : G \to G^*$, for some inner twist $\psi \in \Psi$ (which can be chosen to be defined over F^{ur} by Steinberg's theorem on the vanishing of $H^1(F^{\mathrm{ur}}, G^*(\bar{F}))$), and thereby obtain a σ-stable $G^*(\breve{F})$-conjugacy class of morphisms $\mathbb{D} \to G^*$ defined over $\breve{F}$, respectively a Γ-stable $G^*(\bar{F})$-conjugacy class of morphisms $\mathbb{D} \to G^*$ defined over $\bar{F}$. Fix a Borel pair (T, B) of G^* defined over F. Up to conjugation such a morphism can be arranged to take values in T. Then it corresponds to a map $X^*(T) \to X^*(\mathbb{D}) = \mathbf{Q}$, hence to an element of $X_*(T) \otimes \mathbf{Q}$. Up to further conjugation by the Weyl group of T this element can be made B-dominant. This B-dominant element is then unique. It is therefore Γ-stable. Let $\bar{C}_{T,\mathbf{Q}}$ be the subset of B-dominant elements in $[X_*(T) \otimes \mathbf{Q}]^\Gamma = X_*(T)^\Gamma \otimes \mathbf{Q} = X_*(A_T) \otimes \mathbf{Q}$, where $A_T \subset T$ is the maximal split torus in T. If another Borel pair (T', B') is chosen, there exists $g \in G^*(F)$ such that $g(T, B)g^{-1} = (T', B')$ and such a g is well defined up to right multiplication by $T(F)$, so the induced Γ-equivariant isomorphism $X_*(T) \to X_*(T')$ is independent of g. It further induces an isomorphism $\bar{C}_{T,\mathbf{Q}} \to \bar{C}_{T',\mathbf{Q}}$ and we take $\bar{C}_{\mathbf{Q}}$ to be limit over all possible (T, B). Note that $\bar{C}_{\mathbf{Q}}$ can alternatively be described as the limit over all possible T of the quotients $X_*(A_T) \otimes \mathbf{Q}/W(T)(F)$, where $W(T)$ is the Weyl group of T.

We have thus described both invariants $\kappa([b]) \in \pi_1(G)_\Gamma$ and $\nu_{[b]} \in \bar{C}_{\mathbf{Q}}$ of an element $[b] \in B(G)$. The inclusion $Z(G^*) \to G^*$ induces a map $X_*(Z(G^*)) \otimes \mathbf{Q} \to \bar{C}_{\mathbf{Q}}$. The elements $[b] \in B(G)$ for which $\nu_{[b]}$ lies in the image of this map are called *basic* and the subset of $B(G)$ consisting of basic elements is denoted by $B(G)_{\mathrm{bas}}$. Equivalently, an element $[b]$ is basic if its slope morphism $\mathbb{D} \to G$ takes values in $Z(G)$. Kottwitz shows [Kot85, Proposition 5.6] that $B(G)_{\mathrm{bas}}$ is a section of κ, that is, $\kappa : B(G)_{\mathrm{bas}} \to \pi_1(G)_\Gamma$ is a bijection. Furthermore, the product map $\kappa \times \nu : B(G) \to \pi_1(G)_\Gamma \times \bar{C}_{\mathbf{Q}}$ is injective [Kot97, §4.13].

Let now $\mu \in X_*(T)_\Gamma$. Kottwitz defines in [Kot97, §6] a subset $B(G,\mu) \subset B(G)$ as follows. Let $\mu_1 \in \pi_1(G)_\Gamma$ be the image of μ under $X_*(T)_\Gamma = \pi_1(T)_\Gamma \to \pi_1(G)_\Gamma$. Let $\mu_2 \in X_*(A_T) \otimes \mathbf{Q}$ be the image of μ under the normalized norm map $X_*(T)_\Gamma \to X_*(T)^\Gamma \otimes \mathbf{Q}$. The $W(T)(F)$-orbit of μ_2 contains a unique member of $\bar{C}_{\mathbf{Q}}$, which we take in place of μ_2. Then $B(G,\mu)$ is the subset of $B(G)$ consisting of those $[b]$ for which $\kappa([b]) = \mu_1$ and $\nu_{[b]} \leq \mu_2$. The latter condition can be reformulated by saying that $\nu_{[b]}$ lies in the convex hull of the Weyl orbit of μ_2. The subset $B(G,\mu)$ contains a unique basic element.

4.5 Vector bundles and G-bundles

Here we review the theory of vector bundles and G-bundles on the Fargues-Fontaine curve, developed in the absolute setting in [FF] and [Far15]. As usual $F/\mathbf{Q}_p$ is a finite extension with residue field $\mathbf{F}_q$ and uniformizer π, and $\breve{F}$ is the completion of a maximal unramified extension of F.

Let $\sigma\colon \breve{F} \to \breve{F}$ be the Frobenius automorphism induced by the qth power Frobenius map on $\overline{\mathbf{F}}_q$. Recall that an *isocrystal* is a pair (N, ϕ_N), where N is a finite-dimensional $\breve{F}$-vector space N and $\phi_N\colon N \to N$ is a σ-linear automorphism. By the Dieudonné-Manin classification, every isocrystal (N, ϕ_N) admits a canonical $\mathbf{Q}$-grading: $(N, \phi_N) \cong \bigoplus_{\lambda \in \mathbf{Q}} (N_\lambda, \phi_{N_\lambda})$, where each $(N_\lambda, \phi_{N_\lambda})$ is isoclinic of slope λ. Morphisms between isocrystals preserve this grading.

Let (N, ϕ_N) be an isocrystal. For every perfectoid ring $R/\overline{\mathbf{F}}_q$, we have the vector bundle $\mathcal{O}_{\mathcal{Y}_R} \otimes_{\breve{F}} N$ on $\mathcal{Y}_R$, which comes equipped with a ϕ_R-equivariant automorphism, namely $\phi = \phi_R \otimes \phi_N$. Therefore $\mathcal{O}_{\mathcal{Y}_R} \otimes_{\breve{F}} N$ can be descended to $\mathcal{X}_R = \mathcal{Y}_R/\phi_R$. We will need a schematic version of this descent to the schematic curve X_R, namely

$$\mathcal{E}_{N,\phi_N} = \operatorname{Proj} \bigoplus_{d \geq 0} (H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R}) \otimes_{\breve{F}} N)^{\phi_R \otimes \phi_N = \pi^d}.$$

It is clear that $(N, \phi_N) \mapsto \mathcal{E}_{N,\phi_N}$ is a functor.

Let us discuss the absolute case, when $R = C$ is an algebraically closed perfectoid field. We have a Harder-Narasimhan theory for vector bundles on X_C: for a vector bundle $\mathcal{E}$ on X_C, the slope of $\mathcal{E}$ is $\deg \mathcal{E}/\mathrm{rk}\, \mathcal{E}$. Then every vector bundle admits a canonical $\mathbf{Q}$-filtration by slopes. In fact this $\mathbf{Q}$-filtration is split, so each vector bundle has a $\mathbf{Q}$-grading as well. This applies in particular to the vector bundle $\mathcal{E}_{N,\phi_N}$, in which case the slopes of $\mathcal{E}_{N,\phi_N}$ are -1 times the slopes of (N, ϕ_N).

Theorem 4.5.1 ([FF])**.** *Let $C/\mathbf{F}_q$ be an algebraically closed perfectoid field.*

1. *Every vector bundle on X_C is isomorphic to $\mathcal{E}_{N,\phi_N}$ for an isocrystal (N, ϕ_N).*

2. *Morphisms between vector bundles preserve the $\mathbf{Q}$-filtration (but not necessarily the $\mathbf{Q}$-grading).*

3. *The morphisms between $\mathcal{E}_{M,\phi_M}$ and $\mathcal{E}_{N,\phi_N}$ that preserve the $\mathbf{Q}$-grading are precisely the functorial images of the morphisms between (M, ϕ_M) and (N, ϕ_N).*

We now generalize to G-bundles. Let G be a connected reductive group over F and let Rep_G be the F-linear category of algebraic representations of G. A *G-isocrystal* is a $\otimes$-functor from Rep_G to the category of isocrystals. An element $b \in G(\breve{F})$ determines a G-isocrystal $N_b \colon V \mapsto (V \otimes_F \breve{F}, b(1 \otimes \sigma))$ whose isomorphism class only depends on the class of b in $B(G)$. Recall that $J_b(F)$ is the automorphism group of N_b.

Since isocrystals admit a canonical $\mathbf{Q}$-grading, the element $b \in G(\breve{F})$ induces a $\mathbf{Q}$-grading on $\mathrm{Rep}_{\breve{F}}$, which in turn induces a filtration. Let M (respectively, P) be the stabilizer in $G_{\breve{F}}$ of this granding (respectively, this filtration). Then $P \subset G_{\breve{F}}$ is a parabolic subgroup with Levi factor M. Since morphisms between isocrystals preserve the $\mathbf{Q}$-grading, $J_b(F)$ is a subgroup of $M(\breve{F})$.

A *G-bundle* on a scheme X/F is a $\otimes$-functor from Rep_G to the category of vector bundles on X. We will write $\mathcal{E}_1$ for the trivial G-bundle, which is $V \mapsto V \otimes_F \mathcal{O}_X$. Given an element $b \in G(\breve{F})$, we get a G-bundle $\mathcal{E}_b$ on X_R, via $V \mapsto \mathcal{E}_{N_b(V)}$. (Note that there is no conflict of notation with $\mathcal{E}_1$.) Then the isomorphism class of $\mathcal{E}_b$ only depends on the class of b in $B(G)$.

Theorem 4.5.2 ([Far15])**.** *Let $C/\mathbf{F}_q$ be an algebraically closed perfectoid field. Every G-bundle $\mathcal{E}$ on X_C is isomorphic to $\mathcal{E}_b$ for some $b \in G(\breve{F})$. Thus the set of isomorphism classes of G-bundles on X_C is in bijection with $B(G)$.*

This theorem allows us to define invariants $\kappa(\mathcal{E}) \in \pi_1(G)_\Gamma$ and $\nu_\mathcal{E} \in \bar{C}_\mathbf{Q}$ for an arbitrary G-bundle $\mathcal{E}$ on X_C.

If $(R^\sharp, \iota)$ is an untilt of R, then the G-bundle $\mathcal{E}_b$ on X_R comes equipped with a trivialization along the completion of the divisor $D_{R^\sharp} \hookrightarrow X_R$. This is because the adic version of $\mathcal{E}_b$ is descended from the trivial G-bundle on $\mathcal{Y}_R$, and because $\mathcal{Y}_R$ has a distinguished $R^\sharp$-point lying above (the adic version of) $D_{R^\sharp}$.

We use this to derive a result about automorphisms of a G-bundle on the absolute Fargues-Fontaine curve. Let C/F be an algebraically closed perfectoid field, and let $\infty \in X_{C^\flat}$ be the closed point corresponding to the untilt C of $C^\flat$. Let $b \in G(\breve{F})$. We write $\operatorname{Aut} \mathcal{E}_b$ for the group of all automorphisms of $\mathcal{E}_b$ and $\operatorname{Aut}^{\mathrm{gr}} \mathcal{E}_b$ for the subgroup of those automorphisms that preserve the $\mathbf{Q}$-grading. According to Theorem 4.5.1 the homomorphism $J_b(F) \to \operatorname{Aut} \mathcal{E}_b$ induced by functoriality of $(N, \phi_N) \to \mathcal{E}_{N,\phi_N}$ is an isomorphism of $J_b(F)$ onto $\operatorname{Aut}^{\mathrm{gr}} \mathcal{E}_b$. Note that $\operatorname{Aut} \mathcal{E}_b$ may be much larger that $\operatorname{Aut}^{\mathrm{gr}} \mathcal{E}_b$. For example, if $G = \mathrm{GL}_2$ and $b = \operatorname{diag}(1, p^{-1})$, then $J_b(F) \cong F^\times \times F^\times$, whereas $\operatorname{Aut} \mathcal{E}_b$ is a semidirect product of this group by $H^0(X_{C^\flat}, \mathcal{O}(1))$.

By the remark of the previous paragraph, we have a distinguished trivialization of the stalk $\mathcal{E}_{b,\infty}$ as a G-bundle over $\operatorname{Spec} B_{\mathrm{dR}}^+(C)$. The $\mathbf{Q}$-filtration of $\mathcal{E}_b$ induces a filtration of $\mathcal{E}_{b,\infty}$, whose stabilizer in $G(B_{\mathrm{dR}}^+(C))$ is $P(B_{\mathrm{dR}}^+(C))$. Since automorphisms of $\mathcal{E}_b$ must preserve this $\mathbf{Q}$-filtration, we have a homomorphism $\operatorname{Aut} \mathcal{E}_b \to P(B_{\mathrm{dR}}^+(C))$.

Lemma 4.5.3. *$J_b(F) \to \operatorname{Aut} \mathcal{E}_b$ admits a section, which makes the following diagram commute:*

$$\begin{tikzcd}
\operatorname{Aut} \mathcal{E}_b \arrow[rr] \arrow[d] & & P(B_{\mathrm{dR}}^+(C)) \arrow[d] \\
J_b(F) \arrow[r] & M(\breve{F}) \arrow[r] & M(B_{\mathrm{dR}}^+(C))
\end{tikzcd}$$

Proof. For every object $V \in \operatorname{Rep}_F$, there is the evident section $\operatorname{Aut} \mathcal{E}_{N_b(V)} \to \operatorname{Aut}^{\mathrm{gr}} \mathcal{E}_{N_b(V)}$. Since these constructions are functorial in V, we have a section $\operatorname{Aut} \mathcal{E}_b \to \operatorname{Aut}^{\mathrm{gr}} \mathcal{E}_b \cong \operatorname{Aut} N_b = J_b(F)$ as required. The commutativity of the diagram comes from tracing through the definitions of P and M. $\square$

4.6 G-torsors

Here we remind the reader of some standard material concerning G-torsors and their relation to G-bundles. Let $G \to X$ be a group scheme. For our purposes, a G-torsor on X is a faithfully flat F-morphism $\mathcal{T} \to X$ together with an action $G \times \mathcal{T} \to \mathcal{T}$ lying over the trivial action on X, such that fppf-locally on X we have a G-equivariant isomorphism $\mathcal{T} \cong G \times X$.

If $G' \to X$ is another group scheme there is the evident notion of a (G, G')-bitorsor on X, which receives commuting actions of G and G' and which is a torsor for each. The categories of torsors and bitorsors are groupoids. If $\mathcal{T}$ is a G-torsor and $\mathcal{T}'$ is a (G, G')-bitorsor, we have the contracted product $\mathcal{T} \times_G \mathcal{T}' = (\mathcal{T} \times \mathcal{T}')/G$, a G'-torsor.

Lemma 4.6.1. *Let $G \to X$ be a reductive group scheme. The category of G-bundles on X is equivalent to the category of G-torsors over X.*

Proof. We sketch the equivalence. If $\mathcal{E}$ is a G-bundle on X, let $\mathcal{T}$ be the functor which assigns to an X-scheme Y the set of trivializations of the base change of $\mathcal{E}$ to Y; then $\mathcal{T}$ is representable by a G-torsor over X. Conversely, if $\mathcal{T} \to X$ is a G-torsor, the corresponding G-bundle is $V \mapsto \mathcal{T} \times_G V$. $\square$

We now return to the situation of the Fargues-Fontaine curve X_R. It may be helpful to spell out the equivalence in Lemma 4.6.1 for the G-bundle $\mathcal{E}_b$ on X_R, where $b \in G(\breve{F})$. Before doing so we introduce some notation: let λ_g (respectively, ρ_g) denote left multiplication by g on G (respectively, right muliplication by g^{-1}).

The G-torsor on X_R corresponding to $\mathcal{E}_b$ is

$$\mathcal{T}_b = \operatorname{Proj} \bigoplus_{d \geq 0} (H^0(\mathcal{Y}_R, \mathcal{O}_{\mathcal{Y}_R}) \otimes_{\breve{F}} \breve{F}[G])^{\phi_R \otimes \lambda_b \sigma = \pi^d}, \tag{4.6.1}$$

where $\breve{F}[G]$ is the coordinate ring of $G_{\breve{F}}$. The G-action on $\mathcal{T}_b$ is induced from the action of G on itself by right translation. (Indeed, one checks that $\mathcal{T}_b \times_G V \cong \mathcal{E}_b(V)$ as tensor functors on $V \in \operatorname{Rep}_G$.) It may be useful (if only for psychological reasons) to record the adic version of $\mathcal{T}_b$:

$$\mathcal{T}_b^{\mathrm{ad}} = (\mathcal{Y}_R \times_{\breve{F}} G_{\breve{F}}^{\mathrm{ad}})/(\phi_R \times \lambda_b \sigma)$$

(here G^{ad} means the adic space over F attached to the scheme G).

Now suppose that b is basic. Recall that J_b is the inner form of G corresponding to $\operatorname{ad} b$. For $h \in J_b(\breve{F}) = G(\breve{F})$ we write $\tilde{\mathcal{T}}_h$ for the associated J_b-torsor. It is easy verified that $h \mapsto hb$ induces a bijection $B(J_b) \xrightarrow{\sim} B(G)$. The following lemma shows that this bijection can be upgraded to an equivalence of groupoids.

Lemma 4.6.2. *Let $b \in G(\breve{F})$ be basic.*

1. *There is a natural (G, J_b)-bitorsor structure extending the G-torsor structure on $\mathcal{T}_b$.*

2. *For $h \in G(\breve{F})$ we have an isomorphism of G-torsors $\check{\mathcal{T}}_h \times_{J_b} \mathcal{T}_b \cong \mathcal{T}_{hb}$.*

3. *The groupoids of J_b-torsors and G-torsors on X_R are equivalent, via $\mathcal{T} \mapsto \mathcal{T} \times_{J_b} \mathcal{T}_b$.*

Proof. The isomorphism $\breve{F}[G] \cong \breve{F}[J_b]$ carries $(\operatorname{ad} b)\sigma$ onto σ. Applying (4.6.1) to the element $b^{-1} \in J(\breve{F})$ shows that

$$\check{\mathcal{T}}_{b^{-1}} = \operatorname{Proj} \bigoplus_{d \geq 0} (H^0(\mathcal{Y}_R, \mathcal{O}_{Y_R}) \otimes_{\breve{F}} \breve{F}[G])^{\phi_R \otimes (\lambda_{b^{-1}} \circ \operatorname{ad} b)\sigma = \pi^d}.$$

Using the identity $\lambda_{b^{-1}} \circ \operatorname{ad} b = \rho_b$, we see that inversion on G gives an isomorphism of X_R-schemes $\check{\mathcal{T}}_{b^{-1}} \cong \mathcal{T}_b$. This isomorphism endows $\mathcal{T}_b$ with the structure of a J_b-torsor. The actions of J_b and G on $\mathcal{T}_b$ commute, because they are given by the action of G on $\breve{F}[G]$ by left and right translations, respectively. This completes (1).

For (2), observe that the multiplication morphism μ on G fits into a commutative diagram

$$\begin{tikzcd} G_{\breve{F}} \times G_{\breve{F}} \arrow[r, "\mu"] \arrow[d, "\lambda_h \circ \operatorname{ad} b \times \lambda_b"'] & G_{\breve{F}} \arrow[d, "\lambda_{hb}"] \\ G_{\breve{F}} \times G_{\breve{F}} \arrow[r, "\mu"'] & G_{\breve{F}}. \end{tikzcd}$$

In light of (4.6.1), the co-multiplication $\mu : \breve{F}[G] \to \breve{F}[G] \otimes_{\breve{F}} \breve{F}[G]$ induces the required isomorphism $\check{\mathcal{T}}_h \times_{J_b} \mathcal{T}_b \cong \mathcal{T}_{hb}$.

For (3), we need to show that $\mathcal{T} \mapsto \mathcal{T} \times_{J_b} \mathcal{T}_b$ is an equivalence of groupoids. By symmetry we have a functor $\mathcal{T} \mapsto \mathcal{T} \times_G \check{\mathcal{T}}_{b^{-1}} \cong \mathcal{T} \times_G \mathcal{T}_b$ going in the other direction. The composition of these functors applied to a G-torsor $\mathcal{T}$ gives

$$(\mathcal{T} \times_G \check{\mathcal{T}}_{b^{-1}}) \times_{J_b} \mathcal{T}_b \cong \mathcal{T} \times_G (\check{\mathcal{T}}_{b^{-1}} \times_{J_b} \mathcal{T}_b) \cong \mathcal{T} \times_G \mathcal{T}_1 \cong \mathcal{T}.$$

The other composition is similar.

□

4.7 Beauville-Laszlo gluing and modifications

Let $R/\mathbf{F}_q$ be a perfectoid ring, and let $(R^\sharp, \iota)$ be an untilt of R to F. Recall that it determines a Cartier divisor $D_{R^\sharp} \hookrightarrow X_R$. We have noted that the completion of X_R along $D_{R^\sharp}$ is $\operatorname{Spf} B^+_{\mathrm{dR}}(R^\sharp)$. Let $B_e(R) = H^0(X_R \backslash D_{R^\sharp}, \mathcal{O}_{X_R})$.

Given a G-bundle $\mathcal{E}$, let $\mathcal{E}_e$ be its restriction to $X_R \backslash D_{R^\sharp} = \operatorname{Spec} B_e(R)$, and let $\mathcal{E}^+_{\mathrm{dR}}$ be its completion along $D_{R^\sharp}$, so that $\mathcal{E}^+_{\mathrm{dR}}$ is a G-bundle over $\operatorname{Spf} B^+_{\mathrm{dR}}(R^\sharp)$. By the following gluing lemma, $\mathcal{E}$ is determined by $\mathcal{E}_e$ and $\mathcal{E}^+_{\mathrm{dR}}$.

Lemma 4.7.1. *[BL95] The category of G-bundles on X_R is equivalent to the category of triples $(\mathcal{E}_e, \mathcal{E}^+_{\mathrm{dR}}, \iota)$, where $\mathcal{E}_e$ is a G-bundle over $B_e(R)$, $\mathcal{E}^+_{\mathrm{dR}}$ is a G-bundle over $B^+_{\mathrm{dR}}(R^\sharp)$, and $\iota\colon \mathcal{E}_e \otimes_{B_e(R)} B_{\mathrm{dR}}(R^\sharp) \xrightarrow{\sim} \mathcal{E}^+_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}}(R^\sharp)} B_{\mathrm{dR}}(R^\sharp)$ is an isomorphism.*

We now consider a pair $(\mathcal{E}, \beta)$ consisting of a G-bundle $\mathcal{E}$ on X_R and a trivialization of $\mathcal{E}$ over $B^+_{\mathrm{dR}}(R^\sharp)$. In terms of the gluing lemma this corresponds to a pair $(\mathcal{E}_e, \iota)$, where $\mathcal{E}_e$ is a G-bundle over $B_e(R)$ and $\iota : \mathcal{E}_e \otimes_{B_e(R)} B_{\mathrm{dR}}(R^\sharp) \xrightarrow{\sim} \mathcal{E}_{1, B_{\mathrm{dR}}(R^\sharp)}$ – to obtain a triple as in the gluing lemma we take for $\mathcal{E}^+_{\mathrm{dR}}$ the trivial $B^+_{\mathrm{dR}}(R^\sharp)$-lattice $\mathcal{E}_{1, B^+_{\mathrm{dR}}(R^\sharp)}$ inside of $\mathcal{E}_{1, B_{\mathrm{dR}}(R^\sharp)}$. An isomorphism $(\mathcal{E}_e, \iota) \to (\mathcal{E}'_e, \iota')$ is given by a pair (α_e, h) with $\alpha_e : \mathcal{E}_e \xrightarrow{\sim} \mathcal{E}'_e$ and $h \in G(B^+_{\mathrm{dR}}(R^\sharp))$ satisfying $\iota' \circ (\alpha_e \otimes \mathrm{id}_{B_{\mathrm{dR}}(R^\sharp)}) = h \circ \iota$.

Definition 4.7.2. Let $(\mathcal{E}, \beta)$ be a G-bundle on X_R trivialized over $B_{\mathrm{dR}}(R^\sharp)$, corresponding to the pair $(\mathcal{E}_e, \iota)$. Let $g \in G(B_{\mathrm{dR}}(R^\sharp))$. The *modification of $(\mathcal{E}, \beta)$ at $R^\sharp$ via g* is the pair $(\mathcal{E}[g], \beta)$ corresponding to the pair $(\mathcal{E}_e, g^{-1}\iota)$.

Note that the isomorphism class of $\mathcal{E}[g]$ only depends on the class of g in $\mathrm{Gr}_G(R)$, and this is the motivation for the use of g^{-1} instead of g.

In this paper we are particularly interested in the G-bundles $\mathcal{E}_b$ for $b \in G(\breve{F})$. The untilt $(R^\sharp, \iota)$ provides a point of $\mathcal{Y}_R$ and thus, as explained earlier, a canonical trivialization of $\mathcal{E}_b$ over $B^+_{\mathrm{dR}}(R^\sharp)$. For every $g \in G(B_{\mathrm{dR}}(R^\sharp))$ we thus obtain the modified bundle $\mathcal{E}_b[g]$, whose isomorphism class depends only on the image of g in $\mathrm{Gr}_G(R)$ In the special when $g = L_\nu = \nu(\xi)$ for a cocharacter $\nu \in X_*(T)$ of an F-rational maximal torus $T \subset G$ we will write $(\mathcal{E}[\nu], \beta)$ for the modification of a G-bundle $(\mathcal{E}, \beta)$ by g, and in particular $\mathcal{E}_b[\nu]$ for the modification $\mathcal{E}_b[g]$.

Lemma 4.7.3. *Let $\mathcal{E}$ be a G-bundle on X_C, let $T \subset G$ be a maximal torus, let $\nu \in X_*(T)$ be a cocharacter, and let $\widehat{\nu} \in X^*(\widehat{T})$ be the corresponding character. In the group $X^*(Z(\widehat{G})^\Gamma)$ we have*

$$\kappa(\mathcal{E}[\nu]) = \kappa(\mathcal{E}) - \widehat{\nu}|_{Z(\widehat{G})^\Gamma}.$$

Proof. The proof is a devissage argument based on the functoriality of the Kottwitz map κ and the following easily established fact: If $f\colon G \to H$ is a homomorphism of reductive groups, write f_* for the functor carrying G-bundles to H-bundles; then for a G-bundle E we have

$$f_*(\mathcal{E}[\nu]) \cong (f_*\mathcal{E})[f \circ \nu]. \tag{4.7.1}$$

Step 0: $G = \mathbf{G}_\mathrm{m}$. In this case $\kappa\colon B(\mathbf{G}_\mathrm{m}) \to \mathbf{Z}$ is an isomorphism, which agrees with the degree map on vector bundles; the claim reduces to the fact that modifying a line bundle by $t \mapsto t^n$ reduces its degree by n.

Step 1: $G = T = \mathrm{Res}_{E/F}\,\mathbf{G}_\mathrm{m}$ **for a finite extension** E/F. In this case $X^*(\widehat{T}) = X_*(T)$ is the group ring $\mathbf{Z}[\Gamma_{E/F}]$. The norm maps $N : T \to \mathbf{G}_\mathrm{m}$ and $N : \mathbf{Z}[\Gamma_{E/F}] \to \mathbf{Z}$ fit into the commutative diagram

$$\begin{tikzcd} B(T) \arrow[r, "N"] \arrow[d, "\kappa"'] & B(\mathbf{G}_\mathrm{m}) \arrow[d, "\kappa"] \\ X^*(\widehat{T}^\Gamma) \arrow[r, "N"'] & \mathbf{Z} \end{tikzcd}$$

and all four maps are isomorphisms. The claim follows from (4.7.1) and Step 0.

Step 2: $G = T$ **is a torus.** Let E/F be the splitting field of T and M a free $\mathbf{Z}[\Gamma_{E/F}]$-module together with a Γ-equivariant surjection $M \to X_*(T)$. If S is the torus with $X_*(S) = M$ then S is a product of tori of the form $\mathrm{Res}_{E/F}\mathbf{G}_\mathrm{m}$ and we have a surjection $S \to T$ with connected kernel, and hence [Kot85, §1.9] a surjection $B(S) \to B(T)$, as well as a surjection $X_*(S)_\Gamma \to X_*(T)_\Gamma$. We have $\mathcal{E} \cong \mathcal{E}_b$ for some $b \in B(T)$; let $b_S \in B(S)$ be a lift of b and let $\nu_S \in X_*(S)$ be a lift of ν. The claim follows from (4.7.1) and Step 1 applied to S.

Step 3: G_der **is simply connected.** According to [Kot97, §7.5] the map κ is given by $B(G) \to B(D) \to X^*(\widehat{D}^\Gamma) = X^*(Z(\widehat{G})^\Gamma)$, where $D = G/G_\mathrm{der}$, and the claim follows from (4.7.1) and Step 2 applied to D.

Step 4: General G**.** Let $1 \to K \to \widetilde{G} \to G \to 1$ be a z-extension. Again we have a surjection $B(\widetilde{G}) \to B(G)$ as well as surjections $X_*(\widetilde{T}) \to X_*(T)$ for any maximal torus $\widetilde{T} \subset \widetilde{G}$ with image $T \subset G$. This allows us to lift both b and ν to elements $\widetilde{b} \in \widetilde{G}(\breve{F})$ and $\widetilde{\nu} : \mathbf{G}_\mathrm{m} \to \widetilde{G}$. The claim follows from (4.7.1) and Step 3 applied to $\widetilde{G}$. $\square$

4.8 The admissible locus, and spaces of shtukas

Definition 4.8.1. Let $b \in G(\breve{F})$ and let $\mu \in X_*(T)$. The *b-admissible locus* in $\mathrm{Gr}_{G,\leq\mu}$ is the subfunctor $\mathrm{Gr}_{G,\leq\mu}^{b\text{-adm}} \subset \mathrm{Gr}_{G,\leq\mu}$ assigning to a perfectoid C-algebra R the set of $g \in \mathrm{Gr}_{G,\leq\mu}(R)$ such that for every geometric point $x\colon \operatorname{Spa} C' \to \operatorname{Spa} R$, $x^*\mathcal{E}_b[g]$ is isomorphic to the trivial G-bundle $\mathcal{E}_1$ on $X_{(C')^\flat}$.

Proposition 4.8.2. $\mathrm{Gr}_{G,\leq\mu}^{b\text{-}adm} \subset \mathrm{Gr}_{G\leq\mu}$ *is an open subfunctor, and thus is a diamond. It is empty if* $b \notin B(G,\mu)$.

Proof. The locus of $g \in \mathrm{Gr}_{G\leq\mu}$ where $\mathcal{E}_b[g]$ is semistable (that is, where $\mathcal{E}_b[g]$ corresponds to a class in $B(G)_{\mathrm{bas}}$) is open, by the "semicontinuity of the slope polygon" [KL15, Theorem 7.4.5]. Furthermore, the locus where $\kappa(\mathcal{E}_b[\nu]) = 0$ is open (and closed) by [Far, Theorem 2.15]. Since $\kappa\colon B(G)_{\mathrm{bas}} \to \pi_1(G)_\Gamma$ is a bijection, we have (over a geometric point) $\mathcal{E}_b[g] \cong \mathcal{E}_1$ if and only if $\mathcal{E}_b[g]$ is basic and $\kappa(\mathcal{E}_b[g]) = 0$. Therefore $\mathrm{Gr}_{G,\leq\mu}^{b\text{-adm}}$ is open.

The second assertion follows from Lemma 4.7.3. $\square$

If $\mathcal{E}$ is a G-bundle on X_R whose pullback to every geometric point is trivial, then the space of trivializations of $\mathcal{E}$ is a pro-étale $\underline{G(F)}$-torsor over R [KL15, Theorem 9.3.13]. Trivializing $\mathcal{E}_b[g]$ for the family of b-admissible g gives the space of (infinite-level) local G-shtukas.

Definition 4.8.3. Let $\mathcal{M}_{G,b} \to \operatorname{Spd} C$ denote the sheafification of the presheaf which assigns to a perfectoid C-algebra R the set of isomorphisms

$$\alpha\colon \mathcal{E}_b|_{X_{R^\flat}\backslash D_R} \to \mathcal{E}_1|_{X_{R^\flat}\backslash D_R},$$

Recall from §4.5 that $\mathcal{E}_b$ comes equipped with a trivialization over the formal neighborhood of D_R, i.e. an isomorphism $\beta : \mathcal{E}_b \otimes B_{\mathrm{dR}}^+(R) \to \mathcal{E}_1 \otimes B_{\mathrm{dR}}^+(R)$. Then $(\alpha \otimes B_{\mathrm{dR}}(R))^{-1} \circ (\beta \otimes B_{\mathrm{dR}}(R))$ is an automorphism of $\mathcal{E}_b \otimes B_{\mathrm{dR}}(R)$, hence an element of $J_b(B_{\mathrm{dR}}(R)) = G(B_{\mathrm{dR}}(R))$. The assignment sending α to the $G(B_{\mathrm{dR}}^+(R))$-coset of this element gives a morphism $\mathcal{M}_{G,b} \to \mathrm{Gr}_G$ (the Grothendieck-Messing period map). Note that $\mathcal{E}_b[g] \cong \mathcal{E}_1$ via α, so that this morphism factors through the admissible locus. For a cocharacter μ, let $\mathcal{M}_{G,b,\mu} \subset \mathcal{M}_{G,b}$ be the pullback of $\mathrm{Gr}_{G,b,\leq\mu} \subset \mathrm{Gr}_{G,b}$.

The sheaf $\mathcal{M}_{G,b,\mu}$ admits an action of $J_b(F)$ lying over the action on $\mathrm{Gr}_{G,\leq\mu}^{b\text{-adm}}$, via $\alpha \mapsto \alpha \circ g^{-1}$ for $g \in J_b(F)$. It also admits an action of $G(F)$, via $\alpha \mapsto g \circ \alpha$ for $g \in G(F)$. This action is clearly simple and preserves each fiber of the Grothendieck-Messing period map. The Beauville-Laszlo gluing lemma 4.7.1 implies that this action is transitive on each fiber, thus the Grothendieck-Messing period map is a $G(F)$-torsor. We record this observation as follows.

Proposition 4.8.4. *$\mathcal{M}_{G,b,\mu} \to \mathrm{Gr}^{b\text{-}adm}_{G,\leq\mu}$ is a $G(F)$-torsor. In particular $\mathcal{M}_{G,b,\mu}$ is a locally spatial diamond.*

In situations where there exists a tower of Rapoport-Zink spaces attached to (G, b, μ), the inverse limit along the tower will be $\mathcal{M}_{G,b,\mu}$; see [SW13] for the case of $G = \mathrm{GL}_n$ and μ minuscule. In general, the $\mathcal{M}_{G,b,\mu}/K$ (for $K \subset G(F)$ compact open) will play the role of the tower of "local Shimura varieties" expected by [RV14].

Example 4.8.5. When $G = T$ is a torus, $\mathrm{Gr}_T \cong X_*(T)$. If $\mu \in X_*(T)$, then $\mathrm{Gr}_{T,\mu} = \{\mu\}$. We have an isomorphism $B(T) \cong X_*(T)_\Gamma$ (where $\Gamma = \mathrm{Gal}(\overline{F}/F)$), and $B(T, \mu) = \{b\}$ for the class $b \in B(T)$ identified with the image of μ in $X_*(T)_\Gamma$. Then $\mathcal{M}_{T,b,\mu}$ is a principal homogenous space for $T(F)$. Note that $J_b(F) = T(F)$, and the actions of $J_b(F)$ and $T(F)$ on $\mathcal{M}_{T,b,\mu}$ agree.

4.9 Duality for spaces of shtukas in the basic case

For $b \in G(\breve{F})$ basic, [RV14, Conjecture 5.8] predicts an isomorphism between local Shimura varities attached to the groups G and J_b. Indeed, there is an isomorphism $\mathcal{M}_{G,b,\mu} \cong \mathcal{M}_{J_b,b^*,\mu^*}$, as we explain below. This is a generalization of the duality theorem in [SW13, Theorem 7.2.3], which treats the case $G = \mathrm{GL}_n$ and μ minuscule.

Recall the map $G(\breve{F}) \to G(\breve{F}) = J_b(\breve{F})$ given by $h \mapsto h^* = hb^{-1}$. Let $\breve{b} = b^{-1} = 1^*$, considered as an element of $J_b(\breve{F})$. For a cocharacter μ of G, define $\breve{\mu}$ as the composite of μ^{-1} with $G_{\overline{F}} \cong J_{b,\overline{F}}$.

Proposition 4.9.1. *Let $b \in G(\breve{F})$ be basic. There is an isomorphism $\mathcal{M}_{G,b,\mu} \to \mathcal{M}_{J_b,\breve{b},\breve{\mu}}$ which respects the actions of $G(F) \times J_b(F)$.*

Proof. By Lemma 4.6.1, $\mathcal{M}_{G,b}$ is isomorphic to the sheafification of the presheaf which assigns to a perfectoid C-algebra R the set of isomorphisms of G-torsors $\alpha\colon \mathcal{T}_b \to \mathcal{T}_1$ over $X_{R^\flat}\backslash D_R$. Given such an α, we obtain an isomorphism of J_b-torsors $\mathcal{T}_b \times_G \mathcal{T}_b \to \mathcal{T}_1 \times_G \mathcal{T}_b$. Applying Lemma 4.6.2, this amounts to an isomorphism $\alpha^*\colon \breve{\mathcal{T}}_1 \to \breve{\mathcal{T}}_{\breve{b}}$. Then $\alpha \mapsto \breve{\alpha} = (\alpha^*)^{-1}$ induces an isomorphism of sheaves $\mathcal{M}_{G,b} \to \mathcal{M}_{J_b,\breve{b}}$. To finish the proof, one has to observe that α is a modification of type $\leq \mu$ if and only if $\breve{\alpha}$ is a modification of type $\leq \breve{\mu}$; this is a matter of unraveling the definitions. $\square$

4.10 Definition of $\mathcal{H}(G,b,\mu)[\pi]$, and Assumption 3

For a finite local ring Λ which is ℓ^n-torsion for some $n \geq 1$, we have the intersection complex $\mathrm{IC}_{\mu,\Lambda} \in D(\mathrm{Gr}_{G,b,\leq\mu,\text{ét}},\Lambda)$. Let us write $\mathrm{IC}_{\mu,\Lambda}$ again for the pullback of this to $\mathcal{M}_{G,b,\mu}$. Then we have the compactly supported cohomology $H^i_c(\mathcal{M}_{G,b,\mu},\mathrm{IC}_{\mu,\Lambda})$, a Λ-module with an action of $J_b(F)\times G(F)$.

Definition 4.10.1. Fix a finite extension $E/\mathbf{Q}_\ell$. For each open compact subgroup $K \subset G(F)$ we define

$$R\Gamma_c(G,b,\mu,K) = \varprojlim_n R\Gamma_c(\mathcal{M}_{G,b,\mu}/K, \mathrm{IC}_{\mu,\mathcal{O}_E/\ell^n}) \otimes_{\mathcal{O}_E} E.$$

For a smooth irreducible admissible representation ρ of $J_b(F)$ with coefficients in E we define

$$H^i(G,b,\mu)[\rho] = \varinjlim_K \mathrm{Ext}^i_{J_b(F)}(R\Gamma_c(G,b,\mu,K),\rho),$$

a smooth representation of $G(F)$.

It will be helpful to reinterpret these representations in terms of objects living on the proper diamond $\mathrm{Gr}_{G,\leq\mu}$. Let $\Lambda = \mathcal{O}_E/\ell^n$, and let $\pi\colon G(F) \to \mathrm{GL}(V)$ be a smooth admissible representation of $G(F)$ on a free Λ-module V. Let $f\colon \mathrm{Gr}^{b\text{-adm}}_{G,\mu} \to [*/\underline{G(F)}]$ correspond to the pro-étale $\underline{G(F)}$-torsor $\mathcal{M}_{G,b,\mu} \to \mathrm{Gr}^{b\text{-adm}}_{G,\mu}$. Also let $j\colon \mathrm{Gr}^{b\text{-adm}}_{G,\leq\mu} \to \mathrm{Gr}_{G,\leq\mu}$ denote the inclusion.

Definition 4.10.2. We define an object $\mathcal{F}_\pi \in D(\mathrm{Gr}_{G,\leq\mu,\text{ét}},\Lambda)$ by

$$\mathcal{F}_\pi := j_! f^*\mathcal{L}_\pi \otimes \mathrm{IC}_{\mu,\Lambda}\,.$$

Since $\mathcal{M}_{G,b,\mu} \to \mathrm{Gr}_{G,\leq\mu}$ and $\mathrm{IC}_{\mu,\Lambda}$ are $J_b(F)$-equivariant, $\mathcal{F}_\pi$ descends to a sheaf on $[\mathrm{Gr}_{G,\leq\mu}/J_b(F)]_{\text{ét}}$, which we still call $\mathcal{F}_\pi$.

Assumption 3. $\mathcal{F}_\pi$ is strongly reflexive on $[\mathrm{Gr}_{G,\leq\mu}/J_b(F)]_{\text{ét}}$.

The following lemma reduces the main theorem to proving a certain trace identity.

Lemma 4.10.3. *Work under Assumptions 1-3. In order to prove Theorem 1.0.4, it suffices to show the following. Let $\phi : W_F \to {}^LG$ be a discrete Langlands parameter, and let $\pi \in \Pi_\phi(G)$. Assume that V contains a $G(F)$-invariant lattice V_0; let π_n be the representation of $G(F)$ on V_0/ℓ^n. Let $f \in C^\infty_c(J_b(F),\Lambda)$ be supported on the regular elliptic locus. Then*

$$\mathrm{tr}(f|H^*(\mathrm{Gr}_{G,\leq\mu},\mathcal{F}_{\pi_n})) = (-1)^d \sum_{\rho\in\Pi_\phi(J_b)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho},r_\mu)\,\mathrm{tr}\,\rho(f). \tag{4.10.1}$$

Proof. First we claim that Theorem 1.0.4 is invariant under replacing a representation π of $G(F)$ with $\pi \otimes \chi$, where χ is a character of $G^{\mathrm{ab}}(F)$ and G^{ab} is the maximal abelian quotient of G. We have a $G(F) \times J_b(F)$-equivariant morphism $\mathcal{M}_{G,b,\mu} \to \mathcal{M}_{G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}}}$, where b^{ab} and μ^{ab} are the images of b and μ respectively. From Example 4.8.5, $\mathcal{M}_{G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}}}$ is a principal homogenous space for $G^{\mathrm{ab}}(F)$, on which $G(F)$ and $J_b(F)$ both act through their common map to $G^{\mathrm{ab}}(F)$. This implies that each $H^i(\mathcal{M}_{G,b,\mu}, IC_{\mu,n})$ is induced from a representation of the group $(G(F) \times J_b(F))^1$ consisting of pairs with the same image in $G^{\mathrm{ab}}(F)$. This in turn implies that whenever a representation ρ of $J_b(F)$ is contained in $H^*(G,b,\mu)[\pi]$, then $\rho \otimes \chi$ is contained in $H^*_c(G,b,\mu)[\pi \otimes \chi]$ for each character of $G^{\mathrm{ab}}(F)$, which is the claim.

We fix an isomorphism $\mathbf{C} \to \bar{\mathbf{Q}}_l$ and use it to interpret π as having $\bar{\mathbf{Q}}_l$-coefficients. By Corollary A.2.2 and the above argument we may assume that π has a $G(F)$-invariant lattice. Let π_n be the reduction of a $G(F)$-invariant lattice in π modulo ℓ^n. Let $K \subset J_b(F)$ be a compact open subgroup. Under Assumption (3), $\mathcal{F}_{\pi_n}$ is reflexive on $[\mathrm{Gr}_{G,b,\mu}/\underline{K}]_{\text{ét}}$. By Theorem **??**(3) we have an isomorphism

$$\begin{aligned} & \mathrm{RHom}(R\Gamma_c([\mathrm{Gr}^{b\text{-adm}}_{G,b,\mu}/\underline{K}]_{\text{ét}}, \mathcal{L}_{\pi_n^\vee} \otimes \mathrm{IC}_{\mu,n}) \\ \cong\ & \mathrm{RHom}_{G(F)}(R\Gamma_c((\mathcal{M}_{G,b,\mu}/\underline{K})_{\text{ét}}, \mathrm{IC}_{\mu,n}), \pi_n). \end{aligned}$$

This is a perfect complex of modules over $\Lambda = \mathcal{O}_E/\ell^n$. By hypothesis we have an expression for the trace of a Hecke operator $f \in C_c^\infty(J_b(F), \Lambda)$ on the right hand side whenever f is K-bi-invariant and supported on the regular elliptic locus. That expression persists when we apply $\varprojlim_n$ and invert ℓ. We can then apply $\varinjlim_K$, so that we have an expression for trace distribution of

$$\varinjlim_K \mathrm{RHom}_{G(F)}(R\Gamma_c((\mathcal{M}_{G,b,\mu}/\underline{K})_{\text{ét}}, \mathrm{IC}_{\mu,E}), \pi),$$

on the regular elliptic locus of $J(F)$. By Proposition 4.9.1 the above is $R\Gamma(J_b, \check{b}, \check{\mu})[\pi]$. We have shown that, for $f \in C_c^\infty(J_b(F), \Lambda)$ supported on the elliptic locus,

$$\mathrm{tr}(f|R\Gamma(J_b, \check{b}, \check{\mu})[\pi]) = (-1)^d \sum_{\rho \in \Pi_{\phi^\vee}(J_b)} \dim \mathrm{Hom}_{S_{\phi^\vee}}(\delta_{\pi^\vee,\rho}, r_\mu) \,\mathrm{tr}\, \rho(f).$$

Since $H^*(J_b, \check{b}, \check{\mu})[\pi]$ is admissible, this shows that

$$H^*(J_b, \check{b}, \check{\mu})[\pi] = (-1)^d \sum_{\rho \in \Pi_{\phi^\vee}(J_b)} [\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi^\vee,\rho}, r_\mu)]\rho$$

as elements of $\mathrm{Groth}(J_b(F))^{\mathrm{ell}}$. This is Theorem 1.0.4 for the triple $(J_b, \check{b}, \check{\mu})$ and the representation π playing the role of (G, b, μ) and ρ. $\square$

4.11 Remarks on Assumption 3

Assumption 3 is related to Scholze's program to interpret Langlands parameters in terms of reflexive sheaves on Bun_G, the moduli stack of G-bundles on the Fargues-Fontaine curve. Let us briefly explain the connection.

Scholze shows that an object $\mathcal{F}$ of $D(\mathrm{Bun}_{G,\text{ét}}, \Lambda)$ is reflexive if and only if for all points $x \in \mathrm{Bun}_G$, (all cohomology groups of) the stalk $\mathcal{F}_x$ is an admissible representation of $\mathrm{Aut}\, x$. In particular there are the points coming from basic isocrystals: for basic $b \in B(G)$, we have the substack $j_b\colon \mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ which classifies G-bundles that are isomorphic to $\mathcal{E}_b$ on every geometric point; then $\mathrm{Bun}_B^b = BJ_b(F)$. For an admissible representation ρ of $J_b(F)$, let L_ρ be the corresponding sheaf on $\mathrm{Bun}_{G,\text{ét}}^b$; then $j_{b!}L_\rho$ is reflexive.

The stack Bun_G comes equipped with a family of Hecke correspondences for each cocharacter μ [Far]:

$$\begin{tikzcd} & \mathrm{Hecke}_{\leq\mu} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spd}\, C. \end{tikzcd}$$

The Hecke operator $\mathcal{H}_\mu\colon D(\mathrm{Bun}_G, \Lambda) \to D(\mathrm{Bun}_G \times \mathrm{Spd}\, C, \Lambda)$ is defined by $\mathcal{F} \mapsto h_{2!}(h_1^*\mathcal{F} \otimes \mathrm{IC}_{\mu,\Lambda})$. Scholze shows that if $\mathcal{F}$ is reflexive, then so is $h_1^*\mathcal{F} \otimes \mathrm{IC}_{\mu,\Lambda}$. (A special case occurs when μ is minuscule, so that $\mathrm{IC}_{\mu,\Lambda}$ is a shift and twist of the constant sheaf; then h_1 is a "smooth" morphism of diamond stacks, and reflexive sheaves are preserved under smooth pullbacks.) Since h_2 is proper, $\mathcal{H}_\mu$ preserves reflexive sheaves.

There is a cartesian diagram

$$\begin{tikzcd} {[\mathrm{Gr}_{G,\leq\mu} / J_b(F)]} \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hecke}_{\leq\mu} \arrow[d, "h_2"] \\ \mathrm{Bun}_G^b \times \mathrm{Spa}\, C \arrow[r, "j_b \times \mathrm{id}"'] & \mathrm{Bun}_G \times \mathrm{Spa}\, C, \end{tikzcd}$$

in which the horizontal maps are open immersions. Since $h_1^* j_{1!} L_\pi \otimes \mathrm{IC}_{\mu,\Lambda})$ is reflexive, so is $\alpha^* h_1^* j_{1!} L_\pi \otimes \mathrm{IC}_{\mu\Lambda}$. We also have a cartesian diagram

$$\begin{tikzcd} {[\mathrm{Gr}_{G,\leq\mu}^{b\text{-adm}} / J_b(F)]} \arrow[r, "\beta"] \arrow[d] & \mathrm{Bun}_G^1 \arrow[d, "j_{1!}"] \\ {[\mathrm{Gr}_{G,\leq\mu} / J_b(F)]} \arrow[r, "h_1 \circ \alpha"'] & \mathrm{Bun}_G, \end{tikzcd}$$

which gives us a base change isomorphism $\alpha^* h_1^* j_{1!} L_\pi \otimes \mathrm{IC}_{\mu\Lambda} \cong j_! \alpha^* L_\pi \otimes \mathrm{IC}_\mu$. The morphism α onto $\mathrm{Bun}_G^1 \cong BG(F)$ corresponds to the $G(F)$-torsor $\mathcal{M}_{G,b,\mu}$, so that $\alpha^* L_\pi \cong \mathcal{L}_\pi$. Therefore $\mathcal{F}_\pi$ is reflexive. (Strong reflexivity requires a further argument.)

5 Proof of the main theorem

5.1 Elliptic fixed points on Gr_G

In this section, C is a complete algebraically closed field containing F. Write B_{dR}^+ and B_{dR} for the corresponding rings of Fontaine. Then B_{dR}^+ is a complete discrete valuation ring containing $\breve{F}$ with residue field C. It also contains a unique copy of $\bar{F}$. We have $\mathrm{Gr}_G(C) = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$.

Remark 5.1.1. We can consider the extended Bruhat-Tits building of G over the discretely valued field B_{dR} and identify the $G(B_{\mathrm{dR}})$-set $\mathrm{Gr}_G(C)$ with a piece of this building as follows. The inclusions $F \to \bar{F} \to B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}$ show that the base change $G \times B_{\mathrm{dR}}^+$ is a split reductive group scheme with generic fiber $G \times B_{\mathrm{dR}}$. Let $\mathcal{B}$ be the (reduced) Bruhat-Tits building of the split reductive group $G \times B_{\mathrm{dR}}$ and let $K = G(B_{\mathrm{dR}}^+)$. Then by [BT84, 5.1.40] there exists a hyperspecial point $\bar{o} \in \mathcal{B}$ such that $K = \mathfrak{G}_{\bar{o}}^0(B_{\mathrm{dR}}^+)$, where $\mathfrak{G}_{\bar{o}}^0$ is the connected parahoric B_{dR}^+-group scheme with generic fiber $G \times B_{\mathrm{dR}}$ associated to the point $\bar{o}$. The point $\bar{o}$ can be characterized by [BT84, 4.6.29] as the unique fixed point of K. Let $\mathcal{B}^{\mathrm{ext}}$ be the extended Bruhat-Tits building of $G \times B_{\mathrm{dR}}$. Recall that $\mathcal{B}^{\mathrm{ext}} = \mathcal{B} \times X_*(A_G)_{\mathbf{R}}$, where A_G is the connected center of G (automatically split over B_{dR}). The group $G(B_{\mathrm{dR}})$ acts on $X_*(A_G)_{\mathbf{R}}$ via the isomorphism $X_*(A_G)_{\mathbf{R}} \to X_*(A'_G)_{\mathbf{R}}$, where A'_G is the maximal abelian quotient of G. Let $o = (\bar{o}, z)$ be any point in $\mathcal{B}^{\mathrm{ext}}$ lying over $\bar{o}$. Then K can be characterized as the full stabilizer of o in $G(B_{\mathrm{dR}})$ – it is clear that K stabilizes o and the converse inclusion follows from the Cartan decomposition $G(B_{\mathrm{dR}}) = KX_*(T)K$ (which relies on $\bar{o}$ being hyperspecial) and the fact that $X_*(T)$ acts on the apartment of T in $\mathcal{B}^{\mathrm{ext}}$ by translations. It follows that the action of $G(B_{\mathrm{dR}})$ on $\mathcal{B}^{\mathrm{ext}}$ provides a $G(B_{\mathrm{dR}})$-equivariant bijection from the coset space $G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ to the orbit of $G(B_{\mathrm{dR}})$ through o.

Proposition 5.1.2. *Let $g \in G(\breve{F})$ be a regular semisimple element, and let $T \subset G_{\breve{F}}$ be its connected centralizer. Then an element of $\mathrm{Gr}_G(C)$ is fixed by g if and only if it is fixed by all of $T(\breve{F})$.*

Proof. If $x \in G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ is a g-fixed point, then its image in $\mathcal{B}^{\mathrm{ext}}$ is a g-fixed point belonging to the orbit of o and we can write $x = ho$ for some

$h \in G(B_{\mathrm{dR}})$. For every root $\alpha : T \to \mathbf{G}_{\mathrm{m}}$ we have $\alpha(g) \in \bar{F}^{\times}$ and hence $\alpha(g) \notin 1 + \ker(\theta)$, where $\ker(\theta)$ is the maximal ideal in B^+_{dR}. According to [Tit79, 3.6.1] the image of x in $\mathcal{B}$ belongs to the apartment $\mathcal{A}$ of T. At the same time, $g \in G(B^+_{\mathrm{dR}}) = K$ also fixed $\bar{o}$, so for the same reason $\bar{o} \in \mathcal{A}$. Thus $\bar{o}$ belongs to both apartments $\mathcal{A}$ and $h^{-1}\mathcal{A}$. Since K acts transitively on the apartments containing $\bar{o}$ [BT84, 4.6.28], we can multiply h on the right by an element of K to ensure that $h^{-1}\mathcal{A} = \mathcal{A}$. By [BT72, 7.4.10] we then have $h \in N(T, G)(B_{\mathrm{dR}})$. Since $\bar{o}$ is hyperspecial, every Weyl reflection is realized in K and hence we may again modify h on the right to achieve $h \in T(B_{\mathrm{dR}})$. We see now that $x = ho$ is fixed by all of $T(\breve{F}) \subset T(B^+_{\mathrm{dR}})$ and that furthermore the coset $x = hG(B^+_{\mathrm{dR}})$ is the image of the coset $hT(B^+_{\mathrm{dR}})$. □

Let Gr^g_G be the fixed point locus of g, in the sense of (3.8.1). Recall from Example 4.1.4 the isomorphism $X_*(T) \to \mathrm{Gr}_T$ sending μ to $L_\mu = \mu(\xi)$.

Corollary 5.1.3. *The morphism $\mathrm{Gr}^g_G \to \mathrm{Gr}_G$ factors through an isomorphism $\mathrm{Gr}^g_G \xrightarrow{\sim} \mathrm{Gr}_T$.*

Corollary 5.1.4. *Let $B = TU$ be a Borel subgroup of $G_{\bar{F}}$ containing T, and let $S_\nu = U(B_{\mathrm{dR}}) \cdot L_\nu \subset \mathrm{Gr}_G$. Then $S_\nu(C)$ has a unique g-fixed element, namely L_ν.*

Proof. Let $n \in U(B_{\mathrm{dR}})$ be such that nL_ν is a g-fixed point of $S_\nu(C)$. Then $nL_\nu = L_\mu$ for some $\mu \in X_*(T)$ by Corollary 5.1.3. Using Remark 5.1.1 we obtain

$$K \ni \mu(\xi)^{-1} n \nu(\xi) = (\mu(\xi)^{-1} n \mu(\xi))(\nu - \mu)(\xi).$$

Now $K \cap B(B_{\mathrm{dR}}) = U(B^+_{\mathrm{dR}}) \rtimes T(B^+_{\mathrm{dR}})$ and $(\nu - \mu)(\xi) \in T(B^+_{\mathrm{dR}})$ implies $\nu = \mu$, since $\xi \in B^+_{\mathrm{dR}}$ is a uniformizer. □

5.2 Calculation of local invariants

Let $T \subset G_{\breve{F}}$ be a maximal torus. Let $g \in T(\breve{F})$ be a regular semisimple element and , let $\mu, \nu \in X_*(T)$.

Let $E/\mathbf{Q}_\ell$ be a finite extension, and let $\Lambda = \mathcal{O}_E/\ell^n$ for some n. Let $\mathrm{IC}_{\mu,\Lambda}$ be the intersection complex on $\mathrm{Gr}_{G,\leq\mu}$ with coefficients in Λ. It is $G(B^+_{\mathrm{dR}})$-equivariant, so in particular it is g-equivariant. Furthermore $\mathrm{IC}_{\mu,\Lambda}$ is strongly reflexive by Assumption 3 applied to the trivial representation π.

Proposition 5.2.1. *The local term of g at L_ν is given by*

$$\mathrm{loc}_{L_\nu}(g, \mathrm{IC}_\mu) = (-1)^{2\rho(\nu)} \dim r_\mu[\nu].$$

Proof. Recall from Subsection 4.2 the facts about $\overline{S}_\nu$ and its cohomology that we are assuming. Let $j\colon S_\nu \hookrightarrow \overline{S}_\nu$ denote the inclusion, and let $i\colon \partial S_\nu \to \overline{S}_v$ denote the inclusion of the complement. Consider the exact sequence of $G(B_{\mathrm{dR}}^+)$-equivariant sheaves on the proper diamond $\overline{S}_\nu \cap \mathrm{Gr}_{\leq\mu}$:

$$0 \to j_! j^* \mathrm{IC}_\mu \to \mathrm{IC}_\mu \to i_* i^* \mathrm{IC}_\mu \to 0. \tag{5.2.1}$$

We will consider the trace of $g \in G(\breve{F})$ acting on the cohomology of each term. The cohomology $H^q(\bar{S}_\nu, j_! j^* \mathrm{IC}_\mu) = H^q_c(S_\nu, \mathrm{IC}_\mu)$ is zero unless $q = 2\rho(\nu)$. Since S_ν admits an action of the algebraic group T, and IC_μ is equivariant for this action, we have a morphism from T onto the constant scheme $\operatorname{End} H^{2\rho(\nu)}(S_\nu, \mathrm{IC}_\mu)$. Since T is connected, this morphism must be constant and the action of T is trivial. Therefore $\mathrm{tr}(g|H^*(S_\nu, j_! j^* \mathrm{IC}_\mu)) = (-1)^{2\rho(\nu)} \dim H_c^{2\rho(\nu)}(S_\nu, \mathrm{IC}_\mu) = (-1)^{2\rho(\nu)} \dim r_\mu[\nu]$.

Since $\overline{S}_\nu$ and ∂S_ν are proper diamonds, Corollary **??** can be used to give an expression for the trace of g on the cohomology of sheaves on either space. By Corollary 5.1.4, the fixed points of g on $\overline{S}_\nu$ (resp., ∂S_ν) are the points $L_{\nu'}$ for $\nu' \leq \nu$ (resp., $\nu' < \nu$). We get

$$\mathrm{tr}(g|R\Gamma(\overline{S}_\nu, \mathrm{IC}_\mu)) = \sum_{\nu' \leq \nu} \mathrm{loc}_{L_{\nu'}}(g, \mathrm{IC}_\mu)$$

and

$$\mathrm{tr}(g|R\Gamma(\overline{S}_\nu, i_* i^* \mathrm{IC}_\mu)) = \mathrm{tr}(g|R\Gamma(\partial S_\nu, \mathrm{IC}_\mu)) = \sum_{\nu' < \nu} \mathrm{loc}_{L_{\nu'}}(g, \mathrm{IC}_\mu).$$

Combining these observations with (5.2.1) gives the result. $\square$

5.3 The Beauville-Laszlo morphism on elliptic fixed points

Let $T \subset J_b$ be a F-rational elliptic maximal torus. We fix a complete algebraically closed field C/F; this determines an (absolute) Fargues-Fontaine curve $X = X(C^\flat)$ and a closed point $\infty \in X$. The completion $\widehat{\mathcal{O}}_{X,\infty}$ is the Fontaine period ring $B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+(C)$; this is a discrete valuation ring with uniformizer ξ, residue field C and fraction field B_{dR}.

Let $\nu\colon \mathbf{G}_{\mathrm{m}} \to T$ be a cocharacter defined over C. Then ν determines a geometric point $L_\nu \in \mathrm{Gr}_G(C)$. Let $\mathcal{E}_b[\nu]$ be the modification of $\mathcal{E}_b$ at ∞ via ν as in Definition 4.7.2. Thus we have isomorphisms

$$\mathcal{E}_b[\nu]|_{X\backslash\{\infty\}} \stackrel{\sim}{\to} \mathcal{E}_b|_{X\backslash\{\infty\}} \tag{5.3.1}$$

$$\mathcal{E}_b[\nu]_\infty \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \stackrel{\sim}{\to} \mathcal{E}_{b,\infty} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \tag{5.3.2}$$

of G-bundles over $X\setminus\{\infty\}$ and B_{dR}, respectively; the second identification carries $\mathcal{E}_b[\nu]_\infty$ onto the image of $\mathcal{E}_{b,\infty}$ under $\nu(\xi)\in T(B_{\mathrm{dR}})$.

The group $T(F)$ acts on $\mathcal{E}_b$ and commutes with $\nu(\xi)$, and so by Lemma 4.7.1 it acts on $\mathcal{E}_b[\nu]$.

Proposition 5.3.1. *Let $b\in G(\breve{F})$ be an element in the unique basic class $[b]\in B(G,\{\nu\})$.*

1. *The modified vector bundle $\mathcal{E}_b[\nu]$ is trivial: there exists an isomorphism $\mathcal{E}_b[\nu]\cong\mathcal{E}_1$. In other words L_ν lies in the b-admissible locus $\mathrm{Gr}_G^{b-\mathrm{adm}}$.*

2. *There exists a maximal F-rational torus $S\subset G$ and an isomorphism $\iota_{b,\nu}\colon T\to S$, such that for $g\in T(F)$, the isomorphism in (1) carries g onto $\iota_{b,\nu}(g)$. The pair $(S,\iota_{b,\nu})$ is well-defined up to conjugacy in $G(F)$.*

3. *The elements $g\in T(F)$ and $\iota_{b,\nu}(g)$ are related whenever g is strongly regular. Thus we have an element $\mathrm{inv}[b](\iota_{b,\nu}(g),g)\in B(S)$. The image of $\mathrm{inv}[b](\iota_{b,\nu}(g),g)$ under the isomorphism $B(S)\cong X^*(\widehat{S}^\Gamma)$ equals the restriction of $\iota_{b,\nu}\circ\nu\in X^*(\widehat{S})$ to $\widehat{S}^\Gamma$.*

Proof. Let $b'\in G(\breve{F})$ be an element whose class in $B(G)$ corresponds to the isomorphism class of $\mathcal{E}_b[\nu]$, and choose an isomorphism

$$\gamma\colon\mathcal{E}_b[\nu]\to\mathcal{E}_{b'}.$$

We first claim that $[b']$ is basic. We have the algebraic group $J_{b'}$, a priori an inner form of a Levi subgroup M^* of G^*, where G^* is as before the quasi-split inner form of G. Showing that $[b']$ is basic is equivalent to showing $M^*=G^*$. Let $g\in T(F)$ be a strongly regular semisimple element, so that $T=\mathrm{Cent}(g,J_b)$, and let $g'\in\mathrm{Aut}\,\mathcal{E}_{b'}$ be the automorphism induced by g via γ: $g'=\gamma g\gamma^{-1}$.

Recall from §4.5 that both $\mathcal{E}_b[\nu]$ and $\mathcal{E}_{b'}$ come equipped with a trivialization over B_{dR}^+. The stalks of g', g, and γ at the point $\infty\colon\mathrm{Spec}\,C\to X$ are thus naturally identified with elements of $G(B_{\mathrm{dR}}^+)$. Among them, $g'_\infty\in P(B_{\mathrm{dR}}^+)$, where $P\subset G$ is a parabolic subgroup with Levi factor $J_{b'}$. Let $\overline{g}'_\infty$ be the image of g_∞ under $P(B_{\mathrm{dR}}^+)\to J_{b'}(B_{\mathrm{dR}}^+)$. By Lemma 4.5.3, $\overline{g}'_\infty\in J_{b'}(F)$. By Lemma A.3.1 $\overline{g}'_\infty$ is conjugate to g'_∞, so $\overline{g}'_\infty$ is conjugate to g_∞ in $G(B_{\mathrm{dR}})$. Since $g,\overline{g}'$ are both regular semi-simple $\bar{F}$-points of G, being conjugate in $G(B_{\mathrm{dR}})$ is the same as being conjugate in $G(\bar{F})$. Their centralizers, being F-rational tori, are thus isomorphic over F. Thus $J_{b'}$ contains a maximal torus that is elliptic for G. Elliptic maximal tori transfer across inner forms [Kot86, §10], which means that the Levi subgroup $M^*\subset G^*$ of which $J_{b'}$ is an inner form contains a maximal torus that is elliptic for G^*. Therefore $M^* = G^*$.

We have shown that $\mathcal{E}_b[\nu]$ is semi-stable, implying that $\operatorname{Aut}\mathcal{E}_{b'} = J_{b'}(F)$ and that $g' \in J_{b'}(F)$. Lemma 4.7.3 shows that $\kappa([b']) = \kappa([b]) - \nu = 0$. Therefore $[b'] = [1]$ by [Kot85, Proposition 5.6], which is (1).

Let us assume that $b' = 1$, so that γ is an isomorphism $\mathcal{E}_b[\nu] \cong \mathcal{E}_1$ and $g' \in G(F) = \operatorname{Aut}\mathcal{E}_1$. Let $S = \operatorname{Cent}(g', G)$. Conjugation by γ induces an isomorphism of B_{dR}-rational tori $\iota_{b,\nu}\colon T \to S$ carrying g onto g'. We claim that this isomorphism respect the F-structures. As argued above there exists $y \in G(\bar{F})$ that conjugates g to g'. By Steinberg's theorem we can even take $y \in G(\breve{F})$. The isomorphism $\operatorname{Ad}(y) : T \to S$ maps the F-point g to the F-point g' and is the only such isomorphism. But $\sigma_{\mathrm{S}} \circ \operatorname{Ad}(y) \circ \sigma_T$ also maps g to g' and thus must equal $\operatorname{Ad}(y)$. In other words, $\operatorname{Ad}(y)$ respects the F-structures of T and S. Finally, $\gamma^{-1}y \in G(B_{\mathrm{dR}})$ centralizes g, so lies in $T(B_{\mathrm{dR}})$, so $\operatorname{Ad}(\gamma)$ and $\operatorname{Ad}(y)$ induce the same isomorphism $T \to S$. This proves (2).

Since $g \in J_b(F)$, we have $g^\sigma = b^{-1}gb$. On the other hand, since $g' \in G(F)$ we have $(g')^\sigma = g'$. Therefore the element $b_S := yby^{-\sigma}$ commutes with g' and so lies in $S(\breve{F})$. Recall that the class of b_S in $B(S)$ is the invariant $\operatorname{inv}[b](g', g)$.

The element y^{-1} induces an isomorphism $\mathcal{E}_{b_S} \to \mathcal{E}_b$, and also an isomorphism between modifications $\mathcal{E}_{b_S}[\iota_{b,\nu} \circ \nu] \to \mathcal{E}_b[\nu]$. Composing this with γ gives an isomorphism $\gamma y^{-1}\colon \mathcal{E}_{b_S}[\iota_{b,\nu} \circ \nu] \to \mathcal{E}_1$. Since $\gamma y^{-1} \in S(B_{\mathrm{dR}})$, this isomorphism descends to an isomorphism of S-bundles. By Lemma 4.7.3, the image of b_S in $B(S) \cong X^*(\widehat{S}^\Gamma)$ is $\iota_{b,\nu} \circ \nu|_{\widehat{S}^\Gamma}$, which establishes (3).

□

5.4 A character identity

Fix an element $b \in G(F^{\mathrm{nr}})$ for which $[b] \in B(G)$ is basic, an elliptic F-rational maximal torus $T \subset J_b$, and a cocharacter $\mu \in X_*(T) = X^*(\widehat{T})$. Assume that $[b]$ is the unique basic class in $B(G, \{\mu\})$.

There is a canonical $\widehat{G}$-conjugacy class of embeddings $\widehat{T} \to \widehat{G}$, of which we fix an arbitrary representative and identify $\widehat{T}$ with its image in $\widehat{G}$. Consider the irreducible representation r_μ of $\widehat{G}$ with highest weight μ.

If $\nu \in X^*(\widehat{T})$ is any weight of r_μ then $\nu|_{Z(\widehat{G})} = \mu|_{Z(\widehat{G})}$ and consequently $[b] \in B(G, \{\nu\})$. Proposition 5.3.1 shows that there exists an isomorphism $\mathcal{E}_b[\nu] \cong \mathcal{E}_1$ and an isomorphism $\iota_{b,\nu}\colon T \to S$ onto a F-rational maximal torus $S \subset G$, which translates the action of $T(F)$ on $\mathcal{E}_b[\nu]$ into the action of $S(F)$ on $\mathcal{E}_1$.

Proposition 5.4.1. *Let ϕ be a discrete L-parameter for G. Let $g \in T(F)$ be a regular element. For any $\pi \in \Pi_\phi(G)$ we have*

$$e(G) \sum_{\nu \in X^*(\widehat{T})} \Theta_\pi(\iota_{b,\nu}(g)) \dim r_\mu[\nu] = e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)\Theta_\rho(g). \tag{5.4.1}$$

Proof. We will combine the results of Proposition 5.3.1 with the character relations reviewed in Subsection 2.4. Let $s \in S_\phi$ be a semi-simple element, and let $\dot{s} \in S_\phi^+$ be a lift of it. Then we have the refined endoscopic datum $\dot{\mathfrak{e}} = (H, \mathcal{H}, \dot{s}, \eta)$ defined in (2.4.1); we choose as in that section a z-pair $\mathfrak{z} = (H_1, \eta_1)$. Then

$$\begin{aligned}
&e(G) \sum_{\pi' \in \Pi_\phi(G)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\pi'}(\dot{s})\Theta_{\pi'}(\iota_{b,\nu}(g)) \\
\overset{(2.4.3)}{=} &\sum_{h_1 \in H_1(F)/\mathrm{st}} \Delta(h_1, \iota_{b,\nu}(g)) S\Theta_{\phi^s}(h_1) \\
\overset{(2.4.4)}{=} &\sum_{h_1 \in H_1(F)/\mathrm{st}} \Delta(h_1, g) \left\langle \mathrm{inv}[b](g, \iota_{b,\nu}(g)), s_{h,g} \right\rangle S\Theta_{\phi^s}(h_1) \\
\overset{(5.3.1)(3)}{=} &\sum_{h_1 \in H_1(F)/\mathrm{st}} \Delta(h_1, g) \nu(s_{h,g})^{-1} S\Theta_{\phi^s}(h_1).
\end{aligned}$$

We multiply on either side by $\dim r_\mu[\nu]$ and sum over $\nu \in X_*(T)$ to obtain

$$\begin{aligned}
&e(G) \sum_{\nu \in X^*(\widehat{T})} \sum_{\pi' \in \Pi_\phi(G)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\pi'}(\dot{s})\Theta_{\pi'}(\iota_{b,\nu}(g)) \dim r_\mu[\nu] \\
= &\sum_{h_1 \in H_1(F)/\mathrm{st}} \Delta(h_1, g) \sum_{\nu \in X^*(\widehat{T})} \nu(s_{h,g}^{-1}) \dim r_\mu[\nu] S\Theta_{\phi^s}(h_1) \\
= &\sum_{h_1 \in H_1(F)/\mathrm{st}} \Delta(h_1, g)\, \mathrm{tr}\, r_\mu(s_{h,g}^{-1}) S\Theta_{\phi^s}(h_1) \\
\overset{(*)}{=} &\mathrm{tr}\, \check{r}_\mu(s^\natural) \sum_{h_1 \in H_1(F)/\mathrm{st}} \Delta(h_1, g) S\Theta_{\phi^s}(h_1) \\
\overset{(2.4.2)}{=} &\mathrm{tr}\, \check{r}_\mu(s^\natural) e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \mathrm{tr}\, \tau_{z,\mathfrak{w},\rho}(\dot{s})\Theta_\rho(g),
\end{aligned}$$

where $(*)$ holds since the image of $s_{h,g}$ under any admissible embedding $\widehat{T} \to \widehat{G}$ is conjugate to $s^\natural$ in $\widehat{G}$ and r_μ is a representation of $\widehat{G}$. Recall here that $s^\natural \in S_\phi$ is the image of $\dot{s}$ under (2.3.2).

Multiply both sides of above equation by $\operatorname{tr} \check{\tau}_{z,\mathfrak{w},\pi}(\dot{s})$. As functions of $\dot{s} \in S_\phi^+$ both sides then become invariant under $Z(\hat{\bar{G}})^+$ and thus become functions of the finite quotient $\bar{S}_\phi = S_\phi^+/Z(\hat{\bar{G}})^+ = S_\phi/Z(\widehat{G})^\Gamma$. Now apply $|\bar{S}_\phi|^{-1}\sum_{\bar{s}\in\bar{S}_\phi}$ to both sides to obtain an equality between

$$|\bar{S}_\phi|^{-1}e(G)\sum_{\bar{s}\in\bar{S}_\phi}\sum_{\nu\in X^*(\widehat{T})}\sum_{\pi'\in\Pi_\phi(G)}\operatorname{tr}\check{\tau}_{z,\mathfrak{w},\pi}(\dot{s})\operatorname{tr}\tau_{z,\mathfrak{w},\pi'}(\dot{s})\Theta_{\pi'}(\iota_{b,\nu}(g))\dim r_\mu[\nu]$$

and

$$|\bar{S}_\phi|^{-1}\sum_{\bar{s}\in\bar{S}_\phi}\operatorname{tr}\check{r}_\mu(s^\natural)e(J_b)\sum_{\rho\in\Pi_\phi(J_b)}\operatorname{tr}\check{\tau}_{z,\mathfrak{w},\pi}(\dot{s})\operatorname{tr}\tau_{z,\mathfrak{w},\rho}(\dot{s})\Theta_\rho(g),$$

where in both formulas $\dot{s}$ is an arbitrary lift of $\bar{s}$. Executing the sum over $\bar{s}$ in the first of the two expressions gives

$$e(G)\sum_{\nu\in X^*(\widehat{T})}\Theta_\pi(\iota_{b,\nu}(g))\dim r_\mu[\nu],$$

which is the left side of Eq. (5.4.1). To treat the second expression we note that by definition $\check{\tau}_{z,\mathfrak{w},\pi}\otimes\tau_{z,\mathfrak{w},\rho}(\dot{s}) = \delta_{\pi,\rho}(s^\natural)$. Furthermore, the composition of the map (2.3.2) with the natural projection $S_\phi \to S_\phi/Z(\widehat{G})^\Gamma$ is equal to the natural projection $S_\phi^+ \to S_\phi^+/Z(\hat{\bar{G}})^+ = S_\phi/Z(\widehat{G})^\Gamma = \bar{S}_\phi$. Thus $s^\natural$ is simply a lift of $\bar{s}$ to S_ϕ. This implies that

$$|\bar{S}_\phi|^{-1}\sum_{\bar{s}\in\bar{S}_\phi}\operatorname{tr}\check{r}_\mu(s^\natural)\delta_{\pi,\rho}(s^\natural) = \dim\operatorname{Hom}_{\mathbf{C}}(\delta_{\pi,\rho}, r_\mu)^{\bar{S}_\phi} = \dim\operatorname{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu).$$

□

5.5 Application of the Lefschetz-Verdier fixed-point formula

Recall the object $\mathcal{F}_\pi \in D_{\text{ét}}(\mathrm{Gr}_{G,\le\mu},\Lambda)$ from Definition 4.10.2. The cohomology $R\Gamma(\mathrm{Gr}_{G,\le\mu},\mathcal{F}_\pi)$ is an admissible derived $\Lambda J_b(F)$-module, with trace distribution χ relative to a Haar measure ν. We will use the Lefschetz-Verdier fixed-point formula to compute this trace when f is supported on the locus of strongly regular elliptic elements in $J_b(G)$, with the intention of applying Lemma 4.10.3.

Let $g \in J_b(F)$ be a strongly regular elliptic element. Then $\mathrm{Gr}_{G,\le\mu}^g$ is a finite set contained in $\mathrm{Gr}_{G,\le\mu}^{b\text{-adm}}$. Applying Theorem 3.16.1 shows that χ is

ν-smooth and

$$\begin{aligned}
\frac{d\chi}{d\nu}(g) &= \sum_{x \in \mathrm{Gr}^g_{G,\leq\mu}} \mathrm{loc}_x(u_g)\Theta_\pi(i_x(g)) \\
&\overset{5.2.1}{=} e(G)e(J_b) \sum_{\nu \in X^*(\widehat{T})} \Theta_\pi(\iota_{b,\nu(g)}) \dim r_\mu[\nu_x] \\
&\overset{5.4.1}{=} \sum_{\rho \in \Pi_\phi(J_b)} \dim \mathrm{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu)\Theta_\rho(g).
\end{aligned}$$

This concludes the proof of Theorem 1.0.4 (via its reduction in Lemma 4.10.3), because the Harish-Chandra traces of regular elliptic elements on both sides of that theorem agree.

A Elementary lemmas

A.1 A calculation of the Kottwitz sign

In this appendix, F is any local field of characteristic zero and G is a connected reductive group defined over F. We will give a formula for the Kottwitz sign $e(G)$ in terms of the dual group $\widehat{G}$. Fix a quasi-split inner form G^* and an inner twisting $\psi : G^* \to G$. Let $h \in H^1(\Gamma, G^*_{\mathrm{ad}})$ be the class of $\sigma \mapsto \psi^{-1}\sigma(\psi)$. Via the Kottwitz homomorphism [Kot86, Theorem 1.2] the class h corresponds to a character $\nu \in X^*(Z(\widehat{G}_{\mathrm{sc}})^\Gamma)$.

Choose an arbitrary Borel pair $(\widehat{T}_{\mathrm{sc}}, \widehat{B}_{\mathrm{sc}})$ of $\widehat{G}_{\mathrm{sc}}$ and let $2\rho \in X_*(\widehat{T}_{\mathrm{sc}})$ be the sum of the $\widehat{B}_{\mathrm{sc}}$-positive coroots. The restriction map $X^*(\widehat{T}_{\mathrm{sc}}) \to X^*(Z(\widehat{G}_{\mathrm{sc}}))$ is surjective and we can lift ν to $\dot{\nu} \in X^*(\widehat{T}_{\mathrm{sc}})$ and form $\langle 2\rho, \dot{\nu}\rangle \in \mathbf{Z}$. A different lift $\dot{\nu}$ would differ by an element of $X^*(\widehat{T}_{\mathrm{ad}})$, and since $\rho \in X_*(\widehat{T}_{\mathrm{ad}})$ we see that the image of $\langle 2\rho, \dot{\nu}\rangle$ in $\mathbf{Z}/2\mathbf{Z}$ is independent of the choice of lift $\dot{\nu}$. We thus write $\langle 2\rho, \nu\rangle \in \mathbf{Z}/2\mathbf{Z}$. Since any two Borel pairs in $\widehat{G}_{\mathrm{sc}}$ are conjugate $\langle 2\rho, \nu\rangle$ does not depend on the choice of $(\widehat{T}_{\mathrm{sc}}, \widehat{B}_{\mathrm{sc}})$.

Lemma A.1.1.

$$e(G) = (-1)^{\langle 2\rho, \nu\rangle}.$$

Proof. We fix Γ-invariant Borel pairs $(T_{\mathrm{ad}}, B_{\mathrm{ad}})$ in G^*_{ad} and $(\widehat{T}_{\mathrm{sc}}, \widehat{B}_{\mathrm{sc}})$ in $\widehat{G}_{\mathrm{sc}}$. Then we have the identification $X^*(T_{\mathrm{ad}}) = X_*(\widehat{T}_{\mathrm{sc}})$. Let $(T_{\mathrm{sc}}, B_{\mathrm{sc}})$ be the preimage in G^*_{sc} of $(T_{\mathrm{ad}}, B_{\mathrm{ad}})$.

By definition the Kottwitz sign is the image of h under

$$H^1(\Gamma, G^*_{\mathrm{ad}}) \xrightarrow{\delta} H^2(\Gamma, Z(G^*_{\mathrm{sc}})) \xrightarrow{\rho} H^2(\Gamma, \{\pm 1\}) \longrightarrow \{\pm 1\},$$

where $\rho \in X^*(T_{\rm sc})$ is half the sum of the $B_{\rm sc}$-positive roots and its restriction to $Z(G^*_{\rm sc})$ is independent of the choice of $(T_{\rm ad}, B_{\rm ad})$. By functoriality of the Tate-Nakayama pairing this is the same as pairing $\delta h \in H^2(\Gamma, Z(G^*_{\rm sc}))$ with $\rho \in H^0(\Gamma, X^*(Z(G^*_{\rm sc})))$. The canonical pairing $X^*(T_{\rm ad}) \otimes X^*(\widehat{T}_{\rm sc}) \to \mathbf{Z}$ induces the perfect pairing $X^*(T_{\rm sc})/X^*(T_{\rm ad}) \otimes X^*(\widehat{T}_{\rm sc})/X^*(\widehat{T}_{\rm ad}) \to \mathbf{Q}/\mathbf{Z}$ and hence the isomorphism $X^*(Z(G^*_{\rm sc})) \to \mathrm{Hom}_{\mathbf{Z}}(X^*(Z(\widehat{G}_{\rm sc})), \mathbf{Q}/\mathbf{Z}) = Z(\widehat{G}_{\rm sc})$, where the last equality uses the exponential map. Under this isomorphism $\rho \in X^*(Z(G^*_{\rm sc}))^\Gamma$ maps to the element $(-1)^{2\rho} \in Z(\widehat{G}_{\rm sc})^\Gamma$ obtained by mapping $(-1) \in \mathbf{C}^\times$ under $2\rho \in X^*(T_{\rm ad}) = X_*(\widehat{T}_{\rm sc})$. The lemma now follows from [Kot86, Lemma 1.8]. □

A.2 Integral supercuspidal representations

In this appendix, F is a non-archimedean local field of residual characteristic p and G is a connected reductive group defined over F. We denote by $G^{\rm ab}$ the maximal abelian quotient of G, a torus. Let $l \neq p$ be a prime. We shall record an immediate consequence of work of Vigneras.

Lemma A.2.1. *Let π be an irreducible representation of $G(F)$ on a $\bar{\mathbf{Q}}_l$-vector space. There exists a character $\chi : G^{\rm ab}(F) \to \bar{\mathbf{Q}}_l^\times$ such that the central character of $\pi \otimes \chi$ takes values in $\bar{\mathbf{Z}}_l$.*

Proof. Let A_G be the maximal split central torus in G. Consider the map $\mathrm{val}_{A_G} : A_G(F) \to X_*(A_G)$ given by $\langle \mathrm{val}_{A_G}(x), \alpha\rangle = \mathrm{val}_F(\alpha(x))$ for all $\alpha \in X^*(A_G)$, where $\mathrm{val}_F : F^\times \to \mathbf{Z}$ is the normalized valuation. This map fits into the exact sequence

$$1 \to A_G(F)_0 \to A_G(F) \to X_*(A_G) \to 0$$

where $A_G(F)_0$ is the maximal bounded subgroup of $A_G(F)$. Let ω_π be the central character of π and let $\mathrm{val}_l : \bar{\mathbf{Q}}_l^\times \to \mathbf{Q}$ be the normalized valuation of $\bar{\mathbf{Q}}_l$. The composition $\mathrm{val}_l \circ \omega_\pi$ is trivial on $A_G(F)_0$. Thus the restriction of this composition to $A_G(F)$ becomes a character $X_*(A_G) \to \mathbf{Q}$. Its image is a sublattice of $\mathbf{Q}$ and we can choose $N \in \mathbf{N}$ so that this image is contained in $\frac{1}{N}\mathbf{Z}$.

Let $X^*(G)_F = X^*(G)^\Gamma = X^*(G^{\rm ab})^\Gamma$ be the group of F-rational characters of G, $X_*(G)_F = \mathrm{Hom}_{\mathbf{Z}}(X^*(G)_F, \mathbf{Z})$ and let $\mathrm{val}_G : G^{\rm ab}(F) \to X_*(G)_F$ be defined just as in the case of A_G. It need not be surjective and we let $\Lambda(G)$ be its image. The restriction to $A_G(F)$ of val_G is not necessarily equal to val_{A_G}. The composition $\mathrm{val}_G \circ \mathrm{val}_{A_G}^{-1}$ gives an inclusion of lattices $X_*(A_G) \to \Lambda(G)$ with finite cokernel. We choose an extension of $\mathrm{val}_l \circ \omega_\pi : X_*(A_G) \to \frac{1}{N}\mathbf{Z}$ to a homomorphism $\Lambda(G) \to \frac{1}{M}\mathbf{Z}$ for a suitable multiple M of N and let χ' be the composition of this extension with val_G. Then $\chi' : G^{\mathrm{ab}}(F) \to \frac{1}{M}\mathbf{Z}$ is a group homomorphism whose restriction to $A_G(F)$ coincides with $\mathrm{val}_l \circ \omega_\pi$.

Choose an element $y \in \bar{\mathbf{Q}}_l^\times$ with $\mathrm{val}_l(y) = -\frac{1}{M}$ and let $\chi(g) = y^{M\chi'(g)}$. The central character of $\pi \otimes \chi$ is $\omega_\pi \cdot \chi$ and by construction its restriction to $A_G(F)$ takes values in $\bar{\mathbf{Z}}_l^\times$. Since the quotient $Z_G(F)/A_G(F)$ is compact, $\omega_\pi \cdot \chi$ must take values in $\bar{\mathbf{Z}}_l^\times$. □

Corollary A.2.2. *Let π be an irreducible supercuspidal representation of $G(F)$ on a $\bar{\mathbf{Q}}_l$ vector space. There exists a character $\chi : G^{\mathrm{ab}}(F) \to \bar{\mathbf{Q}}_l^\times$ such that $\pi \otimes \chi$ has a $\bar{\mathbf{Z}}_l$-lattice invariant under $G(F)$.*

Proof. This follows immediately from the above Lemma and [Vig96, II.4.12]. □

A.3 Some group theory

Let G be a connected reductive group over an algebraically closed field, $P \subset G$ a parabolic subgroup with Levi decomposition $P = MN$.

Lemma A.3.1. *If $m \in M$ and $n \in N$ are such that $mn \in P$ is regular semi-simple, then it is G-conjugate to m.*

Proof. Let $t = mn$. Being a semi-simple element of the algebraic group P, it is contained in a maximal torus $T \subset P$. Since P is a parabolic subgroup of G, T is also a maximal torus of G. Since T is contained in P, it normalizes N, and $T \rtimes N$ is solvable, hence contained in a Borel subgroup B of G. If U is the unipotent radical of B, then $N \subset U$. Then $m = tn^{-1}$ is conjugate to t by an element of U, according to [Hum95, 2.4]. □

A.4 Some commutative algebra lemmas

Lemma A.4.1. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$. Let $\kappa = R/\mathfrak{m}$ be the residue field and $\Lambda = R/\mathfrak{m}^k$ for some $k > 0$. For a Λ-module M we have the dual module $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$ and the natural morphisms $M \to M^{**}$ and $(M^* \otimes M) \to (M \otimes M^*)^*$.*

The morphism $M \to M^{**}$ is an isomorphism if and only if M is finitely generated.

Proof. For the "if" direction of the first point we note that the structure theorem for R-modules implies that a finitely generated Λ-module is a direct sum of finitely many cyclic Λ-modules, and each cyclic Λ-module is isomorphic to its own double dual.

Conversely assume that $M \to M^{**}$ is an isomorphism. We induct on k. For $k = 1$, Λ is a field and this is well-known. For general k we consider $N = M/\mathfrak{m}M$. Λ is an Artinian serial ring and hence injective as a module over itself. Thus the dualization functor is exact, and we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\cong"'] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathfrak{m}M)^{**} \arrow[r] & M^{**} \arrow[r] & N^{**} \arrow[r] & 0,
\end{tikzcd}$$

which shows that the right-most vertical map is surjective and the left-most vertical map is injective.

We have an isomorphism of Λ-modules $\mathfrak{m}^{m-1}\Lambda \to \kappa$, from which we obtain

$$N^* = \mathrm{Hom}_\Lambda(N, \Lambda) = \mathrm{Hom}_\Lambda(N, \mathfrak{m}^{m-1}\Lambda) \cong \mathrm{Hom}_\kappa(N, \kappa).$$

Thus N^{**} is also the double dual of N in the category of κ-vector spaces, and it is easy to check that the right-most vertical map is the canonical map in that category. Thus, this map is an isomorphism, and N is fintely generated as a κ-vector space.

By the Snake Lemma, the left-most vertical arrow in the diagram is an isomorphism. We can apply the inductive hypothesis to the $(\Lambda/\mathfrak{m}^{m-1})$-module $\mathfrak{m}M$ and conclude that it is finitely generated. Thus so is M.

$\square$

Lemma A.4.2. *Let Λ be an arbitrary ring, and let $D(\Lambda)$ be the derived category of Λ-modules. For an object M of $D(\Lambda)$, let $\mathbf{D}M = \mathrm{RHom}(M, \Lambda[0])$.*

1. *Assume that Λ is self-injective. The natural morphism $M \to \mathbf{DD}M$ is an isomorphism if and only if each $H^i(M)$ is finitely generated.*

2. *The following are equivalent:*

   (a) *The natural maps $M \to \mathbf{DD}M$ and $\mathbf{D}M \otimes M \to \mathbf{D}(M \otimes \mathbf{D}M)$ are isomorphisms.*

   (b) *The natural map $M \otimes \mathbf{D}M \to \mathrm{RHom}(M, M)$ is an isomorphism.*

   (c) *M is strongly dualizable: that is, for any object N, $N \otimes \mathbf{D}M \to \mathrm{RHom}(M, N)$ is an isomorphism.*

   (d) *M is a compact object; that is, the functor $N \mapsto \mathrm{RHom}(M, N)$ commutes with colimits.*

(e) *M is a perfect complex; that is, M is isomorphic to a bounded complex of finitely generated projective Λ-modules.*

(Throughout, the $\otimes$ means derived tensor product.)

Proof. For the first statement, the injectivity of Λ implies that $H^i(\mathbf{D}M) \cong H^{-i}(M)^*$, so that $H^i(\mathbf{DD}M) \cong H^i(M)^{**}$. Therefore $M \to \mathbf{DD}M$ is an isomorphism if and only if each $H^i(M) \to H^i(M)^{**}$ is an isomorphism. By Lemma A.4.1, this is equivalent to each $H^i(M)$ being finitely generated.

We now turn to the second statement. For (a) $\Longrightarrow$ (b), assume that $M \to \mathbf{DD}M$ and $\mathbf{D}M \otimes M \to \mathbf{D}(M \otimes \mathbf{D}M)$ are isomorphisms. Then $\mathrm{RHom}(M,M) \cong \mathrm{RHom}(M, \mathbf{DD}M) \cong \mathrm{RHom}(M \otimes \mathbf{D}M, \Lambda) \cong \mathbf{D}(M \otimes \mathbf{D}M) \cong \mathbf{D}M \otimes M$.

For (b) $\Longrightarrow$ (c), the identity map on M induces a morphism $\varepsilon\colon \Lambda[0] \to \mathrm{RHom}(M,M) \stackrel{\sim}{\to} M \otimes \mathbf{D}M$ (the coevaluation map). The required inverse to $N \otimes \mathbf{D}M \to \mathrm{RHom}(M,N)$ is

$$\mathrm{RHom}(M,N) \stackrel{\mathrm{id}\otimes\varepsilon}{\to} \mathrm{RHom}(M,N) \otimes M \otimes \mathbf{D}M \to N \otimes \mathbf{D}M.$$

For (c) $\Longrightarrow$ (d), we use the fact that $\otimes$ commutes with colimits.

For (d) $\Longrightarrow$ (e), we use the fact that compact objects of $D(\Lambda)$ are perfect [Sta17, Tag 07LT].

Finally, for (e) implies (a), we can write M as a bounded complex of finitely generated projective Λ-modules. Then duals and derived tensor products can be computed on the level of chain complexes. We are reduced to showing, for finitely generated projective Λ-modules A and B, that $A \to A^{**}$ and $A^* \otimes B \to (A \otimes B^*)^*$ are isomorphisms. After localizing on R, we may assume that A and B are free of finite rank (since duals commute over direct sums), where these statements are easy to check. □

We thank David Hansen and Bhargav Bhatt for their help with the above proof.

B A primer on reflexive sheaves, by David Hansen

In this appendix we discuss some basic examples and non-examples of reflexive sheaves, mostly in the context of classical rigid geometry. Although not strictly necessary in the main text of the paper, we hope these results might partially illuminate the hypotheses of reflexivity and strong reflexivity which recur throughout the paper. We also note that some closely related ideas were worked out almost simultaneously by Gaisin and Welliaveetil [GW17].

B.1 Results

Throughout what follows we fix an algebraically closed nonarchimedean base field C which we assume (for simplicity) is of mixed characteristic $(0,p)$. By an *adic space* we shall mean an adic space X over $S = \mathrm{Spa}(C, \mathcal{O}_C)$ which is locally of $+$-weakly finite type, separated, taut and finite-dimensional. By a *rigid analytic space* we shall mean an adic space of the aforementioned type which is locally of topologically finite type and reduced; note that this last condition is harmless, since replacing a rigid space by its nilreduction leaves the étale site unchanged.

Fix a prime power ℓ^n with $\ell \neq p$, and set $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$. For any separated taut finite-dimensional morphism $f : X \to Y$ of adic spaces which is locally of $+$-weakly finite type, Huber [Hub96, §5.5 & §7.1] defined a functor $Rf_! : D(X_{\text{ét}}, \Lambda) \to D(Y_{\text{ét}}, \Lambda)$ admitting a right adjoint $Rf^!$. In particular, if X is an adic space with structure morphism $f : X \to S$, we may consider the dualizing complex $\kappa_X \overset{\text{def}}{=} Rf^!\Lambda$ and the duality functor

$$\begin{array}{rcl} D(X_{\text{ét}}, \Lambda) & \to & D(X_{\text{ét}}, \Lambda) \\ \mathcal{F} & \mapsto & \mathbf{D}\mathcal{F} \overset{\text{def}}{=} \underline{\mathrm{RHom}}(\mathcal{F}, \kappa_X). \end{array}$$

Definition B.1.1. An object $\mathcal{F} \in D(X_{\text{ét}}, \Lambda)$ is *reflexive* if the natural biduality map

$$\mathcal{F} \to \mathbf{D}\mathbf{D}\mathcal{F}$$

is an isomorphism.

As in the main text of the paper, this property is clearly étale-local on X, and is preserved under pullback along smooth maps and derived pushforward along proper maps. Moreover, reflexivity satisfies a 2-out-of-3 property: if $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to$ is an exact triangle in $D(X_{\text{ét}}, \Lambda)$ such that two terms in the triangle are reflexive, then all three terms are reflexive. We also note that if $i : Z \to X$ is a closed embedding and $\mathcal{F} \in D(Z_{\text{ét}}, \Lambda)$ is such that $i_*\mathcal{F}$ is reflexive, then $\mathcal{F}$ is reflexive. Finally, we observe that if $\mathcal{F}$ is bounded with reflexive cohomology sheaves, then $\mathcal{F}$ is reflexive itself.

One can make an analogous definition in the world of classical algebraic geometry, and it's a standard fact that *constructible* sheaves are reflexive in that setting. We remind the reader that if $\mathcal{X}$ is a separated finite type C-scheme with associated rigid analytic space X, then pullback along the natural map of sites $\mu : X_{\text{ét}} \to \mathcal{X}_{\text{ét}}$ does not preserve constructibility, essentially because Zariski-open subsets of X are not quasicompact. Instead, the μ-pullback of a constructible sheaf on $\mathcal{X}_{\text{ét}}$ is an example of a *Zariski-constructible* sheaf on $X_{\text{ét}}$. There is also an intrinsic notion of constructible sheaf in the rigid analytic world, which is of a rather different flavor. Our first order of business is to check that these examples are all reflexive:

Proposition B.1.2. *If X is a rigid analytic space, then any object $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ with constructible or Zariski-constructible cohomology sheaves is reflexive.*

In §B.3 below, we sketch a direct proof that constructible sheaves are reflexive. The idea is to first show that the constant sheaf Λ is reflexive on any rigid space X, which we then upgrade to the reflexivity of $j_!\Lambda$ where $j : U \to X$ is any separated étale map with affinoid source. For the reflexivity of Λ, we reduce to the smooth case using resolution of singularities.

However, it is more conceptual to deduce Proposition B.1.2 from a general criterion for reflexivity which was explained to us by Peter Scholze. To state this result, recall that for any (reduced) affinoid rigid space $U = \mathrm{Spa}(A, A^\circ)$ with its natural formal model $\mathfrak{U} = \mathrm{Spf}(A^\circ)$ over $\mathrm{Spf}(\mathcal{O}_C)$, there is a natural map of sites $\lambda_U : U_{\text{ét}} \to \mathfrak{U}_{\text{ét}}$ which induces a "nearby cycles" map $R\lambda_{U*} : D^b(U_{\text{ét}}, \Lambda) \to D^b(\mathfrak{U}_{\text{ét}}, \Lambda)$.

Proposition B.1.3 (Scholze)**.** *Let X be a rigid analytic space. Suppose $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ has the property that for every affinoid rigid space $U = \mathrm{Spa}(A, A^\circ)$ with an étale map $a : U \to X$, the nearby cycles $R\lambda_{U*}a^*\mathcal{F}$ are constructible. Then $\mathcal{F}$ is reflexive.*

Combining this with a result of Huber, we deduce

Corollary B.1.4. *If X is a rigid analytic space and $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ has* quasi-constructible *or* oc-quasi-constructible *cohomology sheaves in the sense of [Hub98b, Hub98c], then $\mathcal{F}$ is reflexive.*

In the setting of sheaves on a finite type C-scheme, it may be true that reflexivity and constructibility coincide. One can thus ask whether reflexivity on rigid spaces is characterized by some variant of constructibility; however, this seems unlikely:

Proposition B.1.5. *There is an example of a reflexive sheaf on $\mathrm{Spa}\, C \langle T_1, \ldots, T_6 \rangle$ with (some) infinite-dimensional stalks.*

Finally, we illustrate the failure of the Lefschetz fixed-point formula with an example of a reflexive sheaf which is not strongly reflexive.

Proposition B.1.6. *Let $X = \mathrm{Spa}(C \langle T \rangle, \mathcal{O}_C \langle T \rangle)$ be the one-dimensional rigid disk, and let $\overline{X} = \mathrm{Spa}(C \langle T \rangle, \mathcal{O}_C + T \cdot C \langle T \rangle^{\circ\circ})$ be its canonical adic compactification, with $j : X \to \overline{X}$ the natural inclusion. Then the sheaves $\Lambda_{\overline{X}}$ and $j_!\Lambda_X$ are reflexive but* not *strongly reflexive.*

This stands in contrast to the situation in classical algebraic geometry, where any constructible sheaf on a finite type C-scheme is strongly reflexive. In the course of building this example, we determine the dualizing complex of $\overline{X}$; rather strangely, it turns out that $\kappa_{\overline{X}} \simeq j_! \Lambda_X[2](1)$. In particular, the dualizing complex of $\overline{X}$ is not overconvergent, and some of its stalks vanish identically, in stark contrast with the case of rigid analytic spaces, cf. Proposition B.3.4. Morally, the failure of $\kappa_{\overline{X}}$ to overconverge on the locus lying over the topological fixed points of $T \mapsto T+1$ is "responsible" for the failure of the Lefschetz formula for this automorphism.

B.2 Nearby cycles and reflexivity

In this section we deduce Proposition B.1.3 from the following result, elaborating on a sketch explained to us by Peter Scholze.

Proposition B.2.1. *Let $X = \mathrm{Spa}(A, A^\circ)$ be an affinoid rigid analytic space over $\mathrm{Spa}(C, \mathcal{O}_C)$ as before; set $\mathfrak{X} = \mathrm{Spf}(A^\circ)$, so we get a nearby cycles map $R\lambda_{X*} : D^b(X_{\text{ét}}, \Lambda) \to D^b(\mathfrak{X}_{\text{ét}}, \Lambda)$ as in the introduction. Then there is a natural equivalence $R\lambda_{X*}\mathbf{D}_X \cong \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}$ compatible with étale localization and with the biduality maps, where $\mathbf{D}_X$ and $\mathbf{D}_{\mathfrak{X}}$ denote the natural Verdier duality functors on $X_{\text{ét}}$ and $\mathfrak{X}_{\text{ét}}$.*

In most other settings where a nearby cycles functor is defined, commutation with Verdier duality is well-known (cf. [Ill94] and [Mas16], for example). However, the present situation is somewhat unique in that $R\lambda_{X*}$ admits a useful left adjoint, which we'll exploit heavily in the proof of Proposition B.2.1.

Proof of Proposition B.1.3. Fix $\mathcal{F} \in D^b(X_{\text{ét}}, \Lambda)$ satisfying the conditions of the proposition. We need to show that the cone of the biduality map $\beta : \mathcal{F} \to \mathbf{D}_X \mathbf{D}_X \mathcal{F}$ is acyclic. Given any étale map $a : U = \mathrm{Spa}(B, B^\circ) \to X$, the constructibility hypothesis in the proposition guarantees that the biduality map

$$R\lambda_{U*} a^* \mathcal{F} \to \mathbf{D}_{\mathfrak{U}} \mathbf{D}_{\mathfrak{U}} R\lambda_{U*} a^* \mathcal{F}$$

is an isomorphism by Théorème 4.3 in [Del77, Th. de finitude]. We then see that the biduality map $a^*\mathcal{F} \to \mathbf{D}_U \mathbf{D}_U a^* \mathcal{F} \cong a^* \mathbf{D}_X \mathbf{D}_X \mathcal{F}$ induces a map

$$R\lambda_{U*} a^* \mathcal{F} \to R\lambda_{U*} \mathbf{D}_U \mathbf{D}_U a^* \mathcal{F} \cong \mathbf{D}_{\mathfrak{U}} R\lambda_{U*} \mathbf{D}_U a^* \mathcal{F} \cong \mathbf{D}_{\mathfrak{U}} \mathbf{D}_{\mathfrak{U}} R\lambda_{U*} a^* \mathcal{F} \cong R\lambda_{U*} a^* \mathcal{F}$$

whose composition is the identity; here the first two isomorphisms are obtained by applying Proposition B.2.1 twice, and the third isomorphism is given by the inverse of the biduality map for $\mathfrak{U}$. In particular, the map

$$R\lambda_{U*}a^*\beta : R\lambda_{U*}a^*\mathcal{F} \to R\lambda_{U*}a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}$$

is an isomorphism. Passing to derived global sections on $\mathfrak{U}$ gives an isomorphism

$$R\Gamma(U, \mathcal{F}|_U) \cong R\Gamma(\mathfrak{U}, R\lambda_{U*}a^*\mathcal{F}) \cong R\Gamma(\mathfrak{U}, R\lambda_{U*}a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}) \cong R\Gamma(U, \mathbf{D}_X\mathbf{D}_X\mathcal{F}|_U),$$

so in particular $R\Gamma(U, \mathrm{Cone}(\beta)) \simeq 0$ for all affinoid étale maps $U \to X$. Since the stalks of the cohomology sheaves of $\mathrm{Cone}(\beta)$ can be computed as colimits of $H^i\left(R\Gamma(U_j, \mathrm{Cone}(\beta))\right)$ over suitable cofiltered inverse systems of affinoid étale maps $U_j \to X$, we deduce that $\mathrm{Cone}(\beta)$ is acyclic, as desired. □

This criterion is very useful in practice:

Proof of Proposition B.1.2 and Corollary B.1.4 . In [Hub98b, Hub98c], Huber defines classes of étale sheaves which he calls quasi-constructible and oc-quasi-constructible, which are preserved under arbitrary pullback and with the property that any constructible (resp. Zariski-constructible) sheaf is quasi-constructible (resp. oc-quasi-constructible). Moreover, he proves that the nearby cycles of such sheaves are always constructible, cf [Hub98b, Prop. 3.11] and [Hub98c, Prop. 2.12]. Combining these results with Proposition B.1.3, we get the result. □

The proof of Proposition B.2.1 requires some work. Throughout the rest of this subsection, we fix X and $\mathfrak{X}$ as in the statement of the result, and we let $f : X_{\text{ét}} \to \mathrm{Spa}(C, \mathcal{O}_C)_{\text{ét}}$ and $\mathfrak{f} : \mathfrak{X}_{\text{ét}} \to \mathrm{Spec}(\mathcal{O}_C/p)_{\text{ét}}$ denote the natural morphisms of étale sites. Note that we can identify the derived categories $D(\mathrm{Spa}(C, \mathcal{O}_C)_{\text{ét}}, \Lambda)$ and $D(\mathrm{Spec}(\mathcal{O}_C/p)_{\text{ét}}, \Lambda)$ with the derived category $D(\Lambda)$ of Λ-modules. The key non-formal ingredient in the argument is the existence of a natural equivalence $Rf_! \cong R\mathfrak{f}_! R\lambda_{X*}$, which can be proved as in [Hub98c, Lemma 2.13].

Proof. We first construct a natural transformation $\gamma_X : R\lambda_{X*}\mathbf{D}_X \to \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}$. To do this, observe that for any $A \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ and $B \in D(X_{\text{ét}}, \Lambda)$, we have a natural series of morphisms

$$
\begin{aligned}
\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, R\lambda_{X*}\mathbf{D}_X B) &\overset{(1)}{\cong} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A, \mathbf{D}_X B) \\
&\overset{(2)}{\cong} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} B, Rf^! \Lambda) \\
&\overset{(3)}{\cong} \mathrm{Hom}_{D(\Lambda)}(Rf_!(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \\
&\overset{(4)}{\cong} \mathrm{Hom}_{D(\Lambda)}(R\mathfrak{f}_! R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \\
&\overset{(5)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), R\mathfrak{f}^! \Lambda) \\
&\overset{(6)}{\to} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A \otimes^{\mathbf{L}} R\lambda_{X*}B, R\mathfrak{f}^! \Lambda) \\
&\overset{(7)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}B)
\end{aligned}
$$

obtained as follows: (1) follows from adjointness of $R\lambda_{X*}$ and λ_X^*; (2) and (7) are tensor-hom adjunction; (3) follows from adjointness of $Rf_!$ and $Rf^!$; (4) follows from the natural equivalence $Rf_! \cong R\mathfrak{f}_! R\lambda_{X*}$ explained above; (5) follows from adjointness of $R\mathfrak{f}_!$ and $R\mathfrak{f}^!$; and (6) is dual to the natural "projection map" $A \otimes^{\mathbf{L}} R\lambda_{X*}B \to R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B)$ obtained as the adjoint to the composition

$$\lambda_X^*(A \otimes^{\mathbf{L}} R\lambda_{X*}B) \cong \lambda_X^* A \otimes^{\mathbf{L}} \lambda_X^* R\lambda_{X*}B \to \lambda_X^* A \otimes^{\mathbf{L}} B,$$

cf. [Sta17, Tag 0943]. The composition of these morphisms induces a map

$$\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, R\lambda_{X*}\mathbf{D}_X B) \to \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}B)$$

which is functorial in A and B, so we obtain the desired natural transformation from the Yoneda lemma. Next we observe that when A is perfect, the map (6) is an isomorphism by [Sta17, Tag 0943]. In particular, taking $A = \Lambda[n]$ for varying n, we see that γ_X induces an isomorphism $R\Gamma(\mathfrak{X}, R\lambda_{X*}\mathbf{D}_X B) \overset{\sim}{\to} R\Gamma(\mathfrak{X}, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}B)$ for any B.

Now let $\mathfrak{Y} = \mathrm{Spf}(B^\circ)$ be any affine formal scheme with an étale map $\mathfrak{j} : \mathfrak{Y} \to \mathfrak{X}$, and let $j : Y = \mathrm{Spa}(B, B^\circ) \to X$ denote the induced étale map on rigid generic fibers. We then claim that formation of γ_X is compatible with étale localization, in the sense that the natural diagram

$$
\begin{tikzcd}
\mathfrak{j}^* R\lambda_{X*}\mathbf{D}_X B \arrow[r, "\sim"] \arrow[d, "\mathfrak{j}^*\gamma_X"] & R\lambda_{Y*}\mathbf{D}_Y j^* B \arrow[d, "\gamma_Y"] \\
\mathfrak{j}^* \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}B \arrow[r, "\sim"] & \mathbf{D}_{\mathfrak{Y}} R\lambda_{Y*} j^* B
\end{tikzcd}
$$

commutes; here the horizontal isomorphisms are induced by the natural isomorphisms $\mathfrak{j}^*\mathbf{D}_{\mathfrak{X}} \cong \mathbf{D}_{\mathfrak{Y}}\mathfrak{j}^*$ and $j^*\mathbf{D}_X \cong \mathbf{D}_Y j^*$ together with the (easy) base change isomorphism $\mathfrak{j}^* R\lambda_{X*} \cong R\lambda_{Y*}j^*$. Granted the commutativity of this diagram, passing to derived global sections on $\mathfrak{Y}$ induces a commutative diagram

$$\begin{tikzcd} R\Gamma\left(\mathfrak{Y}, (R\lambda_{X*}\mathbf{D}_X B)|_{\mathfrak{Y}}\right) \arrow[r, "\sim"] \arrow[d] & R\Gamma(\mathfrak{Y}, R\lambda_{Y*}\mathbf{D}_Y j^*B) \arrow[d, "\wr"] \\ R\Gamma\left(\mathfrak{Y}, (\mathbf{D}_{\mathfrak{X}} R\lambda_{X*}B)|_{\mathfrak{Y}}\right) \arrow[r, "\sim"] & R\Gamma(\mathfrak{Y}, \mathbf{D}_{\mathfrak{Y}} R\lambda_{Y*}j^*B) \end{tikzcd}$$

where, crucially, the righthand vertical arrow is an isomorphism by arguing as in the previous paragraph with γ_Y in place of γ_X. Going around the diagram, we see that γ_X induces an isomorphism $R\Gamma(\mathfrak{Y}, R\lambda_{X*}\mathbf{D}_X B) \xrightarrow{\sim} R\Gamma(\mathfrak{Y}, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*}B)$ for any affine étale map $\mathfrak{Y} \to \mathfrak{X}$, and therefore γ_X is an equivalence, as desired.

It remains to check the commutativity of the aforementioned square. This follows from a rather horrible diagram chase. More precisely, choose any $C \in D(\mathfrak{Y}_{\text{ét}}, \Lambda)$, and set $A = \mathfrak{j}_!C$; we then need to check that the diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, R\lambda_{X*}\mathbf{D}_X B) \arrow[r, "\sim", "\text{i.}"'] \arrow[d] & \mathrm{Hom}_{D(\mathfrak{Y}_{\text{ét}},\Lambda)}(C, R\lambda_{Y*}\mathbf{D}_Y j^* B) \arrow[d] \\
\mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A, \mathbf{D}_X B) \arrow[r, "\text{ii.}"'] \arrow[d] & \mathrm{Hom}_{D(Y_{\text{ét}},\Lambda)}(\lambda_Y^* C, \mathbf{D}_Y j^* B) \arrow[d] \\
\mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} B, Rf^! \Lambda) \arrow[r, "\text{iii.}"'] \arrow[d] & \mathrm{Hom}_{D(Y_{\text{ét}},\Lambda)}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B, j^! Rf^! \Lambda) \arrow[d] \\
\mathrm{Hom}_{D(\Lambda)}(Rf_!(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \arrow[r, "\text{iv.}"'] \arrow[d] & \mathrm{Hom}_{D(\Lambda)}(Rf_! j_!(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \Lambda) \arrow[d] \\
\mathrm{Hom}_{D(\Lambda)}(R\mathfrak{f}_! R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \arrow[r, "\text{v.}"'] \arrow[d] & \mathrm{Hom}_{D(\Lambda)}(R\mathfrak{f}_! \mathsf{j}_! R\lambda_{Y*}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \Lambda) \arrow[d] \\
\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), R\mathfrak{f}^! \Lambda) \arrow[r, "\text{vi.}"'] \arrow[d] & \mathrm{Hom}_{D(\mathfrak{Y}_{\text{ét}},\Lambda)}(R\lambda_{Y*}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \mathsf{j}^! R\mathfrak{f}^! \Lambda) \arrow[d] \\
\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A \otimes^{\mathbf{L}} R\lambda_{X*} B, R\mathfrak{f}^! \Lambda) \arrow[r, "\text{vii.}"'] \arrow[d] & \mathrm{Hom}_{D(\mathfrak{Y}_{\text{ét}},\Lambda)}(C \otimes^{\mathbf{L}} R\lambda_{Y*} j^* B, \mathsf{j}^! R\mathfrak{f}^! \Lambda) \arrow[d] \\
\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B) \arrow[r, "\sim", "\text{viii.}"'] & \mathrm{Hom}_{D(\mathfrak{Y}_{\text{ét}},\Lambda)}(C, \mathbf{D}_{\mathfrak{Y}} R\lambda_{Y*} j^* B)
\end{tikzcd}
$$

commutes, functorially in C and B. Here the top and bottom horizontal arrows correspond to the horizontal isomorphisms in our original square, and the composition of all left and right vertical maps define $\mathsf{j}^*\gamma_X$ and γ_Y, respectively, by Yoneda. Let us explain the intermediate horizontal arrows together with some of the commutativity checks, leaving a few details to the reader. The idea is to repeatedly use adjointness of the pairs $(j_!, j^* = j^!)$ and $(\mathsf{j}_!, \mathsf{j}^* = \mathsf{j}^!)$, together with the base change isomorphism $\mathsf{j}^* R\lambda_{X*} \cong R\lambda_{Y*} j^*$ and its adjoint incarnation $\lambda_X^* \mathsf{j}_! \cong j_! \lambda_Y^*$. In particular, applying the latter to $A = \mathsf{j}_! C$ gives a natural isomorphism $\lambda_X^* A \cong j_! \lambda_Y^* C$. Combining this isomorphism with the adjunction of $j_!$ and j^* induces the arrow labeled ii.; on the other hand, tensoring this isomorphism with B and applying the projection formula for $j_!$ gives

$$\lambda_X^* A \otimes^{\mathbf{L}} B \cong j_!(\lambda_Y^* C \otimes j^* B),$$

which induces arrows iii. and iv. (here we've again used the adjunction of $j_!$ and j^*).

Next, by Lemma B.2.2 below, there is a natural equivalence $\tau : \mathfrak{j}_! R\lambda_{Y*} \to R\lambda_{X*} j_!$, which moreover is compatible with $R\mathfrak{f}_!$ in the sense that the composite map

$$R\mathfrak{f}_! \mathfrak{j}_! R\lambda_{Y*} \overset{R\mathfrak{f}_!\tau}{\to} R\mathfrak{f}_! R\lambda_{X*} j_! \cong R f_! j_!$$

induces the natural equivalence

$$R(\mathfrak{f} \circ \mathfrak{j})_! R\lambda_{Y*} \cong R(f \circ j)_!.$$

Applying this transformation to $\lambda_Y^* C \otimes^{\mathbf{L}} j^* B$ induces a map

$$\mathfrak{j}_! R\lambda_{Y*}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B) \to R\lambda_{X*} j_!(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B) \cong R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B),$$

which gives rise to arrows v. and vi. via suitable adjunctions; commutativity of the square spanned by arrows iv. and v. follows from the aforementioned compatibility of this transformation with $R\mathfrak{f}_!$, and commutativity of the square spanned by arrows v. and vi. is straightforward. Next, we observe that the previous map together with the projection maps

$$A \otimes^{\mathbf{L}} R\lambda_{X*} B \overset{\pi_X}{\to} R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B)$$

and

$$C \otimes^{\mathbf{L}} R\lambda_{Y*} j^* B \overset{\pi_Y}{\to} R\lambda_{Y*}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B)$$

fit together sits in a commutative square

$$\begin{tikzcd}
R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B) & \mathfrak{j}_! R\lambda_{Y*}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B) \arrow[l] \\
A \otimes R\lambda_{X*} B \arrow[u, "\pi_X"] & \mathfrak{j}_!(C \otimes R\lambda_{Y*} j^* B) \arrow[l] \arrow[u, "\mathfrak{j}_! \pi_Y"']
\end{tikzcd}$$

where the lower horizontal arrow is given by the inverse of the composition

$$A \otimes R\lambda_{X*} B = \mathfrak{j}_! C \otimes R\lambda_{X*} B \cong \mathfrak{j}_! \left(C \otimes \mathfrak{j}^* R\lambda_{X*} B\right) \cong \mathfrak{j}_!(C \otimes R\lambda_{Y*} j^* B).$$

Applying $\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(-, R\mathfrak{f}^! \Lambda)$ to this square and using the adjunction of $\mathfrak{j}_!$ and $\mathfrak{j}^!$ on the righthand column, we get arrows vi. and vii. together with the commutativity of the relevant square. $\square$

In the course of these arguments, we used the following lemma.

Lemma B.2.2. *Let X and $\mathfrak{X}$ be as above, and let $\mathfrak{j} : \mathfrak{Y} = \mathrm{Spf}(B^\circ) \to \mathfrak{X}$ be an étale map of affine formal schemes, with $j : Y = \mathrm{Spa}(B, B^\circ) \to X$ the induced map on rigid generic fibers. Then the natural transformation $\tau : \mathfrak{j}_! R\lambda_{Y*} \to R\lambda_{X*} j_!$ defined as the adjoint to the composition*

$$R\lambda_{Y*} \to R\lambda_{Y*} j^* j_! \cong \mathfrak{j}^* R\lambda_{X*} j_!$$

is an equivalence, and is compatible with $R\mathfrak{f}_!$ in the sense that the composite map

$$R\mathfrak{f}_! \mathfrak{j}_! R\lambda_{Y*} \overset{R\mathfrak{f}_!\tau}{\to} R\mathfrak{f}_! R\lambda_{X*} j_! \cong R f_! j_!$$

coincides with the natural equivalence

$$R(\mathfrak{f} \circ \mathfrak{j})_! R\lambda_{Y*} \cong R(f \circ j)_!.$$

Proof. By a standard argument (cf. [Sta17, Tag 0AN8]), we can find an étale ring map $A^\circ \to B_0$ such that $B^\circ = \lim_{\leftarrow} B_0/p^n B_0$ as A°-algebras. By Zariski's main theorem, we can find a module-finite ring map $A^\circ \to D$ fitting into a diagram

$$\begin{tikzcd} \mathrm{Spec}(B_0) \arrow[r, "i"] \arrow[dr, "j"'] & \mathrm{Spec}(D) \arrow[d, "h"] \\ & \mathrm{Spec}(A^\circ) \end{tikzcd}$$

where i is an open immersion. Passing to p-adic completions induces a corresponding diagram

$$\begin{tikzcd} \mathfrak{Y} = \mathrm{Spf}(B^\circ) \arrow[r, "\mathfrak{i}"] \arrow[dr, "\mathfrak{j}"'] & \overline{\mathfrak{Y}} = \mathrm{Spf}(D) \arrow[d, "\mathfrak{h}"] \\ & \mathfrak{X} = \mathrm{Spf}(A^\circ) \end{tikzcd}$$

of p-adic formal schemes; here we used the fact that $D \cong \lim_{\leftarrow} D/p^n D$, which holds since $A^\circ \to D$ is module-finite and A° is p-adically separated and complete and Noetherian outside its ideal of definition, cf. [FGK11, Proposition 6.1.1(1)]. Passing to a similar diagram on the generic fibers and going to étale sites, we get a commutative diagram

$$\begin{tikzcd}
& Y_{\text{ét}} \arrow[rr, "\lambda_Y"] \arrow[dl, "i"'] \arrow[dd, "j" near start] & & \mathfrak{Y}_{\text{ét}} \arrow[dl, "\mathfrak{i}"'] \arrow[dd, "\mathfrak{j}"] \\
\overline{Y}_{\text{ét}} \arrow[rr, "\lambda_{\overline{Y}}" near end] \arrow[dr, "h"'] & & \overline{\mathfrak{Y}}_{\text{ét}} \arrow[dr, "\mathfrak{h}"] & \\
& X_{\text{ét}} \arrow[rr, "\lambda_X"] & & \mathfrak{X}_{\text{ét}}
\end{tikzcd}$$

where the $\lambda_\bullet$'s are the evident nearby cycles maps. Note that $h_* \cong Rh_*$ and $\mathfrak{h}_* \cong R\mathfrak{h}_*$ since both morphisms are finite. We then compute that

$$\begin{aligned} \mathfrak{j}_! R\lambda_{Y*} &= \mathfrak{h}_* \mathfrak{i}_! R\lambda_{Y*} \\ &\cong \mathfrak{h}_* R\lambda_{\overline{Y}} i_! \\ &\cong R\lambda_{X*} h_* i_! \\ &= R\lambda_{X*} j_! \end{aligned}$$

where the first isomorphism follows from [Hub96, Corollary 3.5.11.ii] and the second isomorphism is induced by the natural equivalence $\mathfrak{h}_* \lambda_{\overline{Y}*} = \lambda_{X*} h_*$. $\square$

Although not strictly necessary, let us develop a little more theory around the nearby cycles map $R\lambda_{X*}$.

Proposition B.2.3. *Let $X = \operatorname{Spa}(A, A^\circ)$ and $\mathfrak{X} = \operatorname{Spf}(A^\circ)$ be as above. Then $R\lambda_{X*}$ sends reflexive objects to reflexive objects, and the functor $R\lambda_X^! \overset{\text{def}}{=} \mathbf{D}_X \lambda_X^* \mathbf{D}_{\mathfrak{X}}$ defines a "weak" right adjoint of $R\lambda_{X*}$ on reflexive objects, in the sense that it induces functorial isomorphisms*

$$R\lambda_{X*} \underline{\mathrm{RHom}}_X(B, R\lambda_X^! C) \cong \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, C)$$

and

$$\operatorname{Hom}_{D(\mathfrak{X}_{\text{ét}}, \Lambda)}(R\lambda_{X*} B, C) \cong \operatorname{Hom}_{D(X_{\text{ét}}, \Lambda)}(B, R\lambda_X^! C)$$

for any objects $B \in D(X_{\text{ét}}, \Lambda)$ and $C \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$ with C reflexive.

Setting $C = \kappa_{\mathfrak{X}}$ recovers Proposition B.2.1 as a special case of this result, although we use Proposition B.2.1 in the proof. We call $R\lambda_X^!$ a "weak" right adjoint because we're not sure how it behaves on non-reflexive objects, or whether it preserves reflexivity; amusingly, the argument doesn't require the latter fact.

Proof. The local result implies the global result upon applying $H^0(R\Gamma(\mathfrak{X}, -))$. For the local result, we calculate that

$$\begin{aligned}
\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, R\lambda_{X*}\underline{\mathrm{RHom}}_X(B, R\lambda_X^! C)) &\overset{(1)}{\cong} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A, \underline{\mathrm{RHom}}_X(B, R\lambda_X^! C)) \\
&\overset{(2)}{\cong} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A, \underline{\mathrm{RHom}}_X(\lambda_X^* \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_X B)) \\
&\overset{(3)}{\cong} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} \lambda_X^* \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_X B) \\
&\overset{(4)}{\cong} \mathrm{Hom}_{D(X_{\text{ét}},\Lambda)}(\lambda_X^* (A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C), \mathbf{D}_X B) \\
&\overset{(5)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C, R\lambda_{X*} \mathbf{D}_X B) \\
&\overset{(6)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B) \\
&\overset{(7)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(\mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B)) \\
&\overset{(8)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, \mathbf{D}_{\mathfrak{X}} \mathbf{D}_{\mathfrak{X}} C)) \\
&\overset{(9)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}},\Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, C))
\end{aligned}$$

functorially in $A \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$. Here (1) and (5) follow from adjointness of λ_X^* and $R\lambda_{X*}$; (3) and (7) follow from tensor-hom adjunction; (4) is trivial; (6) follows from Proposition B.2.1; (9) follows from the reflexivity of C; and, finally, in (2) and (8) we've used the fact that for $\bullet = X, \mathfrak{X}$ and any objects $C, D \in D(\bullet_{\text{ét}}, \Lambda)$ we have

$$\underline{\mathrm{RHom}}_{\bullet}(C, \mathbf{D}_{\bullet} D) \cong \underline{\mathrm{RHom}}_{\bullet}(D, \mathbf{D}_{\bullet} C)$$

functorially in C and D, which is an easy exercise in tensor-hom adjunction. $\square$

B.3 Constructible sheaves on rigid spaces

In this section we sketch an alternative proof that constructible sheaves on rigid spaces are reflexive, which is in some ways more naive. The first non-formal input is the following claim.

Proposition B.3.1. *If X is a rigid analytic space, then the constant sheaf Λ is reflexive.*

Note that by definition, Λ is reflexive if and only if the natural map $\Lambda \to \underline{\mathrm{RHom}}(\kappa_X, \kappa_X)$ is a isomorphism.

Proof. The result clearly holds when X is smooth. For the general case, we argue by induction on the dimension of X. Thus, fix an integer $d \geq 1$. Assume the result holds for all rigid spaces of dimension $< d$, and let X be a d-dimensional (separated taut) rigid analytic space. We can assume that $X = \mathrm{Spa}(A, A^\circ)$ is affinoid and reduced. The ring A is an excellent Noetherian ring, so by Temkin [Tem12] we can find a projective birational morphism $f : \mathcal{X}' \to \mathrm{Spec}(A)$ where $\mathcal{X}'$ is a regular C-scheme, such that f is an isomorphism over the regular locus of its target. This analytifies to a proper surjective map of rigid spaces

$$\pi : X' \to X$$

such that $X' \to S$ is smooth. In particular, $\Lambda_{X'} = \pi^*\Lambda_X$ is a reflexive sheaf on $X'_{\text{ét}}$, so $R\pi_*\Lambda_{X'}$ is reflexive by stability under proper pushforward. Now, writing K for the cone of the adjunction map

$$\alpha : \Lambda_X \to R\pi_*\pi^*\Lambda_X \cong R\pi_*\Lambda_{X'},$$

the 2-out-of-3 property shows that Λ_X is reflexive if K is reflexive.

For reflexivity of K, consider the diagram

$$\begin{tikzcd}
U \arrow[r, "j'"] \arrow[d, "\wr"'] & X' \arrow[d, "\pi"] & Z' \arrow[l, "i'"'] \arrow[d, "\tau"'] \\
U \arrow[r, "j"] & X & Z \arrow[l, "i"']
\end{tikzcd}$$

where U is the smooth locus in X with closed complement Z, and both squares are cartesian. Since $\pi|_{\pi^{-1}(U)}$ is an isomorphism, $j^*\alpha$ is an isomorphism in $D(U_{\text{ét}}, \Lambda)$, so $j^*K \simeq 0$; applying the usual exact triangle $j_!j^* \to \mathrm{id} \to i_*i^*$, we get an isomorphism $K \simeq i_*i^*K$. Since i_* is a closed immersion and thus proper, we're now reduced to showing that i^*K is a reflexive object of $D(Z_{\text{ét}}, \Lambda)$. This pullback can be computed as

$$\begin{aligned}
i^*K &= i^*\mathrm{Cone}\left(\Lambda_X \to R\pi_*\pi^*\Lambda_X\right) \\
&\cong \mathrm{Cone}\left(i^*\Lambda_X \to i^*R\pi_*\pi^*\Lambda_X\right) \\
&\cong \mathrm{Cone}\left(\Lambda_Z \to R\tau_*i'^*\pi^*\Lambda_X\right) \\
&\cong \mathrm{Cone}\left(\Lambda_Z \to R\tau_*\tau^*\Lambda_Z\right) \\
&\cong \mathrm{Cone}\left(\Lambda_Z \to R\tau_*\Lambda_{Z'}\right),
\end{aligned}$$

where the third line follows by proper base change. Since Z and Z' are both of dimension $< d$, the sheaves Λ_Z and $\Lambda_{Z'}$ are reflexive by the induction hypothesis, and then $R\tau_*\Lambda_{Z'}$ is reflexive as well since τ is proper. Applying the 2-out-of-3 property again, we deduce that i^*K is reflexive, as desired. $\square$

Corollary B.3.2. *If X is a rigid space and $j : U \to X$ is the inclusion of a Zariski-open subset with Zariski-closed complement $i : Z \to X$, then $j_!\Lambda$ and $i_*\Lambda$ are reflexive.*

Proof. Writing $i : Z \to X$ for the inclusion of the closed complement, the claim for $i_*\Lambda$ is immediate from preservation of reflexivity under proper pushforward. The exact triangle $j_!\Lambda \to \Lambda \to i_*\Lambda \to$ and the 2-out-of-3 property then imply reflexivity of $j_!\Lambda$. □

Proposition B.3.3. *Let X be an adic space, and let $U \subset X$ be an open constructible subset with closure $\overline{U} \subset X$; let $j : U \to X$ and $\overline{j} : \overline{U} \to X$ denote the evident inclusions. Then for any overconvergent sheaf $\mathcal{F} \in Sh(X_{\text{ét}}, \Lambda)$, we have natural identifications*

$$Rj_*j^*\mathcal{F} \cong j_*j^*\mathcal{F} \cong \overline{j}_*\overline{j}^*\mathcal{F}$$

in $Sh(X_{\text{ét}}, \Lambda) \subset D(X_{\text{ét}}, \Lambda)$.

Here we say a sheaf $\mathcal{F} \in Sh(X_{\text{ét}}, \Lambda)$ is *overconvergent* if for any specialization of geometric points $\overline{x} \rightsquigarrow \overline{y}$, the associated map on stalks $\mathcal{F}_{\overline{y}} \to \mathcal{F}_{\overline{x}}$ is an isomorphism, cf. [Hub96, Definition 8.2.1]. We also say that $\mathcal{F} \in D(X_{\text{ét}}, \Lambda)$ is overconvergent if it has overconvergent cohomology sheaves.

Proof. By [Hub96, Lemma 2.2.6], the functor j_* is exact, so the first isomorphism is clear. For the second, let $h : U \to \overline{U}$ be the evident open embedding, so

$$Rj_*j^*\mathcal{F} \cong \overline{j}_*Rh_*h^*\overline{j}^*\mathcal{F} \cong \overline{j}_*h_*h^*\overline{j}^*\mathcal{F}.$$

Here we used the fact that $h_* = Rh_*$ by another application of [Hub96, Lemma 2.2.6]. Now we need to see that $\overline{j}^*\mathcal{F} \cong h_*h^*\overline{j}^*\mathcal{F}$. Since $\overline{j}^*\mathcal{F}$ is overconvergent, this reduces us to checking that the natural map $\alpha : \mathcal{G} \to h_*h^*\mathcal{G}$ is an isomorphism for $\mathcal{G}$ any overconvergent sheaf on $\overline{U}$. By [Hub96, Proposition 8.2.3], the sheaf $h_*h^*\mathcal{G}$ is overconvergent, so it suffices to check that α induces an isomorphism on stalks over any rank one (geometric) point. But every rank one point of $\overline{U}$ is contained in U, so this is trivial. □

Proposition B.3.4. *If X is a rigid analytic space, then the dualizing complex κ_X is overconvergent.*

Proof. The proof is "dual" to the proof of Proposition B.3.1. More precisely, the result clearly holds for smooth X; for a general X, we take a (global) resolution $\pi : X' \to X$ and consider the adjunction map

$$R\pi_!\kappa_{X'} \cong R\pi_!\pi^!\kappa_X \to \kappa_X.$$

One now argues by induction as in the proof of Proposition B.3.1, making crucial use of the following facts:

i. Overconvergence satisfies a 2-out-of-3 property.

ii. Overconvergence is preserved by derived pushforward along proper maps [Hub96, Corollary 8.2.4]. □

Granted these results, we now deduce the following key intermediate case.

Proposition B.3.5. *Let X be a rigid analytic space, and let $j : U \to X$ be the inclusion of an open constructible subset U. Then $j_!\Lambda$ is reflexive.*

The argument which follows is easily adapted to prove the more general statement that $j_!M$ is reflexive, where M is any finitely generated constant sheaf of Λ-modules on $X_{\text{ét}}$.

Proof. Set $\overline{V} = X \smallsetminus U$, and let V be the interior of $\overline{V}$; write $h : V \to X$ and $\overline{h} : \overline{V} \to X$ for the evident inclusions. Note that $\overline{U} = X \smallsetminus V$. In particular, writing $\overline{j} : \overline{U} \to X$ for the evident inclusion, we get a canonical exact triangle

$$h_!h^*\kappa_X \to \kappa_X \to \overline{j}_*\overline{j}^*\kappa_X \to .$$

By Propositions B.3.3 and B.3.4, the canonical map

$$\overline{j}_*\overline{j}^*\kappa_X \to Rj_*j^*\kappa_X = Rj_*\kappa_U$$

is an isomorphism. Moreover, $Rj_*\kappa_U \cong \mathbf{D}j_!\Lambda_U$, and $h^*\kappa_X = \kappa_V$. Thus we can rewrite the above triangle as

$$h_!\kappa_V \to \kappa_X \to \mathbf{D}j_!\Lambda_U \to,$$

so dualizing this gives an exact triangle

$$\mathbf{D}^2j_!\Lambda_U \to \mathbf{D}\kappa_X \to \mathbf{D}h_!\kappa_V \to .$$

Since $\mathbf{D}\kappa_X \cong \Lambda_X$ and $\mathbf{D}h_!\kappa_V \cong Rh_*\mathbf{D}\kappa_V \cong Rh_*\Lambda_V$, we can rewrite the latter triangle as

$$\mathbf{D}^2j_!\Lambda_U \to \Lambda_X \to Rh_*\Lambda_V \to .$$

This sits in a commutative diagram of exact triangles

$$\begin{tikzcd}
j_!\Lambda_U \arrow[r] \arrow[d] & \Lambda_X \arrow[r] \arrow[d] & \overline{h}_*\Lambda_{\overline{V}} \arrow[r] \arrow[d] & {} \\
\mathbf{D}^2j_!\Lambda_U \arrow[r] & \Lambda_X \arrow[r] & Rh_*\Lambda_V \arrow[r] & {}
\end{tikzcd}$$

where the lefthand vertical map is the biduality map, the central vertical map is the identity, and the righthand vertical map is the canonical map $a : \overline{h}_*\Lambda_{\overline{V}} \to Rh_*\Lambda_V$. By [Hub98b, Theorem 3.7], the map a is an isomorphism. Therefore the biduality map $j_!\Lambda_U \to \mathbf{D}^2 j_!\Lambda_U$ is an isomorphism, as desired. □

Corollary B.3.6. *Let $f : U \to X$ be any étale map of affinoid rigid spaces, and let M be a constant sheaf of finitely generated Λ-modules on $U_{\text{ét}}$. Then $f_!M$ is reflexive.*

Proof. The claim is local on X. However, locally on X, we can factor f as the composite of an open embedding $j : U \to W$ and a finite étale map $g : W \to X$, cf. [Hub96, Lemma 2.2.8]. By the previous proposition, $j_!M$ is a reflexive sheaf on $W_{\text{ét}}$, and g is finite, hence proper, so $g_* = g_!$ preserves reflexivity. Therefore $f_!M = g_!j_!M$ is reflexive. □

Now, fix an affinoid rigid space X. Let us say a constructible sheaf $\mathcal{G}$ on X is *elementary* if there exists an affinoid rigid space U and an étale map $j : U \to X$ such that $\mathcal{G} \simeq j_!\Lambda$. Note that any finite direct sum of elementary sheaves is elementary. By the previous corollary, any elementary sheaf is reflexive. Moreover, any bounded complex of elementary sheaves is reflexive; this follows by an easy induction on the length of the complex, using the exact triangle associated with the "stupid" truncation functors together with the 2-out-of-3 property. Arguing as in the schemes case, one easily checks that any constructible sheaf $\mathcal{F}$ admits a surjection $s_0 : \mathcal{G}^0 \to \mathcal{F}$ from an elementary sheaf $\mathcal{G}^0$. The kernel of s_0 is again constructible, so we may choose a surjection $s_{-1} : \mathcal{G}^{-1} \to \ker s_0$ with $\mathcal{G}^{-1}$ elementary; iterating this procedure, we can find an isomorphism $\mathcal{F} \simeq \mathcal{G}^\bullet = [\cdots \overset{s_{-2}}{\to} \mathcal{G}^{-1} \overset{s_{-1}}{\to} \mathcal{G}^0]$ in the derived category, where all the $\mathcal{G}^i$'s are elementary. Set $\mathcal{H}_n = \ker s_{-n}$; playing with truncations, we get an exact triangle

$$\tau_{\leq -1}\sigma_{\geq -n}(\mathcal{G}^\bullet) \simeq \mathcal{H}_n[n] \to \sigma_{\geq -n}(\mathcal{G}^\bullet) \to \tau_{\geq 0}\sigma_{\geq -n}(\mathcal{G}^\bullet) \simeq \mathcal{F} \to$$

for any $n \geq 1$. Note that $\sigma_{\geq -n}(\mathcal{G}^\bullet)$ is a bounded complex of elementary sheaves, and hence is reflexive. Since any exact triangle induces a corresponding exact triangle whose terms are the cones of the evident biduality maps, we get an isomorphism

$$\mathrm{Cone}(\mathcal{F} \to \mathbf{D}^2\mathcal{F}) \simeq \mathrm{Cone}(\mathcal{H}_n \to \mathbf{D}^2\mathcal{H}_n)[n+1]$$

for any n. Using the cohomological dimension bounds proved in [Hub96], one easily checks that there is an integer N depending only on X such that the ith cohomology sheaf of $\mathrm{Cone}(\mathcal{H}_n \to \mathbf{D}^2\mathcal{H}_n)$ is zero for any $i \notin [-N, N]$ (in fact, $N = 1 + 2\dim X$ is sufficient). Taking n arbitrarily large, we then see that the cohomology sheaves of $\mathrm{Cone}(\mathcal{F} \to \mathbf{D}^2\mathcal{F})$ are zero in any given degree. Therefore $\mathrm{Cone}(\mathcal{F} \to \mathbf{D}^2\mathcal{F})$ is acyclic, as desired.

B.4 Some counterexamples

In this section we prove Propositions B.1.5 and B.1.6.

Proof of Proposition B.1.5. We construct an explicit example as follows. Let $X = \mathrm{Gr}(2,5)$ be the usual rigid analytic Grassmannian over C, which we regard as parametrizing modifications of the bundle $\mathcal{O}(2/5)$ on the Fargues-Fontaine curve at the distinguished point. Let X^{adm} be the admissible locus inside X. According to an unpublished computation of the author and Jared Weinstein, the closed subdiamond $Z \subset X^{\Diamond}$ corresponding to the closed subset $|X| \smallsetminus |X^{\mathrm{adm}}|$ can be explicitly described by an isomorphism

$$Z \simeq \left(\mathrm{Spa}\left(\mathcal{O}_C[[T^{1/p^{\infty}}]]\right) \smallsetminus V(pT)\right)^{\Diamond} / \underline{D_{1/3}^{\times}},$$

for a certain free action of $\underline{D_{1/3}^{\times}}$ on the diamond $\left(\mathrm{Spa}\left(\mathcal{O}_C[[T^{1/p^{\infty}}]]\right) \smallsetminus V(pT)\right)^{\Diamond}$; here $D_{1/3}$ denotes the division algebra over $\mathbf{Q}_p$ of invariant $1/3$. In particular, there is a natural smooth map $Z \to [\mathrm{pt}/\underline{D_{1/3}^{\times}}]$. Let π be any infinite-dimensional admissible representation of $D_{1/3}^{\times}$, and let $\mathcal{L}_\pi$ be the corresponding pro-étale local system on Z. The sheaf $\mathcal{L}_\pi$ is then reflexive, since it's the pullback of a reflexive sheaf on $[\mathrm{pt}/\underline{D_{1/3}^{\times}}]$ along a smooth map, and therefore its pusforward along the closed embedding $i : Z \to X^{\Diamond}$ is a reflexive sheaf on $X_{\text{ét}} \cong X^{\Diamond}_{\text{ét}}$. Now, choose any open affinoid subset $j : U \to X$ meeting Z together with a finite map $f : U \to \mathrm{Spa}\, C\langle T_1, \ldots, T_6\rangle$. The sheaf $\mathcal{F} \overset{\mathrm{def}}{=} f_* j^* i_* \mathcal{L}_\pi$ is then an example of the type we seek. $\square$

This example is closely related to Example 3.15.1.

We now turn to Proposition B.1.6. For brevity, we prove that $\mathcal{F} = \Lambda_{\overline{X}}$ is reflexive but not strongly reflexive; the case of $j_! \Lambda_X$ is dual. The only non-formal ingredient we need is

Proposition B.4.1. *The sheaves $\Lambda_{\overline{X}}$ and $j_! \Lambda_X$ are reflexive, and the dualizing complex of $\overline{X}$ coincides with $j_! \Lambda_X[2](1)$ where $j : X \to \overline{X}$ is the natural inclusion.*

Proof. We first show that $j_!\Lambda_X$ is reflexive. To see this, let $i : \overline{X} \to \mathbf{P}^1_C$ be the natural closed embedding; then $i_* j_! \Lambda_X \cong (i \circ j)_! \Lambda_X$ is a constructible sheaf on $\mathbf{P}^1_C$, and therefore is reflexive. Thus $j_!\Lambda$ is reflexive by our previous remarks. We now calculate

$$\mathbf{D}_{\overline{X}} j_! \Lambda_X \simeq j_* \mathbf{D}_X \Lambda_X \simeq j_* \kappa_X \simeq j_* \Lambda_X[2](1) \simeq \Lambda_{\overline{X}}[2](1),$$

where we've used the smoothness of X to identify κ_X. Since this calculation exhibits $\Lambda_{\overline{X}}$ as the dual of a reflexive sheaf, it is reflexive itself. Applying $\mathbf{D}_{\overline{X}}$ again and using reflexivity, we get

$$j_! \Lambda_X \simeq \mathbf{D}_{\overline{X}} \mathbf{D}_{\overline{X}} j_! \Lambda_X \simeq \mathbf{D}_{\overline{X}} \left(\Lambda_{\overline{X}}[2](1) \right) \simeq \kappa_{\overline{X}}[-2](-1),$$

as desired. $\square$

Consider the cartesian diagram

$$\begin{tikzcd} & X \times X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X \times \overline{X} \arrow[dr, "h_1"'] & & \overline{X} \times X \arrow[dl, "h_2"] \\ & \overline{X} \times \overline{X} & \end{tikzcd}$$

of adic spaces, where all fiber products are taken over $\operatorname{Spa} C$. Using the previous proposition, it is easy to see that

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \simeq h_{1!} \Lambda_{X \times \overline{X}}[2](1).$$

By the symmetry of the situation, we have

$$\mathcal{F} \boxtimes \mathbf{D}\mathcal{F} \simeq h_{2!} \Lambda_{\overline{X} \times X}[2](1),$$

so then

$$\mathbf{D}\left(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}\right) \simeq \mathbf{D}\left(h_{2!} \Lambda_{\overline{X} \times X}[2](1)\right) \simeq h_{2*} \kappa_{\overline{X} \times X}[-2](-1).$$

To calculate $\kappa_{\overline{X} \times X}$, we use that the projection $\mathrm{pr} : \overline{X} \times X \to \overline{X}$ is smooth of relative dimension one, so

$$\kappa_{\overline{X} \times X} = \mathrm{pr}^! K_{\overline{X}} = \mathrm{pr}^* \kappa_{\overline{X}}[2](1) \simeq f_{2!} \Lambda_{X \times X}[4](2)$$

by the previous proposition. Thus

$$\mathbf{D}\,(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \simeq h_{2*} f_{2!} \Lambda_{X \times X}[2](1).$$

It is now clear that $\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}$ and $\mathbf{D}\,(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$ cannot be isomorphic. For example, let U be the open subspace of $\overline{X} \times \overline{X}$ defined by the conditions $|T_1| \leq |T|_2 \neq 0$. Then

$$H^{-2}\,(R\Gamma(U, \mathbf{D}\mathcal{F} \boxtimes \mathcal{F})) \simeq h_{1!} \Lambda_{X \times \overline{X}}(U) = 0,$$

since $U \nsubseteq X \times \overline{X}$, while on the other hand

$$H^{-2}(R\Gamma(U, \mathbf{D}\,(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}))) \simeq (f_{2!} \Lambda_{X \times X})(U \cap (\overline{X} \times X)) \simeq \Lambda,$$

which one easily checks using the fact that $U \cap (\overline{X} \times X)$ is a nonempty connected open subset of $X \times X$.

References

[BL95] Arnaud Beauville and Yves Laszlo, *Un lemme de descente*, Comptes Rendus de l'Acadmie des Sciences. Srie I. Mathmatique **320** (1995), no. 3, 335340.

[Bor98] Mikhail Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50. MR 1401491

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251. MR 0327923 (48 #6265)

[BT84] ______, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316 (86c:20042)

[Del77] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 41$\o er$2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. MR 0463174

[Fal94] Gerd Faltings, *The trace formula and Drinfeld's upper halfplane*, Duke Math. J. **76** (1994), no. 2, 467–481. MR 1302321

[Far] L. Fargues, *Geometrization of the local Langlands correspondence: an overview.*

[Far15] ———, *G-torseurs en théorie de Hodge p-adique*, preprint, http://webusers.imj-prg.fr/~laurent.fargues/Gtorseurs.pdf, 2015.

[FF] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en theorie de hodge p-adique.*

[FGK11] Kazuhiro Fujiwara, Ofer Gabber, and Fumiharu Kato, *On Hausdorff completions of commutative rings in rigid geometry*, J. Algebra **332** (2011), 293–321. MR 2774689

[Fon13] Jean-Marc Fontaine, *Perfectoïdes, presque pureté et monodromie-poids (d'après Peter Scholze)*, Astérisque (2013), no. 352, Exp. No. 1057, x, 509–534, Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058. MR 3087355

[FS] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, in preparation.

[GW17] Ildar Gaisin and John Welliaveetil, *Constructibility and reflexivity in non-Archimedean geometry*, preprint, 2017.

[Har15] Michael Harris, *Mathematics without apologies: Portrait of a problematic vocation*, Science Essentials, Princeton University Press, 2015.

[Hub96] Roland Huber, *étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hub98a] R. Huber, *A comparison theorem for l-adic cohomology*, Compositio Math. **112** (1998), no. 2, 217–235. MR 1626021

[Hub98b] ———, *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 359–403. MR 1620118

[Hub98c] ———, *A finiteness result for the compactly supported cohomology of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 313–357. MR 1620114

[Hum95] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43, American Mathematical Society, Providence, RI, 1995. MR 1343976

[Ill94] Luc Illusie, *Autour du théorème de monodromie locale*, Astérisque (1994), no. 223, 9–57, Périodes p-adiques (Bures-sur-Yvette, 1988). MR 1293970

[Kala] Tasho Kaletha, *The local Langlands conjectures for non-quasi-split groups*, Proceedings of the 2014 Simons symposium on the trace formula, To appear.

[Kalb] Tasho Kaletha, *Rigid inner forms of real and p-adic groups*, Ann. of Math., to appear.

[Kalc] ———, *Rigid inner forms vs isocrystals*, J. Eur. Math. Soc. (JEMS), to appear.

[Kal13] ———, *Genericity and contragredience in the local Langlands correspondence*, Algebra Number Theory **7** (2013), no. 10, 2447–2474. MR 3194648

[KL15] Kiran S. Kedlaya and Ruochuan Liu, *Relative p-adic Hodge theory: foundations*, Astérisque (2015), no. 371, 239. MR 3379653

[Kot] Robert E. Kottwitz, *$B(G)$ for all local and global fields*, arXiv:1401.5728.

[Kot83] ———, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 289–297. MR 697075 (84i:22012)

[Kot84a] ———, *Shimura varieties and twisted orbital integrals*, Math. Ann. **269** (1984), no. 3, 287–300. MR 761308

[Kot84b] ———, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650. MR 757954 (85m:11080)

[Kot85] ———, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220. MR 809866 (87i:14040)

[Kot86] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR 858284 (88d:22027)

[Kot97] ———, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339. MR 1485921 (99e:20061)

[KS99] Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190. MR 1687096 (2000k:22024)

[Laf02] Vincent Lafforgue, *Chtoucas pour les groupes rductifs et paramtrisation de langlands globale*, https://arxiv.org/abs/1209.5352, 2102.

[Mas16] David Massey, *Natural commuting of vanishing cycles and the Verdier dual*, Pacific J. Math. **284** (2016), no. 2, 431–437. MR 3544308

[Mie] Yoichi Mieda, *Lefschetz trace formula and ℓ-adic cohomology of Rapoport-Zink tower for gsp(4)*, http://www.ms.u-tokyo.ac.jp/~mieda/pdf/RZ-LTF.pdf.

[Mie12] Yoichi Mieda, *Lefschetz trace formula and ℓ-adic cohomology of Lubin-Tate tower*, Math. Res. Lett. **19** (2012), no. 1, 95–107. MR 2923178

[Mie14a] ———, *Geometric approach to the local Jacquet-Langlands correspondence*, Amer. J. Math. **136** (2014), no. 4, 1067–1091. MR 3245187

[Mie14b] ———, *Lefschetz trace formula for open adic spaces*, J. Reine Angew. Math. **694** (2014), 85–128. MR 3259040

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143. MR 2342692

[NP01] B. C. Ngô and P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, J. Algebraic Geom. **10** (2001), no. 3, 515–547.

[Rap95] Michael Rapoport, *Non-Archimedean period domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 423–434. MR 1403942

[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), no. 2, 153–181. MR 1411570

[RV14] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326. MR 3271247

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996. MR 1393439

[Sch12] Peter Scholze, *Perfectoid spaces*, Publ. math. de l'IHS (2012), no. 1, 245–313.

[Sch17] ______, *The étale cohomology of diamonds*, ARGOS Seminar in Bonn, 2017.

[She14] Xu Shen, *Cell decomposition of some unitary group rapoportzink spaces*, Math. Ann. **360** (2014), 825899.

[Sta17] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2017.

[Str05] Matthias Strauch, *On the Jacquet-Langlands correspondence in the cohomology of the Lubin-Tate deformation tower*, Astérisque (2005), no. 298, 391–410, Automorphic forms. I. MR 2141708

[Str08] ______, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217** (2008), no. 3, 889–951. MR 2383890

[SW13] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237. MR 3272049

[SW14] ______, *Lectures on p-adic geometry*, Lecture notes at https://math.berkeley.edu/∼jared/Math274/ScholzeLectures.pdf, 2014.

[Tem12] Michael Temkin, *Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case*, Duke Math. J. **161** (2012), no. 11, 2207–2254. MR 2957701

[Tit79] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69. MR 546588 (80h:20064)

[Var07] Yakov Varshavsky, *Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara*, Geom. Funct. Anal. **17** (2007), no. 1, 271–319. MR 2306659

[Vig96] Marie-France Vignéras, *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$*, Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1395151

[Wei16] Jared Weinstein, $G_{\mathbf{Q}_p}$ *as a geometric fundamental group*, Int. Math. Res. Not. (2016).

[Zhu17] Xinwen Zhu, *Affine Grassmannians and the geometric Satake in mixed characteristic*, Annals of Math. **185** (2017), 403–492.