

# On the Kottwitz conjecture for moduli spaces of local shtukas

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## Abstract

Kottwitz’s conjecture describes the contribution of a supercuspidal representation to the cohomology of a local Shimura variety in terms of the local Langlands correspondence. Using a Lefschetz-Verdier fixed-point formula, we prove a weakened and generalized version of Kottwitz’s conjecture. The weakening comes from ignoring the action of the Weil group and only considering the actions of the groups  $G$  and  $J_b$  up to non-elliptic representations. The generalization is that we allow arbitrary connected reductive groups  $G$  and non-minuscule coweights  $\mu$ . Thus our work concerns the cohomology of the moduli space of local shtukas introduced by Scholze.

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## 1 Introduction

Let  $F$  be a finite extension of the field  $\mathbf{Q}_p$  of  $p$ -adic numbers, and let  $\check{F}$  be the completion of the maximal unramified extension of  $F$ , relative to a fixed algebraic closure  $\bar{F}$ . Let  $\sigma \in \text{Aut}(\check{F}/F)$  be the arithmetic Frobenius element. Let  $G$  be a connected reductive group defined over  $F$ ,  $[b] \in B(G)$  a  $\sigma$ -conjugacy class of elements of  $G(\check{F})$ , and  $\{\mu\}$  a conjugacy class of cocharacters  $\mathbf{G}_m \rightarrow G$  defined over  $\bar{F}$ . Assume that  $\{\mu\}$  is minuscule and that  $[b] \in B(G, \{\mu\})$ . The triple  $(G, [b], \{\mu\})$  is called a local Shimura datum [RV14, §5]. In loc. cit. the authors conjecture the existence of an associated tower  $\mathcal{M}_{G,b,\mu,K}$  of rigid analytic spaces over  $\check{E}$ , indexed by open compact subgroups  $K \subset G(F)$ . Here  $E$  is the field of definition of the conjugacy class  $\{\mu\}$ , a finite extension of  $F$ . The isomorphism class of the tower should only depend on the classes  $[b]$  and  $\{\mu\}$ . The theory of Rapoport-Zink spaces [RZ96] provides instances of such a tower together with a moduli interpretation, as the generic fiber of a deformation space of  $p$ -divisible groups. In general these towers were constructed in [SW14, §24].

The *Kottwitz conjecture* [Rap95, Conjecture 5.1], [RV14, Conjecture 7.3] relates the cohomology of the  $\mathcal{M}_{G,b,\mu,K}$  to the local Langlands correspondence, in the case that  $[b]$  is basic. Let us review the precise statement. Let  $B(G)_{\text{bas}} \subset B(G)$  be the set of basic  $\sigma$ -conjugacy classes. Assume that  $[b] \in B(G)_{\text{bas}}$  and choose a representative  $b \in [b]$ . Let  $J_b$  be the associated inner form of  $G$ . Note that since  $B(G, \{\mu\})$  contains a unique basic element,  $[b]$  is uniquely determined by  $\{\mu\}$ .

The tower  $\mathcal{M}_{G,b,\mu,K}$  receives commuting actions of  $J_b(F)$  and  $G(F)$ . The action of  $J_b(F)$  preserves each  $\mathcal{M}_{G,b,\mu,K}$ , while the action of  $g \in G(F)$  sends  $\mathcal{M}_{G,b,\mu,K}$  to  $\mathcal{M}_{G,b,\mu,gKg^{-1}}$ . There is furthermore a *Weil descent datum* on this tower from  $\check{E}$  down to  $E$ . It need not be effective.

Let  $C$  be the completion of an algebraic closure of  $\check{E}$ . Consider

$$H_c^i(\mathcal{M}_{G,b,\mu,K} \times_{\check{E}} C, \bar{\mathbf{Q}}_\ell),$$

a  $\bar{\mathbf{Q}}_\ell$ -vector spaces equipped with an action of  $J_b(F)$  as well as an action of  $I_E$ , which extends to an action of  $W_E$  due to the Weil descent datum. The

actions of  $J_b(F)$  and  $W_E$  commute. Given an irreducible smooth admissible representation  $\rho$  of  $J_b(F)$  we have the  $\overline{\mathbf{Q}}_\ell$ -vector space

$$H^{i,j}(G, b, \mu)[\rho] = \varinjlim_K \text{Ext}_{J_b(F)}^j(H_c^i(\mathcal{M}_{G,b,\mu,K} \times_{\check{E}} C, \overline{\mathbf{Q}}_\ell), \rho)$$

with a smooth action of  $G(F) \times W_E$ , leading to the virtual  $G(F) \times W_E$ -representation

$$H^*(G, b, \mu)[\rho] := \sum_{i \in \mathbf{Z}} (-1)^{i+j} H^{i,j}(G, b, \mu)[\rho](-d),$$

where  $d = \dim \mathcal{M}_{G,b,\mu} = \langle 2\rho_G, \mu \rangle$ , and  $\rho_G$  is half the sum of the positive roots of  $G$ . The isomorphism class of  $H^*(G, b, \mu)[\rho]$  only depends on  $(G, [b], \{\mu\})$  and  $\rho$ .

The Kottwitz conjecture describes  $H^*(G, b, \mu)[\rho]$  in terms of the local Langlands correspondence. The complex dual groups of  $G$  and  $J_b$  are canonically identified and we write  $\widehat{G}$  for either, and let  ${}^L G = \widehat{G} \rtimes \Gamma$  be the corresponding  $L$ -group. The basic form of the local Langlands conjecture predicts that the set of isomorphism classes of essentially square-integrable representations of  $G(F)$  (resp.,  $J_b(F)$ ) is partitioned into  $L$ -packets  $\Pi_\phi(G)$  (resp.,  $\Pi_\phi(J_b)$ ), each such packet indexed by a discrete Langlands parameter  $\phi : W_F \times \text{SL}_2(\mathbf{C}) \rightarrow {}^L G$ . When  $\phi$  is trivial on  $\text{SL}_2(\mathbf{C})$  it is expected that the packets  $\Pi_\phi(G)$  and  $\Pi_\phi(J_b)$  consist entirely of supercuspidal representations.

Let  $S_\phi = \text{Cent}(\phi, \widehat{G})$ . For any  $\pi \in \Pi_\phi(G)$  and  $\rho \in \Pi_\phi(J_b)$  the refined form of the local Langlands conjecture implies the existence of an algebraic representation  $\delta_{\pi,\rho}$  of  $S_\phi$ , which can be thought of as measuring the relative position of  $\pi$  and  $\rho$ . The conjugacy class  $\{\mu\}$  of cocharacters gives dually a conjugacy class of weights of  $\widehat{G}$  and we denote by  $r_{\{\mu\}}$  the irreducible representation of  $\widehat{G}$  of highest weight  $\mu$ . There is a natural extension of  $r_{\{\mu\}}$  to  ${}^L G_E$ , the  $L$ -group of the base change of  $G$  to  $E$  [Kot84a, Lemma 2.1.2]. Write  $r_{\{\mu\}} \circ \phi_E$  for the representation of  $S_\phi \times W_E$  given by

$$r_{\{\mu\}} \circ \phi_E(s, w) = r_{\{\mu\}}(s \cdot \phi(w)).$$

Let  $\text{Groth}(G(F) \times W_E)$  be the Grothendieck group of the category of  $(G(F) \times W_E)$ -modules over  $\overline{\mathbf{Q}}_\ell$  which are admissible as a  $G(F)$ -module and smooth as a  $W_E$ -module.

**Conjecture 1.0.1** (Kottwitz). *Let  $\phi : W_F \rightarrow {}^L G$  be a discrete Langlands parameter. Given  $\rho \in \Pi_\phi(J_b)$ , each  $H^i(G, b, \mu)[\rho]$  is admissible, and we have*

the following equality<sup>1</sup> in  $\text{Groth}(G(F) \times W_E)$ :

$$H^*(G, b, \mu)[\rho] = (-1)^d \sum_{\pi \in \Pi_\phi(G)} \pi \boxtimes \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_{\{\mu\}} \circ \phi_E) \left(-\frac{d}{2}\right). \quad (1.0.1)$$

In this article we prove a version of this conjecture that is both weaker and more general. The weakening comes from ignoring the Weil-group action, and thus working in  $\text{Groth}(G(F))$  instead of  $\text{Groth}(G(F) \times W_E)$ . Moreover, we only detect the behavior of representations on the set of elliptic conjugacy classes in  $G(F)$ . This means that, while we identify the right hand side as contributing to the left hand side, we are not able to exclude potential contributions to the left hand side of non-elliptic representations (meaning those whose distribution characters are supported away from the locus of regular elliptic elements in  $G(F)$ ).

The generalization is that we remove two conditions that are present in the formulations of Kottwitz's conjecture in [Rap95] and [RV14]. One of them is that  $G$  is a  $B$ -inner form of its quasi-split inner form  $G^*$ . This condition, reviewed in Subsection 2.2, has the effect of making the definition of  $\delta_{\pi, \rho}$  straightforward. To remove it, we use the formulation of the refined local Langlands correspondence [Kal16a, Conjecture G] based on the cohomology sets  $H^1(u \rightarrow W, Z \rightarrow G)$  of [Kal16b]. The definition of  $\delta_{\pi, \rho}$  in this setting is a bit more involved and is given in Subsection 2.3, see Definition 2.3.2.

The second condition that we remove is that the conjugacy class  $\{\mu\}$  consists of minuscule cocharacters. When  $\mu$  is not minuscule, we work with Scholze's spaces of mixed-characteristic shtukas  $\text{Sht}_{G, b, \mu}$  introduced in [SW14, §23]. These are no longer towers of rigid spaces; rather, they belong to Scholze's category of diamonds. Given an irreducible admissible representation  $\rho$  of  $J_b(F)$  with coefficients in  $\overline{\mathbf{Q}}_\ell$  and equipped with an invariant lattice, we define in Subsection 2.4 a virtual representation  $H^*(G, b, \mu)[\rho]$  of  $G(F) \times W_E$ , using the cohomology of the spaces  $\text{Sht}_{G, b, \mu}$  as developed in [Sch17].

We now present our main theorem. It is conditional on the refined local Langlands correspondence for supercuspidal  $L$ -parameters, in the formulation of [Kal16a, Conjecture G]. In particular, it relies crucially on the endoscopic character identities satisfied by  $L$ -packets. These are reviewed in Appendix A. At the moment a construction of the refined correspondence is known for regular supercuspidal parameters when  $p$  does not divide the

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<sup>1</sup>[RV14, Conjecture 7.3] omits the sign  $(-1)^d$ .

order of the Weyl group of  $G$ , and the character identities are known for toral supercuspidal parameters under stricter assumptions on  $p$  [Kal].

The two sides in the equation asserted by the theorem have not been fully explained in this introduction. This will be the subject of Section 2.

**Theorem 1.0.2.** *Assume the refined local Langlands correspondence [Kal16a, Conjecture G]. Let  $\phi: W_F \rightarrow {}^L G$  be a discrete Langlands parameter, with coefficients in  $\overline{\mathbf{Q}}_\ell$ . Let  $\rho \in \Pi_\phi(J_b)$ , and assume that (a twist of)  $\rho$  admits a  $J_b(F)$ -invariant  $\overline{\mathbf{Z}}_\ell$ -lattice. Then each  $H^i(G, b, \mu)[\rho]$  is an admissible representation of  $G(F)$ , and we have the following equality in  $\text{Groth}(G(F))$ :*

$$H^*(G, b, \mu)[\rho] = (-1)^d \sum_{\pi \in \Pi_\phi(G)} [\dim \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_\mu)] \pi,$$

*up to an element of  $\text{Groth}(G(F))$  which has trace 0 on the locus of regular elliptic elements of  $G(F)$ .*

**Remark 1.0.3.** The condition that  $\rho$  admits an invariant  $\overline{\mathbf{Z}}_\ell$ -lattice is necessary because the theory of étale cohomology of diamonds in [Sch17] works with  $\overline{\mathbf{Z}}_\ell$ -coefficients, but not (yet) with  $\overline{\mathbf{Q}}_\ell$ -coefficients, so it appears to be an artifact of the current technology. As it stands, if  $\rho$  is supercuspidal, then some character twist of  $\rho$  admits such a lattice, and this is enough to prove the claimed identity for  $\rho$ .

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## 2 Statement of the main result

### 2.1 Basic notions

Let  $\check{F}$  be the completion of the maximal unramified extension of  $F$ , and let  $\sigma \in \text{Aut } \check{F}$  be the Frobenius automorphism. Let  $G$  be a connected reductive group defined over  $F$ . Fix a quasi-split group  $G^*$  and a  $G^*(\overline{F})$ -conjugacy class  $\Psi$  of inner twists  $G^* \rightarrow G$ . Given an element  $b \in G(\check{F})$ , there is an associated inner form  $J_b$  of a Levi subgroup of  $G^*$  as described in [Kot97, §3.3, §3.4]. Its group of  $F$ -points is given by

$$J_b(F) \cong \left\{ g \in G(\check{F}) \mid \text{Ad}(b)\sigma(g) = g \right\}.$$

Up to isomorphism the group  $J_b$  depends only on the  $\sigma$ -conjugacy class  $[b]$ . It will be convenient to choose  $b$  to be decent [RZ96, Definition 1.8]. Then there exists a finite unramified extension  $F'/F$  such that  $b \in G(F')$ . This allows us to replace  $\check{F}$  by  $F'$  in the above formula. The slope morphism  $\nu : \mathbf{D} \rightarrow G_{\check{F}}$  of  $b$ , [Kot85, §4], is also defined over  $F'$ . The centralizer  $G_{F',\nu}$  of  $\nu$  in  $G_{F'}$  is a Levi subgroup of  $G_{F'}$ . The  $G(F')$ -conjugacy class of  $\nu$  is defined over  $F$ , and then so is the  $G(F')$ -conjugacy class of  $G_{F',\nu}$ . There is a Levi subgroup  $M^*$  of  $G^*$  defined over  $F$  and  $\psi \in \Psi$  that restricts to an inner twist  $\psi : M^* \rightarrow J_b$ , see [Kot97, §4.3].

From now on assume that  $b$  is basic. This is equivalent to  $M^* = G^*$ , so that  $J_b$  is in fact an inner form of  $G^*$  and of  $G$ . Furthermore,  $\Psi$  is an equivalence class of inner twists  $G^* \rightarrow G$  as well as  $G^* \rightarrow J_b$ . This identifies the dual groups of  $G^*$ ,  $G$ , and  $J_b$ , and we write  $\widehat{G}$  for either of them.

Let  $\phi : W_F \times \mathrm{SL}_2(\mathbf{C}) \rightarrow {}^L G$  be a discrete Langlands parameter and let  $S_\phi = \mathrm{Cent}(\phi, \widehat{G})$ . For  $\lambda \in X^*(Z(\widehat{G})^\Gamma)$  write  $\mathrm{Rep}(S_\phi, \lambda)$  for the set of isomorphism classes of algebraic representations of the algebraic group  $S_\phi$  whose restriction to  $Z(\widehat{G})^\Gamma$  is  $\lambda$ -isotypic, and write  $\mathrm{Irr}(S_\phi, \lambda)$  for the subset of irreducible such representations. The class of  $b$  corresponds to a character  $\lambda_b : Z(\widehat{G})^\Gamma \rightarrow \mathbf{C}^\times$  via the isomorphism  $B(G)_{\mathrm{bas}} \rightarrow X^*(Z(\widehat{G})^\Gamma)$  of [Kot85, Proposition 5.6]. Assuming the validity of the refined local Langlands conjecture [Kal16a, Conjecture G] we will construct in the following two subsections for any  $\pi \in \Pi_\phi(G)$  and  $\rho \in \Pi_\phi(J_b)$  an element  $\delta_{\pi,\rho} \in \mathrm{Rep}(S_\phi, \lambda_b)$  that measures the relative position of  $\pi$  and  $\rho$ .

## 2.2 Construction of $\delta_{\pi,\rho}$ in a special case

The statements of the Kottwitz conjecture given in [Rap95, Conjecture 5.1] and [RV14, Conjecture 7.3] make the assumption that  $G$  is a  $B$ -inner form of  $G^*$ . In that case, the construction of  $\delta_{\pi,\rho}$  is straightforward and we shall now recall it.

The assumption on  $G$  means that some  $\psi \in \Psi$  can be equipped with a decent basic  $b^* \in G^*(F^{\mathrm{nr}})$  such that  $\psi$  is an isomorphism  $G_{F^{\mathrm{nr}}}^* \rightarrow G_{F^{\mathrm{nr}}}$  satisfying  $\psi^{-1}\sigma(\psi) = \mathrm{Ad}(b^*)$ . In other words,  $\psi$  becomes an isomorphism over  $F$  from the group  $J_{b^*}$ , now constructed relative to  $G^*$  and  $b^*$ , and  $G$ . Under this assumption, and after choosing a Whittaker datum  $\mathfrak{w}$  for  $G^*$ , the isocrystal formulation of the refined local Langlands correspondence [Kal16a, Conjecture F], which is implied by the rigid formulation [Kal16a, Conjecture

G] according to [Kal18], predicts the existence of bijections

$$\begin{aligned}\Pi_\phi(G) &\cong \text{Irr}(S_\phi, \lambda_{b^*}) \\ \Pi_\phi(J_b) &\cong \text{Irr}(S_\phi, \lambda_{b^*} + \lambda_b)\end{aligned}$$

where we have used the isomorphisms  $B(G)_{\text{bas}} \cong X^*(Z(\widehat{G})^\Gamma) \cong B(G^*)_{\text{bas}}$  of [Kot85, Proposition 5.6] to obtain from  $[b] \in B(G)_{\text{bas}}$  and  $[b^*] \in B(G^*)_{\text{bas}}$  characters  $\lambda_b$  and  $\lambda_{b^*}$  of  $Z(\widehat{G})^\Gamma$ .

These bijections are uniquely characterized by the endoscopic character identities which are part of [Kal16a, Conjecture F]. Write  $\pi \mapsto \tau_{b^*, \mathfrak{w}, \pi}$ ,  $\rho \mapsto \tau_{b^*, \mathfrak{w}, \rho}$  for these bijections and define

$$\delta_{\pi, \rho} := \check{\tau}_{b^*, \mathfrak{w}, \pi} \otimes \tau_{b^*, \mathfrak{w}, \rho}. \quad (2.2.1)$$

While these bijections depend on the choice of Whittaker datum  $\mathfrak{w}$  and the choice of  $b^*$ , we will argue in Subsection 2.3 that for any pair  $\pi$  and  $\rho$  the representation  $\delta_{\pi, \rho}$  is independent of these choices. Of course it does depend on  $b$ , but this we take as part of the given data.

### 2.3 Construction of $\delta_{\pi, \rho}$ in the general case

We now drop the assumption that  $G$  is a  $B$ -inner form of  $G^*$ . Because of this, we no longer have the isocrystal formulation of the refined local Langlands correspondence. However, we do have the formulation based on rigid inner twists [Kal16a, Conjecture G]. What this means with regards to the Kottwitz conjecture is that neither  $\pi$  nor  $\rho$  correspond to representations of  $S_\phi$ . Rather, they correspond to representations  $\tau_\pi$  and  $\tau_\rho$  of a different group  $\pi_0(S_\phi^+)$ . Nonetheless it will turn out that  $\check{\tau}_\pi \otimes \tau_\rho$  provides in a natural way a representation  $\delta_{\pi, \rho}$  of  $S_\phi$ .

In order to make this precise we will need the material of [Kal16b] and [Kal18], some of which is summarized in [Kal16a]. First, we will need the cohomology set  $H^1(u \rightarrow W, Z \rightarrow G^*)$  defined in [Kal16b, §3] for any finite central subgroup  $Z \subset G^*$  defined over  $F$ . As in [Kal18, §3.2] it will be convenient to package these sets for varying  $Z$  into the single set

$$H^1(u \rightarrow W, Z(G^*) \rightarrow G^*) := \varinjlim H^1(u \rightarrow W, Z \rightarrow G^*).$$

The transition maps on the right are injective, so the colimit can be seen as an increasing union.

Next, we will need the reinterpretation, given in [Kot], of  $B(G)$  as the set of cohomology classes of algebraic 1-cocycles of a certain Galois gerbe

$1 \rightarrow \mathbf{D}(\bar{F}) \rightarrow \mathcal{E} \rightarrow \Gamma \rightarrow 1$ . This reinterpretation is also reviewed in [Kal18, §3.1]. Let us make it explicit for basic decent elements  $b' \in G(F^{\text{nr}})$ . There is a uniquely determined 1-cocycle  $\mathbf{Z} \cong \langle \sigma \rangle \rightarrow G(F^{\text{nr}})$  whose value at  $\sigma$  is equal to  $b'$ . By inflation we obtain a 1-cocycle of  $W_F$  in  $G(\bar{F})$ . Since  $b'$  is basic and decent, for some finite Galois extension  $K/F$  splitting  $G$  the restriction of this 1-cocycle to  $W_K$  factors through  $K^\times$  and is a homomorphism  $K^\times \rightarrow Z(G)(K)$ . Moreover, this homomorphism is algebraic and is in fact given by a multiple of the slope morphism  $\nu : \mathbf{D} \rightarrow G$ . In this way we obtain a 1-cocycle valued in  $G(\bar{F})$  of the extension  $1 \rightarrow K^\times \rightarrow W_{K/F} \rightarrow \Gamma_{K/F} \rightarrow 1$ , which we can pull-back along  $\Gamma \rightarrow \Gamma_{K/F}$  and then combine with  $\nu$  to obtain a 1-cocycle of  $\mathcal{E}$  valued in  $G(\bar{F})$ , that is algebraic in the sense that its restriction to  $\mathbf{D}(\bar{F})$  is given by a morphism of algebraic groups, namely  $\nu$ . The reader is referred to [Kot97, §8 and App B] for further details.

Finally, we will need the comparison map

$$B(G)_{\text{bas}} \rightarrow H^1(u \rightarrow W, Z(G) \rightarrow G)$$

of [Kal18, §3.3]. In fact, this comparison map is already defined on the level of cocycles, via pull-back along the diagram [Kal18, (3.13)], and takes the form

$$G(F^{\text{nr}})_{\text{d,bas}} \rightarrow Z^1(u \rightarrow W, Z(G) \rightarrow G) \quad (2.3.1)$$

where on the left we have the decent basic elements in  $G(F^{\text{nr}})$ .

After this short review we turn to the construction of  $\delta_{\pi,\rho} \in \text{Rep}(S_\phi, \lambda_b)$ . Choose any inner twist  $\psi \in \Psi$  and let  $\bar{z}_\sigma := \psi^{-1}\sigma(\psi) \in G_{\text{ad}}^*(\bar{F})$ . Then  $\bar{z} \in Z^1(F, G_{\text{ad}}^*)$  and the surjectivity of the natural map  $H^1(u \rightarrow W, Z(G^*) \rightarrow G^*) \rightarrow H^1(F, G_{\text{ad}}^*)$  asserted in [Kal16b, Corollary 3.8] allows us to choose  $z \in Z^1(u \rightarrow W, Z(G^*) \rightarrow G^*)$  lifting  $\bar{z}$ . Then  $(\psi, z) : G^* \rightarrow G$  is a rigid inner twist. Let  $z_b \in Z^1(u \rightarrow W, Z(G) \rightarrow G)$  be the image of  $b$  under (2.3.1). For psychological reasons, let  $\xi : G_{F'} \rightarrow J_{b,F'}$  denote the identity map. Then  $(\xi \circ \psi, \psi^{-1}(z) \cdot z_b) : G^* \rightarrow J_b$  is also a rigid inner twist.

The  $L$ -packets  $\Pi_\phi(G)$  and  $\Pi_\phi(J_b)$  are now parameterized by representations of a certain cover  $S_\phi^+$  of  $S_\phi$ . While [Kal16a, Conjecture G] is formulated in terms of a finite cover depending on an auxiliary choice of a finite central subgroup  $Z \subset G^*$ , we will adopt here the point of view of [Kal18] and work with a canonical infinite cover. Following [Kal18, §3.3] we let  $Z_n \subset Z(G)$  be the subgroup of those elements whose image in  $Z(G)/Z(G_{\text{der}})$  is  $n$ -torsion, and let  $G_n = G/Z_n$ . Then  $G_n$  has adjoint derived subgroup and connected center. More precisely,  $G_n = G_{\text{ad}} \times C_n$ , where  $C_n = C_1/C_1[n]$  and  $C_1 = Z(G)/Z(G_{\text{der}})$ . It is convenient to identify  $C_n = C_1$  as algebraic tori and take the  $m/n$ -power map  $C_1 \rightarrow C_1$  as the transition map  $C_n \rightarrow C_m$  for

$n|m$ . The isogeny  $G \rightarrow G_n$  dualizes to  $\widehat{G}_n \rightarrow \widehat{G}$  and we have  $\widehat{G}_n = \widehat{G}_{\text{sc}} \times \widehat{C}_1$ . Note that  $\widehat{C}_1 = Z(\widehat{G})^\circ$ . The transition map  $\widehat{G}_m \rightarrow \widehat{G}_n$  is then the identity on  $\widehat{G}_{\text{sc}}$  and the  $m/n$ -power map on  $\widehat{C}_1$ . Set  $\widehat{\widehat{G}} = \varprojlim \widehat{G}_n = \widehat{G}_{\text{sc}} \times \widehat{C}_\infty$ , where  $\widehat{C}_\infty = \varprojlim \widehat{C}_n$ . Elements of  $\widehat{\widehat{G}}$  can be written as  $(a, (b_n)_n)$ , where  $a \in \widehat{G}_{\text{sc}}$  and  $(b_n)_n$  is a sequence of elements  $b_n \in \widehat{C}_1$  satisfying  $b_n = b_m^{\frac{m}{n}}$  for  $n|m$ . In this presentation, the natural map  $\widehat{\widehat{G}} \rightarrow \widehat{G}$  sends  $(a, (b_n)_n)$  to  $a_{\text{der}} \cdot b_1$ , where  $a_{\text{der}} \in \widehat{G}_{\text{der}}$  is the image of  $a \in \widehat{G}_{\text{sc}}$  under the natural map  $\widehat{G}_{\text{sc}} \rightarrow \widehat{G}_{\text{der}}$ .

**Definition 2.3.1.** Let  $Z(\widehat{\widehat{G}})^+ \subset S_\phi^+ \subset \widehat{\widehat{G}}$  be the preimages of  $Z(\widehat{G})^\Gamma \subset S_\phi \subset \widehat{G}$  under  $\widehat{\widehat{G}} \rightarrow \widehat{G}$ .

Given a character  $\lambda : \pi_0(Z(\widehat{\widehat{G}})^+) \rightarrow \mathbf{C}^\times$  (which we will always assume trivial on the kernel of  $Z(\widehat{\widehat{G}})^+ \rightarrow \widehat{G}_n$  for some  $n$ ) let  $\text{Rep}(\pi_0(S_\phi^+), \lambda)$  denote the set of isomorphism classes of representations of  $\pi_0(S_\phi^+)$  whose pull-back to  $\pi_0(Z(\widehat{\widehat{G}})^+)$  is  $\lambda$ -isotypic, and let  $\text{Irr}(\pi_0(S_\phi^+), \lambda)$  be the (finite) subset of irreducible representations. Let  $\lambda_z$  be the character corresponding to the class of  $z$  under the Tate-Nakayama isomorphism

$$H^1(u \rightarrow W, Z(G^*) \rightarrow G^*) \rightarrow \pi_0(Z(\widehat{\widehat{G}})^+)^*$$

of [Kal16b, Corollary 5.4], and let  $\lambda_{z_b}$  be the character corresponding to the class of  $z_b$  in  $H^1(u \rightarrow W, Z(G) \rightarrow G)$ . Then according to [Kal16a, Conjecture G], upon fixing a Whittaker datum  $\mathfrak{w}$  for  $G^*$  there are bijections

$$\begin{aligned} \Pi_\phi(G) &\cong \text{Irr}(\pi_0(S_\phi^+), \lambda_z) \\ \Pi_\phi(J_b) &\cong \text{Irr}(\pi_0(S_\phi^+), \lambda_z + \lambda_{z_b}) \end{aligned}$$

again uniquely determined by the endoscopic character identities. We write  $\pi \mapsto \tau_{z, \mathfrak{w}, \pi}$ ,  $\rho \mapsto \tau_{z, \mathfrak{w}, \rho}$  for these bijections, and  $\tau \mapsto \pi_{z, \mathfrak{w}, \tau}$ ,  $\tau \mapsto \rho_{z, \mathfrak{w}, \tau}$  for their inverses. We form the representation  $\check{\tau}_{z, \mathfrak{w}, \pi} \otimes \tau_{z, \mathfrak{w}, \rho} \in \text{Rep}(\pi_0(S_\phi^+), \lambda_{z_b})$ .

Recall the map [Kal18, (4.7)]

$$S_\phi^+ \rightarrow S_\phi, \quad (a, (b_n)) \mapsto \frac{a_{\text{der}} \cdot b_1}{N_{E/F}(b_{[E:F]})}. \quad (2.3.2)$$

Here  $a_{\text{der}} \in \widehat{G}_{\text{der}}$  is the image of  $a \in \widehat{G}_{\text{sc}}$  under the natural map  $\widehat{G}_{\text{sc}} \rightarrow \widehat{G}_{\text{der}}$  and  $E/F$  is a sufficiently large finite Galois extension. This map is independent of the choice of  $E/F$ . According to [Kal18, Lemma 4.1] pulling back along this map sets up a bijection  $\text{Irr}(\pi_0(S_\phi^+), \lambda_{z_b}) \rightarrow \text{Irr}(S_\phi, \lambda_b)$ . Note

that since  $\phi$  is discrete the group  $S_\phi^{\natural}$  defined in loc. cit. is equal to  $S_\phi$ . The lemma remains valid, with the same proof, if we remove the requirement of the representations being irreducible, and we obtain the bijection  $\text{Rep}(\pi_0(S_\phi^+), \lambda_{z_b}) \rightarrow \text{Rep}(S_\phi, \lambda_b)$ .

**Definition 2.3.2.** Let  $\delta_{\pi, \rho}$  be the image of  $\check{\tau}_{z, \mathfrak{w}, \pi} \otimes \tau_{z, \mathfrak{w}, \rho}$  under the bijection  $\text{Rep}(\pi_0(S_\phi^+), \lambda_{z_b}) \rightarrow \text{Rep}(S_\phi, \lambda_b)$ .

In the situation when  $G$  is a  $B$ -inner form of  $G^*$ , this definition of  $\delta_{\pi, \rho}$  agrees with the one of Subsection 2.2, because then we can take  $z$  to be the image of  $b^*$  under (2.3.1) and then  $\tau_{z, \mathfrak{w}, \pi}$  and  $\tau_{b^*, \mathfrak{w}, \pi}$  are related via (2.3.2), and so are  $\tau_{z, \mathfrak{w}, \rho}$  and  $\tau_{b^*, \mathfrak{w}, \rho}$ , see [Kal18, §4.2].

**Lemma 2.3.3.** *The representation  $\delta_{\pi, \rho}$  is independent of the choices of Whittaker datum  $\mathfrak{w}$  and of a rigidifying 1-cocycle  $z \in Z^1(u \rightarrow W, Z(G^*) \rightarrow G^*)$ .*

*Proof.* Both of these statements follow from [Kal16a, Conjecture G]. For the independence of Whittaker datum, one can prove that the validity of this conjecture implies that if  $\mathfrak{w}$  is replaced by another choice  $\mathfrak{w}'$  then there is an explicitly constructed character  $(\mathfrak{w}, \mathfrak{w}')$  of  $\pi_0(S_\phi/Z(\widehat{G})^\Gamma)$  whose inflation to  $\pi_0(S_\phi^+)$  satisfies  $\tau_{z, \mathfrak{w}, \sigma} = \tau_{z, \mathfrak{w}', \sigma} \otimes (\mathfrak{w}, \mathfrak{w}')$  for any  $\sigma \in \Pi_\phi(G) \cup \Pi_\phi(J_b)$ . See §4 and in particular Theorem 4.3 of [Kal13], the proof of which is valid for a general  $G$  that satisfies [Kal16a, Conjecture G], bearing in mind that the transfer factor we use here is related to the one used there by  $s \mapsto s^{-1}$ . The independence of  $z$  follows from the same type of argument, but now using [Kal18, Lemma 6.2].  $\square$

## 2.4 Spaces of local shtukas and their cohomology

We recall here some material from [SW14] and [Far] regarding the Fargues-Fontaine curve and moduli spaces of local Shtukas.

For any perfectoid space  $S$  of characteristic  $p$ , we have the Fargues-Fontaine curve  $X_S$  [FF], an adic space over  $\mathbf{Q}_p$ . If  $S$  lies over  $k$ , there is a relative version of this curve lying over  $F$ . Since  $F$  is fixed we use the same notation  $X_S$  for this. For  $S = \text{Spa}(R, R^+)$  affinoid with pseudouniformizer  $\varpi$ , the adic space  $X_S$  is defined as follows:

$$\begin{aligned} Y_S &= (\text{Spa } W_{\mathcal{O}_F}(R^+)) \setminus \{p[\varpi] = 0\} \\ X_S &= Y_S / \text{Frob}^{\mathbf{Z}}. \end{aligned}$$

Here  $\text{Frob}$  is the  $q$ th power Frobenius on  $S$ .

For an affinoid perfectoid space  $S$  lying over the residue field of  $F$ , the following sets are in bijection:

1.  $S$ -points of  $\mathrm{Spd} F$
2. Untilts  $S^\sharp$  of  $S$  over  $F$ ,
3. Cartier divisors of  $Y_S$  of degree 1.

Given an untilt  $S^\sharp$ , we let  $D_{S^\sharp} \subset Y_S$  be the corresponding divisor. If  $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$  is affinoid, then the completion of  $Y_S$  along  $D_{S^\sharp}$  is represented by Fontaine's ring  $B_{\mathrm{dR}}^+(R^\sharp)$ . The untilt  $S^\sharp$  determines a Cartier divisor on  $X_S$ , which we still refer to as  $D_{S^\sharp}$ .

There is a functor  $b \mapsto \mathcal{E}^b$  from the category of isocrystals with  $G$ -structure to the category of  $G$ -bundles on  $X_S$  (for any  $S$ ). When  $S$  is a geometric point this functor induces an isomorphism between the sets of isomorphism classes [Far15].

We now recall Scholze's definition of the local shtuka space. It is a set-valued functor on the pro-etale site of perfectoid spaces over  $\mathbf{F}_p$  and is equipped with a morphism to  $\mathrm{Spd} C$ . Thus it can be described equivalently as a set-valued functor on the pro-etale site of perfectoid spaces over  $C$ .

**Definition 2.4.1.** The local shtuka space  $\mathrm{Sht}_{G,b,\mu}$  inputs a perfectoid  $C$ -algebra  $(R, R^+)$ , and outputs the set of isomorphisms

$$\gamma: \mathcal{E}^1|_{X_{R^b} \setminus D_R} \cong \mathcal{E}^b|_{X_{R^b} \setminus D_R}$$

of  $G$ -torsors that are meromorphic along  $D_R$  and bounded by  $\mu$  pointwise on  $\mathrm{Spa} R$ .

Let us briefly recall the condition of being pointwise bounded by  $\mu$ . If  $\mathrm{Spa}(C, O_C) \rightarrow \mathrm{Spa} R$  is a geometric point we obtain via pull-back  $\gamma: \mathcal{E}^1|_{X_{C^b} \setminus \{x_C\}} \rightarrow \mathcal{E}^b|_{X_{C^b} \setminus \{x_C\}}$ , where we have written  $x_C$  in place of  $D_C$  to emphasize that this is a point on  $X_{C^b}$ . The completed local ring of  $X_{C^b}$  at  $x_C$  is Fontaine's ring  $B_{\mathrm{dR}}^+(C)$ . A trivialization of both bundles  $\mathcal{E}^1$  and  $\mathcal{E}^b$  on a formal neighborhood of  $x_C$ , together with  $\gamma$ , leads to an element of  $G(B_{\mathrm{dR}}(C))$ , well defined up to left and right multiplication by elements of  $G(B_{\mathrm{dR}}^+(C))$ . The corresponding element of the double coset space  $G(B_{\mathrm{dR}}^+(C)) \backslash G(B_{\mathrm{dR}}(C)) / G(B_{\mathrm{dR}}^+(C))$  is indexed by a conjugacy class of co-characters of  $G/C$  according to the Cartan decomposition, and we demand that this conjugacy class is dominated by  $\mu$  in the usual order (given by the simple roots of the universal Borel pair).

The space  $\text{Sht}_{G,b,\mu}$  is a locally spatial diamond [SW14, §23]. Since the automorphism groups of  $\mathcal{E}^1$  and  $\mathcal{E}^b$  are the constant group diamonds  $\underline{G}(\mathbf{Q}_p)$  and  $\underline{J}_b(\mathbf{Q}_p)$  respectively, the space  $\text{Sht}_{G,b,\mu}$  is equipped with commuting actions of  $\underline{G}(\mathbf{Q}_p)$  and  $\underline{J}_b(\mathbf{Q}_p)$ , acting by pre- and post-composition on  $\gamma$ .

**Remark 2.4.2.** According to [SW14, Corollary 23.2.2] the above definition recovers the moduli space of local shtukas with one leg and infinite level structure. We have dropped the subscript  $\infty$  used in [SW14] to denote the infinite level structure, because we will not often refer to the space without level structure.

We will use the cohomology theory developed in [Sch17]. For any compact open subgroup  $K \subset G(F)$  the quotient  $\text{Sht}_{G,b,\mu}/K$  is again a locally spatial diamond, denoted by  $\text{Sht}_{G,b,\mu,K}$  in [SW14, §23]. Let  $\Lambda = \mathcal{O}_L$ , where  $L/\mathbf{Q}_\ell$  is a finite extension. The diamond  $\text{Sht}_{G,b,\mu,K}$  is stratified by closed subdiamonds corresponding to cocharacters bounded by  $\mu$ ; let us write  $\text{IC}_\mu$  for the intersection complex with coefficients in  $L$ . (Thus if  $\mu$  is minuscule, then  $\text{IC}_\mu$  is the constant sheaf  $L$ .) We may then consider the cohomology with compact support  $R\Gamma_c(\text{Sht}_{G,b,\mu}/K, \text{IC}_\mu)$ , which lies in the derived category of representations of  $J_b(F)$  with coefficients in  $\Lambda$ .

**Definition 2.4.3.** Let  $\rho$  be an irreducible admissible representation with coefficients in  $L$  equipped with an invariant lattice. Let

$$H^*(G, b, \mu)[\rho] = \varinjlim_{K \subset G(F)} R\text{Hom}_{J_b(F)}(R\Gamma_c(\text{Sht}_{G,b,\mu}/K, \text{IC}_\mu), \rho),$$

where  $K$  runs over the set of open compact subgroups of  $G(F)$ .

This defines an element of the derived category of representations of  $G(F)$  with coefficients in  $\Lambda$ . With some care, an action of the Weil group  $W_E$  can be introduced, but we will not pursue that.

**Remark 2.4.4.** We now discuss the relationship between this definition of  $H^*(G, b, \mu)[\rho]$  and the one given in [RV14].

When  $\mu$  is minuscule,  $\text{Sht}_{G,b,\mu}$  is the diamond  $\mathcal{M}_{G,b,\mu}^\diamond$  associated to the local Shimura variety  $\text{Sht}_{G,b,\mu}$  [SW14, §24.1]. Moreover, in that case,  $\text{IC}_\mu$  is the constant sheaf. In [RV14],  $H^*(G, b, \mu)[\rho]$  is defined as the alternating sum

$$\sum_{i,j \in \mathbf{Z}} (-1)^{i+j} H^{i,j}(G, b, \mu)[\rho],$$

where

$$H^{i,j}(G, b, \mu)[\rho] = \varinjlim_K \text{Ext}_{J_b(F)}^j(R^i\Gamma_c(G, b, \mu, K), \rho).$$

There is a spectral sequence  $H^{i,j}(G, b, \mu)[\rho] \implies H^{i+j}(G, b, \mu)[\rho]$ , so that if one knew that each  $H^{i,j}(G, b, \mu)[\rho]$  were an admissible representation of  $J_b(F)$  which is nonzero for only finitely many  $(i, j)$ , then the admissibility of  $H^{i+j}(G, b, \mu)[\rho]$  would follow and moreover the definition of  $H^*(G, b, \mu)[\rho]$  given in [RV14] would agree with the definition given here.

Rapoport-Viehmann [RV14, Proposition 6.1] proves the admissibility of  $H^{i,j}(G, b, \mu)[\rho]$  under an assumption (Properties 5.3(iii)) that  $\mathcal{M}_{G,b,\mu,K}$  admits a covering by  $J_b(F)$ -translates of an open subset  $U$  obeying a certain condition which guarantees [Hub98a, Theorem 3.3] that the compactly supported  $\ell$ -adic cohomology of  $U$  is finite-dimensional. We do not prove this assumption, nor do we prove that  $H^{i,j}(G, b, \mu)[\rho]$  is admissible for general  $(G, b, \mu)$ .

## 2.5 Remarks on the proof, and relation with prior work

Theorem 1.0.2 is proved by an application of a Lefschetz-Verdier fixed-point formula. Let us illustrate the idea in the Lubin-Tate case, when  $G = \mathrm{GL}_n$ ,  $\mu = (1, 0, \dots, 0)$ , and  $b$  is basic of slope  $1/n$ . In this case  $J_b(F) = D^\times$ , where  $D/F$  is the division algebra of invariant  $1/n$ . The spaces  $\mathcal{M}_K = \mathcal{M}_{G,b,\mu,K}$  are known as the Lubin-Tate tower; we consider these as rigid-analytic spaces over  $C$ , where  $C/\mathbf{Q}_p$  is a complete algebraically closed field. Atop the tower sits the infinite-level Lubin-Tate space  $\mathcal{M} = \varprojlim_K \mathcal{M}_K$  as described in [SW13]. This is a perfectoid space admitting an action of  $\mathrm{GL}_n(F) \times D^\times$ . The Hodge-Tate period map exhibits  $\mathcal{M}$  as a pro-étale  $D^\times$ -torsor over Drinfeld's upper half-space  $\Omega^{n-1}$  (the complement in  $\mathbf{P}^{n-1}$  of all  $\mathbf{Q}_p$ -rational hyperplanes). This map  $\mathcal{M} \rightarrow \Omega^{n-1}$  is equivariant for the action of  $\mathrm{GL}_n(F)$ .

Now suppose  $g \in \mathrm{GL}_n(\mathbf{Q}_p)$  is a regular elliptic element (that is, an element with irreducible characteristic polynomial). Then  $g$  has exactly  $n$  fixed points on  $\Omega^{n-1}$ . For each such fixed point  $x \in (\Omega^{n-1})^g$ , the element  $g$  acts on the fiber  $\mathcal{M}_x$ .

**Key observation.** The action of  $g$  on  $\mathcal{M}_x$  agrees with the action of an element  $g' \in D^\times$ , where  $g$  and  $g'$  are related (meaning they become conjugate over  $\overline{\mathbf{Q}_p}$ ).

Suppose that  $\rho$  is an admissible representation of  $D^\times$  with coefficients in  $\overline{\mathbf{Q}_\ell}$ . There is a corresponding  $\ell$ -adic local system  $\mathcal{L}_\rho$  on  $\Omega_{C,\text{ét}}^{n-1}$ .

A naïve form of the Lefschetz trace formula would predict:

$$\mathrm{tr}(g|H_c^*(\Omega^{n-1}, \mathcal{L}_\rho)) = \sum_{x \in (\Omega^{n-1})^g} \mathrm{tr}(g|\mathcal{L}_{\rho,x}).$$

For each fixed point  $x$ , the key observation above gives  $\mathrm{tr}(g|\mathcal{L}_{\rho,x}) = \mathrm{tr} \rho(g')$ , where  $g$  and  $g'$  are related. By the Jacquet-Langlands correspondence, there exists a discrete series representation  $\pi$  of  $\mathrm{GL}_n(F)$  satisfying  $\mathrm{tr} \pi(g) = (-1)^{n-1} \mathrm{tr} \rho(g')$  (here  $\mathrm{tr} \pi(g)$  is interpreted as a Harish-Chandra character). Therefore in  $\mathrm{Groth}(\mathrm{GL}_n(\mathbf{Q}_p))$  we have an equality

$$H_c^*(\Omega^{n-1}, \mathcal{L}_\rho) = (-1)^{n-1} n\pi,$$

at least up to representations which have trace 0 on regular elliptic elements. The virtual  $\mathrm{GL}_n(\mathbf{Q}_p)$ -representation  $H^*(G, b, \mu)[\rho]$  is dual to the Euler characteristic  $H_c^*(\Omega^{n-1}, \mathcal{L}_{\rho^\vee})$ , where  $\rho^\vee$  is the smooth dual; thus the above is in accord with Theorem 1.0.2.

This argument goes back at least to the 1990s, as discussed in [Har15, Chap. 9], and as far as we know first appears in [Fal94]. The present article essentially pushes this argument as far as it will go: if a suitable Lefschetz formula is valid, then Theorem 1.0.2 is true in general; it can be reduced to an endoscopic character identity relating representations of  $G(F)$  and  $J_b(F)$  (Theorem 3.2.9), which we prove in §3.

Therefore the difficulty in Theorem 1.0.2 lies in proving the validity of the Lefschetz formula. Prior work of Strauch and Mieda proved Theorem 1.0.2 in the case of the Lubin-Tate tower [Str05], [Str08], [Mie12], [Mie14a] and also in the case of a basic Rapoport-Zink space for  $\mathrm{GSp}(4)$  [Mie].

In applying a Lefschetz formula to a non-proper rigid space, care must be taken to treat the boundary. For instance, if  $X$  is the closed unit disc  $\{|T| \leq 1\}$  in the adic space  $\mathbf{A}^1$ , then the automorphism  $T \mapsto T + 1$  has Euler characteristic 1 on  $X$ , despite having no fixed points. The culprit is that this automorphism fixes the single boundary point in  $\overline{X} \setminus X$ . Mieda [Mie14b] proves a Lefschetz formula for an operator on a rigid space, under an assumption that the operator has no topological fixed points on a compactification. Now, in all of the above cases,  $\mathcal{M}_{G,b,\mu,K}$  admits a *cellular decomposition*. This means (approximately) that  $\mathcal{M}_{G,b,\mu,K}$  contains a compact open subset, whose translates by Hecke operators cover all of  $\mathcal{M}_{G,b,\mu,K}$ . This is enough to establish the “topological fixed point” hypothesis necessary to apply Mieda’s Lefschetz formula. Shen [She14] constructs a cellular decomposition for a basic Rapoport-Zink space attached to the group  $U(1, n-1)$ , which paves the way for a similar proof of Theorem 1.0.2

in this case as well. For general  $(G, b, \mu)$ , however, the  $\mathcal{M}_{G,b,\mu,K}$  do not admit a cellular decomposition, and so there is probably no hope of applying the methods of [Mie14b].

We had no idea how to proceed, until we learned of the shift of perspective offered by Fargues' program on the geometrization of local Langlands [Far]. At the center of that program is the stack  $\text{Bun}_G$  of  $G$ -bundles on the Fargues-Fontaine curve. This is a smooth Artin stack in diamonds. There is a bijection  $b \mapsto \mathcal{E}^b$  between Kottwitz' set  $B(G)$  and points of the underlying topological space of  $\text{Bun}_G$ . Thus for every  $b$  there is an inclusion  $i_b: \text{Bun}_G^b \rightarrow \text{Bun}_G$ , where  $\text{Bun}_G^b$  classifies  $G$  bundles which are everywhere isomorphic to  $\mathcal{E}^b$ . When  $b$  is basic,  $i_b$  is an open immersion, and  $\text{Bun}_G^b \cong [* / J_b(F)]$ . In particular, when  $\rho$  is a smooth representation  $J_b(F)$ , there is a corresponding local system  $\mathcal{L}_\rho$  on  $\text{Bun}_G^b$ .

As in geometric Langlands, there is a stack  $\text{Hecke}_{\leq \mu}$  lying over the product  $\text{Bun}_G \times \text{Bun}_G$ , which parametrizes modifications of  $G$ -bundles at one point of the curve, which are bounded by  $\mu$ . For each  $\mu$ , one uses  $\text{Hecke}_{\leq \mu}$  to define a Hecke operator  $T_\mu$  on sheaves on  $\text{Bun}_G$ . The (infinite-level) moduli space of local shtukas  $\text{Sht}_{G,b,\mu}$  appears as the fiber of  $\text{Hecke}_{\leq \mu}$  over  $(\mathcal{E}^b, \mathcal{E}^1)$ .

Forthcoming work of Fargues-Scholze [FS] constructs a semi-simplified Langlands parameter to any irreducible admissible  $\overline{\mathbf{Q}}_\ell$ -representation of  $G(F)$  that admits an invariant lattice. It studies objects in the derived category of  $\overline{\mathbf{Q}}_\ell$ -sheaves on  $\text{Bun}_G$  which satisfy a certain finiteness condition, namely *reflexivity*. A complex of sheaves  $\mathcal{F}$  is *reflexive* if the natural map

$$\mathcal{F} \rightarrow \mathbf{D}\mathbf{D}\mathcal{F} \tag{2.5.1}$$

is an isomorphism, where  $\mathbf{D}$  denotes the Verdier dual. In [FS] there is a crucial criterion for reflexivity for a  $\overline{\mathbf{Q}}_\ell$ -sheaf on  $\text{Bun}_G$ . It is also shown that the Hecke operators  $T_\mu$  preserve reflexivity. In this article we define a notion of *strong reflexivity*, which is reflexivity together with the condition that the natural map

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \rightarrow \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \tag{2.5.2}$$

is also an isomorphism. In §4, we show that strong reflexivity formally implies the Lefschetz fixed-point formula. (In the world of schemes, the appropriate finiteness condition is *constructibility*, which, suitably defined, implies strong reflexivity, as in [Del77]. In the world of rigid-analytic spaces, constructibility does not guarantee strong reflexivity; this explains the failure of the example of the Lefschetz formula for the closed unit disc. These matters are rather subtle, and so we have included a discussion of reflexivity

and strong reflexivity in the Appendix.) Our form of the Lefschetz formula is adapted for the situation where a locally pro- $p$  group  $H$  acts on a proper rigid-analytic variety  $X$ : one obtains a formula for the *trace distribution* of  $H$  on the cohomology of  $X$ , in terms of the fixed-point locus of  $H$  acting on  $X$ .

In §5, we study the Lefschetz formula as it pertains to the affine Grassmannian  $\mathrm{Gr}_G$ , in both the classical and  $B_{\mathrm{dR}}$  settings. The latter object will play the role of projective space  $\mathbf{P}^{n-1}$  in the Lubin-Tate example discussed above.

Finally, in §6, we turn to  $\mathrm{Bun}_G$ . Starting with an admissible representation  $\rho$  of  $J_b(F)$ , we use a result from [FS] that the sheaf  $i_{b*}\mathcal{L}_\rho$  on  $\mathrm{Bun}_G$  is strongly reflexive. The proof of Theorem 1.0.2 consists of an application of our form of the Lefschetz formula to  $i_1^*T_\mu(i_{b*}\mathcal{L}_\rho)$  (a representation of  $G(F)$ ).

### 3 Representation-theoretic preparations

Throughout,  $F/\mathbf{Q}_p$  is a finite extension, and  $G/F$  is a connected reductive group.

#### 3.1 The space of strongly regular conjugacy classes in $G(F)$

The following definitions are important for our work.

- $G_{\mathrm{rs}} \subset G$  is the open subvariety of regular semisimple elements, meaning those whose connected centralizer is a maximal torus.
- $G_{\mathrm{sr}} \subset G$  is the open subvariety of strongly regular semi-simple elements, meaning those regular semisimple elements whose centralizer is connected, i.e. a maximal torus;
- $G(F)_{\mathrm{ell}} \subset G(F)$  is the open subset of strongly regular elliptic elements, meaning those strongly regular semisimple elements in  $G(F)$  whose centralizer is an elliptic maximal torus.

We put  $G(F)_{\mathrm{sr}} = G_{\mathrm{sr}}(F)$  and  $G(F)_{\mathrm{rs}} = G_{\mathrm{rs}}(F)$ . Note that  $G(F)_{\mathrm{ell}} \subset G(F)_{\mathrm{sr}} \subset G(F)_{\mathrm{rs}}$ . The inclusion  $G(F)_{\mathrm{sr}} \subset G(F)_{\mathrm{rs}}$  is dense.

If  $g$  is regular semisimple, then it is necessarily contained in a unique maximal torus  $T$ , namely the neutral component  $\mathrm{Cent}(g, G)^\circ$ , but this is not necessarily all of  $\mathrm{Cent}(g, G)$ . If  $G$  is simply connected, then  $\mathrm{Cent}(g, G)$  is connected; thus in such a group, regular semisimple and strongly regular semisimple mean the same thing.

Observe that if  $g$  is regular semisimple, then  $\alpha(g) \neq 1$  for all roots  $\alpha$  relative to the action of  $T$ . Indeed, if  $\alpha(g) = 1$ , then the root subgroup of  $\alpha$  would commute with  $g$ , and then it would have dimension strictly greater than  $\dim T$ .

All of the sets  $G(F)_{\text{sr}}$ ,  $G(F)_{\text{rs}}$ ,  $G(F)_{\text{ell}}$  are conjugacy-invariant, so we may for instance consider the quotient  $G(F)_{\text{sr}} // G(F)$ , considered as a topological space.

**Lemma 3.1.1.**  *$G(F)_{\text{sr}} // G(F)$  is locally profinite.*

*Proof.* Let  $T \subset G$  be a  $F$ -rational maximal torus, and let  $N(T, G)$  be its centralizer in  $G$ . The set  $H^1(F, N(T, G))$  classifies conjugacy classes of  $F$ -rational tori, as follows: given a  $F$ -rational torus  $T'$ , we must have  $T = xT'x^{-1}$  for some  $x \in G(\overline{F})$ . Then for all  $\sigma \in \text{Gal}(\overline{F}/F)$ ,  $x^\sigma x^{-1}$  normalizes  $T$ . We associate to  $T'$  the class of  $\sigma \mapsto x^\sigma x^{-1}$  in  $H^1(F, N(T, G))$ , and it is a simple matter to see that this defines a bijection as claimed. (In fact  $H^1(F, N(T, G))$  is finite.)

There is a map  $G(F)_{\text{sr}} // G(F) \rightarrow H^1(F, N(T, G))$ , sending the conjugacy class of  $g \in G(F)_{\text{sr}}$  to the conjugacy class of the unique  $F$ -rational torus containing it, namely  $\text{Cent}(g, G)^\circ$ . We claim that this map is locally constant.

To prove the claim, we consider

$$\varphi : G(F) \times T_{\text{sr}}(F) \rightarrow G_{\text{sr}}(F), \quad (g, t) \mapsto gtg^{-1},$$

a morphism of  $p$ -adic analytic varieties. We would like to show that  $\varphi$  is open. To do this, we will compute its differential at the point  $(g, t)$ , by means of a change of variable. Consider the map

$$\psi = L_{gtg^{-1}}^{-1} \circ \varphi \circ (L_g \times L_t).$$

Explicitly, for  $(z, w) \in G(F) \times T(F)$  we have  $\psi(z, w) = gt^{-1}ztwz^{-1}g^{-1}$ .

Let  $\mathfrak{g} = \text{Lie } G$ ,  $\mathfrak{t} = \text{Lie } T$ . The derivative  $d\psi(1, 1) : \mathfrak{g} \times \mathfrak{t} \rightarrow \mathfrak{g}$  is given by the formula

$$d\psi(1, 1)(Z, W) = \text{Ad}(g)[(\text{Ad}(t^{-1}) - \text{id})Z + W].$$

We would like to check that  $d\psi(1, 1)$  is surjective. We may decompose  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ , where  $\mathfrak{t}^\perp$  is the descent to  $F$  of the direct sum of all root subspaces of  $\mathfrak{g}_{\overline{F}}$  for the action of  $T$ .

The element  $t$  is strongly regular. We have observed that  $\alpha(t) \neq 1$  for all roots of  $\mathfrak{g}$  for the action of  $T$ . Therefore  $\text{Ad}(t^{-1}) - \text{id} : \mathfrak{g}/\mathfrak{t} \rightarrow \mathfrak{t}^\perp$  is an isomorphism. It follows that  $d\psi$  is surjective. The derivative of  $\varphi$  at  $(g, t)$  is

$$d\varphi(g, t) = dL_{gtg^{-1}}(gtg^{-1}) \circ d\psi(1, 1) \circ (dL_g(1) \times dL_t(1)).$$

All terms  $dL$  are isomorphisms, so  $d\varphi(g, t)$  is also surjective. Thus  $\varphi$  is a submersion in the sense of Bourbaki VAR §5.9.1, hence it is open by loc. cit. §5.9.4.

Therefore if  $g \in T(F)_{\text{sr}}$  and  $g'$  is sufficiently close to  $g$  in  $G(F)$ , then  $g'$  is conjugate in  $G(F)$  to an element of  $T(F)$ , which proves the claim about the local constancy of  $G(F)_{\text{sr}} // G(F) \rightarrow H^1(F, N(T, G))$ .

The fiber of this map over  $T'$  is  $T'(F)_{\text{sr}}$  modulo the action of the finite group  $N(T', G)(F)/T'(F)$ . This group acts freely on  $T'(F)_{\text{sr}}$ , since by definition  $\text{Cent}(g, G) = T'$  for  $g \in T'(F)_{\text{sr}}$ . Since  $T'(F)$  is locally profinite, so is its quotient by the free action of a finite group.  $\square$

### 3.2 Hecke correspondences

Suppose that  $b \in G(\check{F})$  is basic. The goal of this section is to define a family of explicit Hecke operators, which input a conjugation-invariant function on  $G(F)_{\text{rs}}$  and output a conjugation-invariant function on  $J_b(F)_{\text{rs}}$ . Given a sufficiently strong version of the local Langlands conjectures, we will show that the Hecke operators act predictably on the trace characters attached to irreducible admissible representations.

We begin by recalling the concept of related elements and the definition of their invariant in the isocrystal setting from [Kal14].

**Lemma 3.2.1.** *Suppose  $g \in G(F)$  and  $j \in J_b(F)$  are strongly regular elements which are conjugate over an algebraic closure of  $\check{F}$ . Then they are conjugate over  $\check{F}$ .*

*Proof.* Let  $K$  be an algebraic closure of  $\check{F}$ . Say  $j = zgz^{-1}$  with  $z \in G(K)$ . Let  $T = \text{Cent}(g, G)$ ; then for all  $\tau$  in the inertia group  $\text{Gal}(\overline{F}/F^{\text{nr}})$ ,  $z^{-\tau}z$  commutes with  $g$  and therefore lies in  $T(K)$ . Then  $\tau \mapsto z^{-\tau}z$  is a cocycle in  $H^1(\check{F}, T)$ . Since  $T$  is a connected algebraic group,  $H^1(\check{F}, T) = 0$  [Ste65, Theorem 1.9]. If  $x \in T(K)$  splits the cocycle, then  $y = zx^{-1} \in G(\check{F})$ , and  $j = ygy^{-1}$ , so that  $g$  and  $j$  are related.  $\square$

It is customary to call elements  $g, j$  as in the above lemma *stably conjugate*, or *related*. Suppose we have strongly regular elements  $g \in G(F)_{\text{rs}}$  and  $j \in J_b(F)_{\text{rs}}$  which are related. Let  $T = \text{Cent}(g, G)$ , and suppose  $y \in G(\check{F})$

with  $j = ygy^{-1}$ . The rationality of  $g$  means that  $g^\sigma = g$ , whereas the rationality of  $j$  in  $J_b$  means that  $j^\sigma = b^{-1}jb$ . Combining these statements shows that  $b_0 := y^{-1}by^\sigma$  commutes with  $g$  and therefore lies in  $T(\check{F})$ .

**Definition 3.2.2.** For strongly regular related elements  $g \in G(F)_{\text{rs}}$  and  $j \in J_b(F)_{\text{rs}}$ , the invariant  $\text{inv}[b](g, j)$  is the class of  $y^{-1}by^\sigma$  in  $B(T)$ , where  $y \in G(\check{F})$  satisfies  $j = ygy^{-1}$ .

**Fact 3.2.3.** *The invariant  $\text{inv}[b](g, j) \in B(T)$  only depends on  $b$ ,  $g$ , and  $j$  and not on the element  $y$  which conjugates  $g$  into  $j$ . It depends on the rational conjugacy classes of  $g$  and  $j$  as follows:*

- For  $z \in G(F)$  we have  $\text{inv}[b](\text{(ad } z)(g), j) = (\text{ad } z)(\text{inv}[b](g, j))$ , a class in  $B(\text{(ad } z)(T))$ .
- For  $z \in J_b(F)$  we have  $\text{inv}[b](g, \text{(ad } z)(j)) = \text{inv}[b](g, j)$ .

The image of  $\text{inv}[b](g, j)$  under the composition of  $B(T) \rightarrow B(G)$  and  $\kappa: B(G) \rightarrow \pi_1(G)_\Gamma$  equals  $\kappa(b)$ .

**Definition 3.2.4.** We define a diagram of topological spaces

$$\begin{array}{ccc} & \text{Rel}_b & \\ & \swarrow \quad \searrow & \\ G(F)_{\text{sr}} // G(F) & & J_b(F)_{\text{sr}} // J(F). \end{array} \quad (3.2.1)$$

as follows. The space  $\text{Rel}_b$  is the set of conjugacy classes of triples  $(g, j, \lambda)$ , where  $g \in G(F)_{\text{sr}}$  and  $j \in J_b(F)_{\text{sr}}$  are related, and  $\lambda \in X_*(T)$ , where  $T = \text{Cent}(g, G)$ . It is required that  $\kappa(\text{inv}[b](g, j))$  agrees with the image of  $\lambda$  in  $X_*(T)_\Gamma$ . We consider  $(g, j, \lambda)$  conjugate to  $((\text{ad } z)(g), (\text{ad } z')(j), (\text{ad } z)(\lambda))$  whenever  $z \in G(F)$  and  $z' \in J_b(F)$ . We give  $\text{Rel}_b \subset (G(F) \times J_b(F) \times X_*(G)) / (G(F) \times J_b(F))$  the subspace topology, where  $X_*(G)$  is taken to be discrete.

**Remark 3.2.5.** Given  $g \in G(F)_{\text{sr}}$  and  $\lambda$  a cocharacter of its torus, there is at most one conjugacy class of  $j \in J_b(F)$  with  $(g, j, \lambda) \in \text{Rel}_b$ . In other words,  $g$  and  $\text{inv}[b](g, j)$  determine the conjugacy class of  $j$ . Indeed, suppose  $(g, j, \lambda)$  and  $(g, j', \lambda)$  are both in  $\text{Rel}_b$ . Then  $j = ygy^{-1}$  and  $j' = zgz^{-1}$  for some  $y, z \in G(F^{\text{nr}})$ , and  $y^{-1}by^\sigma$  and  $z^{-1}bz^\sigma$  are  $\sigma$ -conjugate in  $T(\check{F})$ . This means there exists  $t \in T(\check{F})$  such that  $y^{-1}by^\sigma = (zt)^{-1}b(zt)^\sigma$ . We see that  $x = zty^{-1} \in J_b(F)$ , and that  $x$  conjugates  $j$  onto  $j'$ .

**Lemma 3.2.6.** *The map  $\text{Rel}_b \rightarrow G(F)_{\text{sr}} // G(F)$  is a homeomorphism locally on the source. Its image consists of those classes that transfer to  $J_b$ . In particular, the image is open and closed.*

*The analogous statement is true for  $\text{Rel}_b \rightarrow J(F)_{\text{sr}} // J(F)$ .*

*Proof.* The proof of Lemma 3.1.1 shows that  $G(F)_{\text{sr}} // G(F)$  is the disjoint union of spaces  $T(F)_{\text{sr}}/W_T$ , as  $T \subset G$  runs through the finitely many conjugacy classes of  $F$ -rational maximal tori, and  $W_T = N(T, G)(F)/T(F)$  is a finite group. By the above remark,  $\text{Rel}_b$  injects into the disjoint union of the spaces  $T(F)_{\text{sr}}/W_T \times X_*(T)$ , with the map to  $G(F)_{\text{sr}} // G(F)$  corresponding to the projection  $T(F)_{\text{sr}}/W_T \times X_*(T) \rightarrow T(F)_{\text{sr}}/W_T$ . Since  $X_*(T)$  is discrete, this map is a homeomorphism locally on the source. The other statements are evident from the definitions.  $\square$

The definition of  $\text{Rel}_b$  already suggests a means for transferring functions from  $G(F)_{\text{sr}} // G(F)$  to  $J_b(F)_{\text{sr}} // J_b(F)$ , namely, by pulling back from  $G(F)_{\text{sr}} // G(F)$  to  $\text{Rel}_b$ , multiplying by a kernel function, and then pushing forward to  $J_b(F)_{\text{sr}} // J_b(F)$ . The kernel function is necessary because the fibers of  $\text{Rel}_b \rightarrow J_b(F)_{\text{sr}} // J_b(F)$  may be infinite. In fact we will define one such kernel function for each conjugacy class of  $\bar{F}$ -rational cocharacters  $\mu: \mathbf{G}_m \rightarrow G$ .

Let  $\widehat{G}$  be the Langlands dual group. Given a cocharacter  $\mu$ , we obtain a triple  $(\widehat{T}, \widehat{B}, \widehat{\mu})$  up to  $\widehat{G}$ -conjugacy, where  $\widehat{T} \subset \widehat{G}$  is a maximal torus,  $\widehat{B} \subset \widehat{G}$  is a Borel subgroup containing  $\widehat{T}$ , and  $\widehat{\mu}: \widehat{T} \rightarrow \mathbf{G}_m$  is a character that is  $\widehat{B}$ -dominant. Let  $r_\mu$  be the unique irreducible rational representation of the dual group  $\widehat{G}$  whose highest weight with respect to  $(\widehat{T}, \widehat{B})$  is  $\widehat{\mu}$ .

A cocharacter  $\lambda \in X_*(T)$  corresponds to a character  $\widehat{\lambda} \in X^*(\widehat{T})$ . Let  $r_\mu[\lambda]$  be the  $\widehat{\lambda}$ -weight space of  $r_\mu$ . The quantity  $\dim r_\mu[\lambda]$  will give us our kernel function.

We now fix a commutative ring  $\Lambda$  in which  $p$  is invertible. For a topological space  $X$ , we let  $C(X, \Lambda)$  be the space of continuous  $\Lambda$ -valued functions on  $X$ , where  $\Lambda$  is given the discrete topology.

**Definition 3.2.7.** We define the Hecke operator

$$T_{b,\mu}^{G \rightarrow J}: C(G(F)_{\text{sr}} // G(F), \Lambda) \rightarrow C(J_b(F)_{\text{sr}} // J_b(F), \Lambda)$$

by

$$[T_{b,\mu}^{G \rightarrow J} f](j) = \sum_{(g,j,\lambda) \in \text{Rel}_b} f(g) \dim r_\mu[\lambda].$$

Analogously, we define

$$T_{b,\mu}^{J \rightarrow G}: C(J_b(F)_{\text{sr}} // J_b(F), \Lambda) \rightarrow C(G(F)_{\text{sr}} // G(F), \Lambda)$$

by

$$[T_{b,\mu}^{J \rightarrow G} f'](g) = \sum_{(g,j,\lambda) \in \text{Rel}_b} f'(j) \dim r_\mu[\lambda].$$

Since  $r_\mu$  is finite-dimensional, the sum is finite.

**Lemma 3.2.8.** *The operator  $T_{b,\mu}^{G \rightarrow J}$  is the zero operator unless  $[b]$  is the unique basic class in  $B(G, \mu)$ .*

*Proof.* Suppose there exists an  $F$ -rational maximal torus  $T \subset G$  and a cocharacter  $\lambda \in X_*(T)$  such that  $r_\mu[\lambda] \neq 0$ . Then  $\widehat{\mu}$  and  $\widehat{\lambda}$  must agree when restricted to the center  $Z(\widehat{G})$ , which is to say that  $\widehat{\mu}$  and  $\widehat{\lambda}$  have the same image in  $X^*(Z(\widehat{G}))$ . Equivalently, if we conjugate  $\mu$  so as to assume it is a cocharacter of  $T$ , then  $\mu$  and  $\lambda$  have the same image under  $X_*(T) \cong \pi_1(T) \rightarrow \pi_1(G)$ . By Fact 3.2.3 and the functoriality of  $\kappa$  the image of  $\lambda$  in  $\pi_1(G)_\Gamma$  equals  $\kappa(b)$ . We conclude that  $\kappa([b])$  equals the image of  $\mu$  in  $\pi_1(G)_\Gamma$ . This means that  $[b]$  is the unique basic class in  $B(G, \mu)$ .  $\square$

Assume therefore that  $[b]$  is the unique basic class in  $B(G, \mu)$ . We may define a ‘‘truncation’’  $\text{Rel}_{b,\mu} \subset \text{Rel}_b$ , consisting of conjugacy classes of triples  $(g, j, \lambda)$  for which  $\lambda \leq \mu$ . Then the kernel function  $(g, j, \lambda) \mapsto \dim r_\mu[\lambda]$  is supported on  $\text{Rel}_{b,\mu}$ . In the diagram

$$\begin{array}{ccc} & \text{Rel}_{b,\mu} & \\ & \swarrow \quad \searrow & \\ G(F)_{\text{rs}} // G(F) & & J_b(F)_{\text{rs}} // J_b(F), \end{array} \tag{3.2.2}$$

both maps are finite étale over their respective images.

The following theorem is proved in §3.6. It relates the operator  $T_{b,\mu}^{G \rightarrow J}$  to the local Jacquet-Langlands correspondence for  $G$ .

**Theorem 3.2.9.** *Assume that  $b \in B(G, \mu)$  is basic, and that  $\Lambda$  is a field of characteristic 0. Let  $\phi: W_F \rightarrow {}^L G$  be a discrete Langlands parameter with coefficients in  $\Lambda$ , and let  $\rho \in \Pi_\phi(G)$ . Let  $\Theta_\rho \in C(G(F)_{\text{sr}} // G(F), \Lambda)$  be its Harish-Chandra character. Then for any  $g \in G(F)_{\text{sr}}$  that transfers to  $J_b(F)$ , we have*

$$[T_{b,\mu}^{J \rightarrow G} \Theta_\rho](g) = (-1)^d \sum_{\pi \in \Pi_\phi(G)} \dim \text{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu) \Theta_\pi(g).$$

**Remark 3.2.10.** When  $g$  does not transfer to  $J_b(F)$ , the left-hand side is zero by definition, but the right-hand side need not be zero, as the following example shows.

**Example 3.2.11.** Let  $G = \mathrm{GL}_2$ , and let  $\mu: \mathbb{G}_m \rightarrow G$  the co-character sending  $x$  to the diagonal matrix with entries  $(x, 1)$ . We have  $\pi_1(G) = \mathbf{Z}$  as a trivial  $\Gamma$ -module. If  $b \in B(G, \mu)$  is the basic class, then  $J_b(F)$  is the multiplicative group of the quaternion algebra over  $F$ . The  $L$ -packets  $\Pi_\phi(G) = \{\pi\}$  and  $\Pi_\phi(J_b) = \{\rho\}$  are singletons, with  $\pi$  and  $\rho$  supercuspidal. We have  $S_\phi = Z(\widehat{G}) = \mathbf{C}^\times$  and  $\delta_{\pi, \rho}$  is the identity character of  $S_\phi$ . The representation  $r_\mu$  is the standard representation of  $\widehat{G} = \mathrm{GL}_2(\mathbf{C})$  and  $\dim \mathrm{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_\mu) = 2$ . Therefore the right-hand side above equals  $(-1) \cdot 2 \cdot \Theta_\rho(j)$ .

Let  $w\mu$  be the cocharacter sending  $x$  to the diagonal matrix with entries  $(1, x)$ . The map  $\lambda \mapsto r_\mu[\lambda]$  sends  $\mu$  and  $w\mu$  to 1 and all other co-characters to 0. For any strongly regular  $j \in J_b(F)$  there is a unique  $G(F)$ -conjugacy class of strongly regular  $g \in G(F)$  related to  $j$ . Let  $S \subset G$  be the centralizer of one such  $g$ . Then  $X_*(S) \cong \mathbf{Z}[\Gamma_{E/F}]$  for a quadratic extension  $E/F$  and the map  $\pi_1(S)_\Gamma \rightarrow \pi_1(G)_\Gamma$  is an isomorphism. There are two elements  $\lambda, w\lambda \in X_*(S)$  that map to  $\mathrm{inv}[b](g, j)$ . Therefore  $T_{b, \mu}^{G \rightarrow J} f(j) = 2f(g)$  and Theorem 3.2.9 reduces to the Jacquet-Langlands character identity. Note that all strongly regular  $j \in J_b(F)$  transfer to  $G(F)$ . Hence the functions  $T_{b, \mu}^{G \rightarrow J} \Theta_\pi$  and  $(-1) \cdot 2 \cdot \Theta_\rho$  are equal.

We can switch the roles of the two groups and let  $G$  be the non-trivial inner form of  $\mathrm{GL}_2$ . We can still choose the same  $\mu$  and  $b$ . Then  $J_b$  is isomorphic to  $\mathrm{GL}_2$ . The entire analysis remains valid. However, not all  $j \in J_b(F)$  transfer to  $G(F)$ ; only the elliptic ones do. Therefore the functions  $T_{b, \mu}^{G \rightarrow J} \Theta_\pi$  and  $(-1) \cdot 2 \cdot \Theta_\rho$  are not equal, because the former is zero on all non-elliptic  $j$ , while the latter is not.

### 3.3 Interaction with orbital integrals

Once again,  $\Lambda$  is a commutative ring in which  $p$  is invertible. For a topological space  $X$ , we let  $C_c(X, \Lambda)$  be the space of compactly supported complex-valued continuous functions.

Choose a  $\Lambda$ -valued Haar measure  $dx$  on  $G(F)$ . For a function  $f \in C(G(F)_{\mathrm{sr}} // G(F), \Lambda)$ , integration against  $f$  is a  $G(F)$ -invariant distribution on  $G(F)$ . A “stable” variant on the Weyl integration formula can be used to compute this distribution in terms of orbital integrals. For  $\phi \in C_c(G(F), \Lambda)$ , let  $\phi_G \in C(G(F)_{\mathrm{sr}} // G(F), \Lambda)$  be the orbital integral function,

$$\phi_G(y) = \int_{x \in G(F)/G(F)_y} \phi(xyx^{-1}) dx.$$

Our formula for the distribution attached to  $f$  takes the form

$$\int_{x \in G} f(x) \phi(x) dx = \langle f, \phi_G \rangle_G. \quad (3.3.1)$$

We explain now the notation  $\langle f, \phi_G \rangle_G$ . For a function  $g \in C(G(F)_{\text{sr}} // G(F), \Lambda)$ , we define

$$\langle f, g \rangle_G = \sum_T |W(T, G)(F)|^{-1} \int_{t \in T(F)_{\text{sr}}} |D(t)| \sum_{t_0 \sim t} f(t_0) g(t_0) dt,$$

where

- $T$  runs over a set of representatives for the stable classes of maximal tori,
- $W(T, G) = N(T, G)/T$  is the absolute Weyl group,
- $D(t) = \det \left( \text{Ad}(t) - 1 \mid \text{Lie } G / \text{Lie } T \right)$  is the usual Weyl discriminant, and
- $t_0$  runs over the  $G(F)$ -conjugacy classes inside of the stable class of  $t$ .

The Haar measures on all tori are chosen compatibly. Note that the integral over  $T$  need not converge. It will do so in the case  $g = \phi_G$  for  $\phi \in C_c(G(F), \Lambda)$ .

The following lemma shows that the operators  $T_{b,\mu}^{G \rightarrow J}$  and  $T_{b,\mu}^{J \rightarrow G}$  are adjoint with respect to the pairing  $\langle \cdot, \cdot \rangle_G$  and its analogue  $\langle \cdot, \cdot \rangle_{J_b}$ , defined similarly.

**Lemma 3.3.1.** *Given  $f' \in C(J_b(F)_{\text{sr}} // J_b(F), \Lambda)$  and  $g \in C(G(F)_{\text{sr}} // G(F), \Lambda)$  we have*

$$\langle T_{b,\mu}^{J \rightarrow G} f', g \rangle_G = \langle f', T_{b,\mu}^{G \rightarrow J} g \rangle_{J_b},$$

*provided one of the two sides (and hence the other) is finite.*

*Proof.* By definition  $\langle T_{b,\mu}^{J \rightarrow G} f', g \rangle_G$  equals

$$\sum_T |W(T, G)(F)|^{-1} \int_{t \in T(F)_{\text{sr}}} |D(t)| \sum_{t_0 \sim t} \sum_{(t_0, t'_0, \lambda)} r_{\mu, \lambda} f'(t'_0) g(t_0) dt.$$

The first sum runs over a set of representatives for the stable classes of maximal tori in  $G$ . The second sum runs over the set  $t_0$  of  $G(F)$ -conjugacy

classes of elements that are stably conjugate to  $t$ . Let  $T_{t_0}$  denote the centralizer of  $t_0$ . The third sum runs over triples  $(t_0, t'_0, \lambda)$ , where  $t'_0$  is a  $J_b(F)$ -conjugacy class that is stably conjugate to  $t_0$ , and  $\lambda \in X_*(T_{t_0})$  maps to  $\text{inv}(t_0, t'_0) \in X_*(T_{t_0})_\Gamma$ . Note that if  $T$  does not transfer to  $J_b$  then it does not contribute to the sum, because the sum over  $(t_0, t'_0, \lambda)$  is empty. Let  $\mathcal{X}$  be a set of representatives for those stable classes of maximal tori in  $G$  that transfer to  $J_b$ . The above expression becomes

$$\sum_{T \in \mathcal{X}} |W(T, G)(F)|^{-1} \int_{t \in T(F)_{\text{sr}}} |D(t)| \sum_{(t_0, t'_0, \lambda)} r_{\mu, \lambda} f'(t'_0) g(t_0) dt,$$

where now the second sum runs over triples  $(t_0, t'_0, \lambda)$  with  $t_0$  a  $G(F)$ -conjugacy class and  $t'_0$  a  $J_b(F)$ -conjugacy class, both stably conjugate to  $t$ , and  $\lambda \in X_*(T_{t_0})$  mapping to  $\text{inv}(t_0, t'_0) \in X_*(T_{t_0})_\Gamma$ .

Let  $\mathcal{X}'$  be a set of representatives for those stable classes of maximal tori of  $J_b$  that transfer to  $G$ . We have a bijection  $\mathcal{X} \leftrightarrow \mathcal{X}'$ . Fix arbitrarily an admissible isomorphism  $T \rightarrow T'$  for any  $T \in \mathcal{X}$  and  $T' \in \mathcal{X}'$  that correspond under this bijection. It induces an isomorphism  $W(T, G) \rightarrow W(T', J_b)$  of finite algebraic groups. Given  $t \in T(F)_{\text{reg}}$  let  $t' \in T'(F)_{\text{reg}}$  be its image under the admissible isomorphism. Then  $|D(t)| = |D(t')|$ . The above expression becomes

$$\sum_{T' \in \mathcal{X}'} |W(T', J_b)(F)|^{-1} \int_{t' \in T'_{\text{reg}}(F)} |D(t')| \sum_{(t_0, t'_0, \lambda)} r_{\mu, \lambda} f'(t'_0) g(t_0) dt,$$

where now the second sum runs over triples  $(t_0, t'_0, \lambda)$ , where  $t_0$  is a  $G(F)$ -conjugacy class,  $t'_0$  is a  $J_b(F)$ -conjugacy class, both are stably conjugate to  $t'$ , and  $\lambda \in X_*(T_{t_0})$  maps to  $\text{inv}(t_0, t'_0) \in X_*(T_{t_0})_\Gamma$ . Reversing the arguments from the beginning of this proof we see that this expression equals  $\langle f', T_{b, \mu}^{G \rightarrow J} g \rangle_J$ .  $\square$

This lemma hints at the possibility of extending the operator  $T_{b, \mu}^{G \rightarrow J}$  from the space of invariant functions on  $G(F)$  to the space of invariant distributions on  $G(F)$ . Recall that the space of distributions  $\text{Dist}(G(F), \Lambda)$  is the  $\Lambda$ -linear dual of  $C_c(G(F), \Lambda)$ . The subspace of invariant distributions  $\text{Dist}(G(F), \Lambda)^{G(F)}$  is the linear dual of  $C_c(G(F), \Lambda)_{G(F)}$ . Given an invariant distribution  $d$  on  $G(F)$  we would like to define  $T_{b, \mu}^{G \rightarrow J} d$  by the formula

$$\langle T_{b, \mu}^{G \rightarrow J} d, \phi_{J_b} \rangle = \langle d, T_{b, \mu}^{J \rightarrow G} \phi_{J_b} \rangle \quad (3.3.2)$$

for any test function  $\phi \in C_c(J_b(F), \Lambda)$ , where now  $\langle -, - \rangle$  denotes the pairing of canonical duality between  $\text{Dist}(G(F), \Lambda)^{G(F)}$  and  $C_c(G(F), \Lambda)_{G(F)}$ , and

the corresponding pairing for  $J_b(F)$ . But at the moment there is no guarantee that  $T_{b,\mu}^{J \rightarrow G} \phi_{J_b} \in C(G(F)_{\text{sr}} // G(F), \Lambda)$  will be the orbital integral function for a test function on  $G(F)$ . This problem is reminiscent of the problem of endoscopic transfer of functions, where the (stable) orbital integrals of the transfer function are determined explicitly in terms of the transfer factors, but proving the existence of a test function with the prescribed orbital integrals is a highly non-trivial matter. We shall discuss this problem in the following section, where we prove that  $T_{b,\mu}^{J \rightarrow G} \phi_{J_b}$  is indeed the orbital integral of a compactly supported test function. Then we can use (3.3.2) to define an operator

$$T_{b,\mu}^{G \rightarrow J} : \text{Dist}(G(F), \Lambda)^{G(F)} \rightarrow \text{Dist}(J_b(F), \Lambda)^{J_b(F)}$$

that extends the operator

$$T_{b,\mu}^{G \rightarrow J} : C(G(F)_{\text{sr}} // G(F), \Lambda) \rightarrow C(J_b(F)_{\text{sr}} // J_b(F), \Lambda)$$

defined in Definition 3.2.7.

### 3.4 Transfer of conjugation-invariant distributions from $G$ to $J_b$

Let  $b \in G(\check{F})$  and let  $\mu$  be a cocharacter of  $G$ , such that  $[b]$  is the unique basic class in  $B(G, \mu)$ . In this section we upgrade the operator  $T_{b,\mu}$  in Definition 3.2.7 so that it transfers *distributions* rather than functions. This does not follow formally from the picture in (3.2.2). Indeed it requires one extra ingredient, a “lifting” of  $\text{Rel}_{b,\mu}$  to a topological space which lives over  $G(F) \times J_b(F)$ , rather than over the respective sets of conjugacy classes. This upgrade comes at the cost of restricting our attention to the elliptic elements.

**Definition 3.4.1.** Define a set  $\text{Gr}_{\leq \mu, \text{ell}}^1$  as the set of pairs  $(T, \lambda)$ , where  $T \subset G$  is an elliptic rational torus, and  $\lambda \in X_*(T)_{\leq \mu}$ . We give this the weakest topology so that the conjugation action of  $G(F)$  is continuous; thus each pair  $(T, \lambda)$  has a neighborhood isomorphic to  $K/(K \cap T)$ , where  $K \subset G(F)$  is sufficiently small.

We define  $\text{Gr}_{\leq \mu, \text{ell}}^b$  similarly, starting from the group  $J_b$ .

The Gr here stands for Grassmannian; later we will see that  $\text{Gr}_{\leq \mu, \text{ell}}^1$  and  $\text{Gr}_{\leq \mu, \text{ell}}^b$  are the loci of elliptic fixed points in the  $B_{\text{dR}}$ -affine Grassmannian.

**Theorem 3.4.2.** *There exists a locally profinite set  $\text{Sht}_{b,\mu,\text{ell}}$  carrying an action of  $G(F) \times J_b(F)$ , and a diagram*

$$\begin{array}{ccc} & \text{Sht}_{b,\mu,\text{ell}} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \text{Gr}_{\leq \mu,\text{ell}}^b & & \text{Gr}_{\leq \mu,\text{ell}}^1 \end{array}$$

*satisfying the following properties.*

1. *The map  $\pi_1$  is a  $G(F)$ -torsor in topological spaces, and is  $J_b(F)$ -equivariant.*
2. *The map  $\pi_2$  is a  $J_b(F)$ -torsor in topological spaces, and is  $G(F)$ -equivariant.*
3. *Let  $x \in \text{Sht}_{b,\mu,\text{ell}}$ , let  $h_1(x) = (T_x, \lambda_x)$ , and let  $h_2(x) = (T'_x, \lambda'_x)$ . Then there exists  $y \in G(\check{F})$  which conjugates  $(T_x, \lambda_x)$  onto  $(T'_x, \lambda'_x)$ , and then the class  $[y^{-1}by^\sigma] \in B(T_x)$  equals the image of  $\lambda_x$  under  $X_*(T_x) \rightarrow B(T_x)$ . Conjugation by  $y$  induces an isomorphism of  $F$ -rational tori  $\iota_x: T_x \xrightarrow{\sim} T'_x$ ; then for all  $t \in T_x(F)$ , we have  $(t, \iota_x(t)) \cdot x = x$ .*

**Remark 3.4.3.** The first two properties alone imply the following for  $x \in \text{Sht}_{b,\mu,\text{ell}}$  with  $h_1(x) = (T_x, \lambda_x)$ : given  $t \in T_x(F)$ , we have  $h_1((t, 1) \cdot x) = t \cdot h_1(x) = h_1(x)$ , and therefore (since  $h_1$  is an  $J_b(F)$ -torsor) there exists a unique  $t' \in J_b(F)$  such that  $(t, t') \cdot x = x$ . Then  $t \mapsto t'$  is a homomorphism  $T_x(F) \rightarrow J_b(F)$ . Property (3) above asserts that this map factors through an isomorphism  $T_x(F) \xrightarrow{\sim} T'_x(F)$ , which is  $\text{Ad } y$  for some  $y \in G(\check{F})$ , and (finally) that for  $t \in T_x(F)_{\text{sr}}$ , the invariant  $\text{inv}[b](t, t') \in B(T_x)$  agrees with the image of  $\lambda$  under  $X_*(T_x) \rightarrow B(T_x)$ .

We will prove Theorem 3.4.2 in §6.4; it seems to require  $p$ -adic Hodge theory. The space  $\text{Sht}_{b,\mu,\text{ell}}$  is the locus in the local shtuka space  $\text{Sht}_{b,\mu}$  consisting of points which are fixed by elements of  $G(F)_{\text{ell}} \times J_b(F)_{\text{ell}}$ . The example of  $\text{GL}_n$  below is instructive.

**Example 3.4.4** (The Lubin-Tate case.). Suppose  $F = \mathbf{Q}_p$  and  $G = \text{GL}_n(\mathbf{Q}_p)$ . Let  $\mu$  be the (minuscule) cocharacter defined by  $\mu(t) = \text{diag}(t, 1, \dots, 1)$ , and let  $b \in G(\check{\mathbf{Q}}_p)$  be a basic element such that  $[b] \in B(G, \mu)$ . Then  $b$  is the isocrystal associated to a  $p$ -divisible group  $H_0/\overline{\mathbf{F}}_p$  of height  $n$  and dimension 1. This means that Frobenius acts on the Dieudonné module  $M(H_0) \cong \check{\mathbf{Q}}_p^n$  by  $b$ . We can identify  $J_b(F)$  with the group of automorphisms of  $H_0$  (in the

isogeny category). It is the group of units in the central division algebra over  $\mathbf{Q}_p$  of invariant  $1/n$ .

Let  $\mathcal{M}_{b,\mu}$  be Lubin-Tate space at infinite level, considered as a perfectoid space over an algebraically closed perfectoid field  $C$  with residue field  $\overline{\mathbf{F}}_p$ . Its  $C$ -points classify equivalence classes of triples  $(H, \alpha, \iota)$ , where  $H/\mathcal{O}_C$  is a  $p$ -divisible group,  $\alpha: \mathbf{Q}_p^n \rightarrow VH$  is a trivialization of the rational Tate module, and  $\iota: H_0 \rightarrow H \otimes_{\mathcal{O}_C} \overline{\mathbf{F}}_p$  is an isomorphism in the isogeny category. (Equivalence between two such triples is a quasi-isogeny between  $p$ -divisible groups which makes both diagrams commute.) Then  $\mathcal{M}_{b,\mu}$  admits an action of  $G(F) \times J_b(F)$ , via composition with  $\alpha$  and  $\iota$ , respectively.

There are two period maps out of  $\mathcal{M}_{b,\mu}$ : the Gross-Hopkins period map  $\pi_1: \mathcal{M}_{b,\mu} \rightarrow \mathbf{P}(M(H_0)) \cong \mathbf{P}^{n-1}$ , which is a  $J_b(\mathbf{Q}_p)$ -equivariant  $G(\mathbf{Q}_p)$ -torsor, and the Hodge-Tate period map  $\pi_2: \mathcal{M}_{b,\mu} \rightarrow \Omega^{n-1}$ , which is a  $G(\mathbf{Q}_p)$ -equivariant  $J_b(\mathbf{Q}_p)$ -torsor. Here,  $\Omega^{n-1}$  is Drinfeld's half space, equal to the complement in  $\mathbf{P}^{n-1}$  of all  $\mathbf{Q}_p$ -rational hyperplanes.

We now consider the fixed points in this situation under the various actions. Projective space  $\mathbf{P}^{n-1}$  is isomorphic to the quotient  $G/P_\mu$ , where  $P_\mu$  is the parabolic subgroup associated with  $\mu$ ; from this we deduce that the set of points of  $\mathbf{P}^{n-1}$  which are fixed by a maximal torus  $T$  is in natural correspondence with the set of cocharacters in  $X_*(T)$  which are conjugate to  $\mu$ . This set has cardinality  $n$ .

Let  $\mathbf{P}_{\text{ell}}^{n-1}$  be the set of  $C$ -points of  $\mathbf{P}^{n-1}$  which are fixed by an elliptic torus in  $J_b(\mathbf{Q}_p)$ . Now, any maximal torus in  $J_b(\mathbf{Q}_p)$  is elliptic. We see that  $\mathbf{P}_{\text{ell}}^{n-1} \cong \text{Gr}_{\mu,\text{ell}}^b$  is the set of pairs  $(T', \lambda')$ , where  $T' \subset J_b$  is a maximal torus, and  $\lambda' \in X_*(T')$  is conjugate to  $\mu$ .

Similarly, define  $\Omega_{\text{ell}}^{n-1}$  to be the set of  $C$ -points of  $\Omega_{\text{ell}}^{n-1}$  which are fixed by an elliptic torus in  $G(\mathbf{Q}_p)$ . Crucially, the fixed points of an elliptic torus on  $\mathbf{P}^{n-1}$  must lie in  $\Omega_{\text{ell}}^{n-1}$ . This means that  $\Omega_{\text{ell}}^{n-1} \cong \text{Gr}_{\mu,\text{ell}}^1$ .

Finally, let  $\mathcal{M}_{b,\mu,\text{ell}}$  be the set of  $C$ -points  $(H, \alpha, \iota)$  of  $\mathcal{M}_{b,\mu}$  which are fixed by an element  $(g, g') \in G(F)_{\text{ell}} \times J_b(F)_{\text{ell}}$ . This set plays the role of  $\text{Sht}_{b,\mu,\text{ell}}$  in Theorem 3.4.2. The condition that  $(H, \alpha, \iota)$  is fixed by  $(g, g')$  means that there exists an automorphism  $\gamma$  of  $H$  (in the isogeny category) which corresponds to  $g$  on the Tate module and  $g'$  on the special fiber, respectively. We verify now that  $g$  and  $g'$  are related. Let  $B_{\text{cris}} = B_{\text{cris}}(C)$  be the crystalline period ring. There are isomorphisms

$$B_{\text{cris}}^n \rightarrow VH \otimes_{\mathbf{Q}_p} B_{\text{cris}} \rightarrow M(H_0) \otimes B_{\text{cris}},$$

where the first map is induced from  $\alpha$ , and the second map comes from the comparison isomorphism between étale and crystalline cohomology of

$H$  (using  $\iota$  to identify the latter with  $M(H_0)$ ). The composite map carries the action of  $g$  onto that of  $g'$ , which is to say that  $g$  and  $g'$  become conjugate over  $B_{\text{cris}}$ . This implies that  $g$  and  $g'$  are conjugate over an algebraic closure of  $\check{\mathbf{Q}}_p$ , and therefore by Lemma 3.2.1 they are related.

We define ‘‘inertia’’ versions of the sets  $\text{Gr}_{\leq \mu, \text{ell}}^1$ ,  $\text{Gr}_{\leq \mu, \text{ell}}^b$ , and  $\text{Sht}_{b, \mu, \text{ell}}$  by

$$\begin{aligned} I \text{Gr}_{\leq \mu, \text{ell}}^1 &= \left\{ (g, \lambda) \mid g \in G(F)_{\text{ell}}, \lambda \in X_*(T_g)_{\leq \mu} \right\} \\ I \text{Gr}_{\leq \mu, \text{ell}}^b &= \left\{ (g', \lambda') \mid g' \in J_b(F)_{\text{ell}}, \lambda' \in X_*(T_{g'})_{\leq \mu} \right\} \\ I \text{Sht}_{b, \mu, \text{ell}} &= \left\{ (g, g', x) \mid x \in \text{Sht}_{b, \mu, \text{ell}}, g \in T_x(F)_{\text{ell}}, g' \in T'_x(F)_{\text{ell}}, (g, g') \cdot x = x \right\}, \end{aligned}$$

where we have put  $T_g = \text{Cent}(g, G)$  and  $T_{g'} = \text{Cent}(g', J_b)$ .

These sets fit into a diagram

$$\begin{array}{ccc} & I \text{Sht}_{b, \mu, \text{ell}} & \\ & \swarrow I\pi_1 & \searrow I\pi_2 \\ I \text{Gr}_{\leq \mu, \text{ell}}^b & & I \text{Gr}_{\leq \mu, \text{ell}}^1 \\ p_1 \downarrow & & \downarrow p_2 \\ J_b(F)_{\text{ell}} & & G(F)_{\text{ell}} \end{array} \quad (3.4.1)$$

in which  $I\pi_1$  is a  $G(F)$ -torsor,  $I\pi_2$  is a  $J_b(F)$ -torsor, and the maps  $p_1$  and  $p_2$  are finite étale surjections. Note that the quotient  $I \text{Sht}_{b, \mu, \text{ell}} / (G(F) \times J_b(F))$  is isomorphic to the space  $\text{Rel}_{b, \leq \mu}$  defined earlier.

We are going to use the diagram in (3.4.1) to transfer conjugation-invariant distributions from  $J_b(F)_{\text{ell}}$  to  $G_b(F)_{\text{ell}}$ , with the help of the following lemma.

**Lemma 3.4.5.** *Let  $H$  be a locally pro- $p$  group, let  $\Lambda$  be a commutative ring in which  $p$  is invertible, and let  $\nu$  be a  $\Lambda$ -valued Haar measure on  $H$ . Let  $h: \tilde{S} \rightarrow S$  be an  $H$ -torsor in locally profinite sets. The integration-along-fibers map  $C_c(\tilde{S}, \Lambda) \rightarrow C_c(S, \Lambda)$  induces an isomorphism of  $C(S, \Lambda)$ -modules*

$$h_!: C_c(\tilde{S}, \Lambda)_H \rightarrow C_c(S, \Lambda)$$

and, dually, an isomorphism of  $C(S, \Lambda)$ -modules

$$\text{Dist}(\tilde{S}, \Lambda)^H \rightarrow \text{Dist}(S, \Lambda).$$

*Proof.* The  $C(S, \Lambda)$ -modules  $C_c(S, \Lambda)$  and  $C_c(\tilde{S}, \Lambda)$  are smooth in the sense of Definition B.2.1. Therefore Lemma B.2.5 reduces the problem to the case where  $S$  is profinite and the torsor  $\tilde{S} = S \times H$  is split. Then  $C_c(\tilde{S}, \Lambda) = C_c(S, \Lambda) \otimes_{\Lambda} C_c(H, \Lambda)$  and  $C_c(\tilde{S}, \Lambda)_H = C_c(S, \Lambda) \otimes_{\Lambda} C_c(H, \Lambda)_H$ . Now, since  $H$  is a pro- $p$  group and  $p$  is invertible in  $\Lambda$ , the  $\Lambda$ -valued Haar measure on  $H$  is unique. This means exactly that the integration map  $C_c(H, \Lambda)_H \rightarrow \Lambda$  is an isomorphism, so that  $C_c(\tilde{S}, \Lambda)_H \cong C_c(S, \Lambda)$ .  $\square$

Referring to the diagram in (3.4.1), we define a  $\Lambda$ -linear map

$$C_c(G(F)_{\text{ell}}, \Lambda)_{G(F)} \rightarrow C_c(J_b(F)_{\text{ell}}, \Lambda)_{J_b(F)} \quad (3.4.2)$$

by

$$\begin{array}{ccc} C_c(G(F)_{\text{ell}}, \Lambda)_{G(F)} & \xrightarrow{p_2^*} & C_c(I \text{Gr}_{\leq \mu, \text{ell}}^1, \Lambda)_{G(F)} \\ & \xrightarrow{I\pi_{2!}^{-1}} & C_c(I \text{Sht}_{b, \mu, \text{ell}}, \Lambda)_{G(F) \times J_b(F)} \\ & \xrightarrow{I\pi_{1!}} & C_c(I \text{Gr}_{\leq \mu, \text{ell}}^b, \Lambda)_{J_b(F)} \\ & \xrightarrow{\times K_{\mu}} & C_c(I \text{Gr}_{\leq \mu, \text{ell}}^b, \Lambda)_{J_b(F)} \\ & \xrightarrow{p_{1*}} & C_c(J_b(F)_{\text{ell}}, \Lambda)_{J_b(F)}, \end{array}$$

where  $p_2^*$  means pullback,  $p_{1*}$  means pushforward (*i.e.*, sum over fibers), the maps  $I\pi_{i!}$  are as in Lemma 3.4.5, and finally  $K_{\lambda} \in C(I \text{Gr}_{\leq \mu, \text{ell}}^b, \Lambda)^{J_b(F)}$  is the function  $(g, \lambda') \mapsto \dim r_{\mu}[\lambda']$ .

**Proposition 3.4.6.** *Let  $\phi \in C_c(G(F)_{\text{ell}}, \Lambda)$ , and let  $\phi' \in C_c(J_b(F)_{\text{ell}}, \Lambda)$  be any lift of the image of  $\phi$  under the map in (3.4.2). Then*

$$\phi'_J = T_{b, \mu}^{G \rightarrow J} \phi_G.$$

*Proof.* We write down an integral expression for  $\phi'$ , as follows.

Let  $\tilde{\phi} \in C_c(I \text{Sht}_{b, \mu, \text{ell}}, \Lambda)$  be a lift of  $I\pi_{2!}^{-1} p_2^* \phi$ . Then  $\tilde{\phi}$  has the property that

$$\int_{h' \in J_b(F)} \tilde{\phi}((1, h') \cdot (g, g', x)) dh' = \phi(g)$$

for all  $(g, g', x) \in I \text{Sht}_{b, \mu, \text{ell}}$ . Then for  $g' \in J_b(F)_{\text{ell}}$  with centralizer  $T'$ , we have

$$\phi'(g') = \sum_{\lambda' \in X_*(T')_{\leq \mu}} \dim r_{\mu}[\lambda'] \int_{h \in G(F)} \tilde{\phi}((h, 1) \cdot y_{\lambda'}),$$

where for each  $\lambda' \in X_*(T')$  we have chosen a lift  $y_{\lambda'} = (g_{\lambda'}, g', x_{\lambda'})$  of  $(g', \lambda')$ , so that  $\iota_{x_{\lambda'}}(g_{\lambda'}) = g'$ .

We now analyze the orbital integral of  $\phi'$ . For  $g' \in J_b(F)_{\text{ell}}$  as above, it is

$$\begin{aligned} \phi'_J(g') &= \int_{h' \in J_b(F)/T(F)} \phi'(h'g'(h')^{-1}) dh' \\ &= \sum_{\lambda' \in X_*(T')_{\leq \mu}} \dim r_{\mu}[\lambda'] \int_{h' \in J_b(F)/T(F)} \int_{h \in G(F)} \tilde{\phi}((h, h') \cdot y_{\lambda'}) dh dh' \end{aligned}$$

We rewrite the inner integral as a nested integral:

$$\begin{aligned} \phi'_J(g') &= \sum_{\lambda'} \dim r_{\mu}[\lambda'] \int_{h' \in J_b(F)/T'(F)} \int_{h \in G(F)/T(F)} \int_{t \in T(F)} \tilde{\phi}((ht^{-1}, h') \cdot y_{\lambda'}) dt dh dh' \\ &= \sum_{\lambda'} \dim r_{\mu}[\lambda'] \int_{h' \in J_b(F)/T'(F)} \int_{h \in G(F)/T(F)} \int_{t' \in T'(F)} \tilde{\phi}((h, h't') \cdot y_{\lambda'}) dt' dh dh' \end{aligned}$$

where we have used the fact that the isomorphism  $t \mapsto t'$  between  $T(F)$  and  $T'(F)$  satisfies  $(t, t') \cdot y_{\lambda'} = y_{\lambda'}$ . We can now exchange the order of the first two integrals to obtain

$$\begin{aligned} \phi'_J(g') &= \sum_{\lambda'} \dim r_{\mu}[\lambda'] \int_{h \in G(F)/T(F)} \int_{h' \in J_b(F)} \tilde{\phi}((h, h') \cdot y_{\lambda'}) dh' dh \\ &= \sum_{\lambda'} \dim r_{\mu}[\lambda'] \int_{h \in G(F)/T(F)} \phi(hg_{\lambda'}h^{-1}) dh \\ &= \sum_{\lambda'} \dim r_{\mu}[\lambda'] \phi_G(g) \\ &= [T_{b,\mu}^{G \rightarrow J} f](g'). \end{aligned}$$

□

**Definition 3.4.7.** Let

$$T_{b,\mu}^{J \rightarrow G}: \text{Dist}(J_b(F), \Lambda)^{J_b(F)} \rightarrow \text{Dist}(G(F), \Lambda)^{G(F)}$$

be the  $\Lambda$ -linear dual of the map in (3.4.2).

Then  $T_{b,\mu}^{J \rightarrow G}$  extends the transfer of functions defined in Definition 3.2.7. Indeed, suppose  $f \in C(J_b(F)_{\text{ell}}, \Lambda)^{J_b(F)}$  is a conjugation-invariant function, and  $f dg' \in \text{Dist}(J_b(F)_{\text{ell}}, \Lambda)$  is the corresponding distribution. Then for

$\phi \in C_c(G(F)_{\text{ell}}, \Lambda)$ , with transfer  $\phi' \in C_c(J_b(F)_{\text{ell}}, \Lambda)$  as in Proposition 3.4.6, we have

$$\begin{aligned}
\int_{g \in G(F)_{\text{ell}}} \phi(g) T_{b,\mu}^{J \rightarrow G}(f dg') &= \int_{g' \in J_b(F)_{\text{ell}}} f(g') \phi'(g') dg' \\
&= \langle f, \phi'_{J_b} \rangle_{J_b} \\
&= \langle f, T_{b,\mu}^{G \rightarrow J} \phi_G \rangle_{J_b} \\
&= \langle T_{b,\mu}^{J \rightarrow G} f, \phi_G \rangle_{J_b} \\
&= \int_{g \in G(F)_{\text{ell}}} \phi(g) (T_{b,\mu}^{J \rightarrow G} f) dg',
\end{aligned}$$

so that

$$T_{b,\mu}^{J \rightarrow G}(f dg') = T_{b,\mu}^{J \rightarrow G}(f) dg'$$

as desired.

### 3.5 Relation to Kottwitz' conjecture

The goal of the rest of the paper is to prove the following theorem.

**Theorem 3.5.1.** *Let  $\phi: W_F \rightarrow {}^L G$  be a discrete Langlands parameter with coefficients in  $\overline{\mathbf{Q}}_\ell$ , and let  $\rho \in \Pi_\phi(J_b)$  be a element in the corresponding  $L$ -packet. Assume that  $\rho$  contains an  $J_b(F)$ -invariant  $\overline{\mathbf{Z}}_\ell$ -lattice. Let  $(\text{tr } \rho)_{\text{ell}} \in \text{Dist}(J_b(F)_{\text{ell}}, \overline{\mathbf{Q}}_\ell)$  be the trace distribution of  $J_b(F)_{\text{ell}}$  on  $\rho$ . Then the  $H^i(G, b, \mu)[\rho]$  are admissible representations of  $G(F)$ , and the trace distribution of  $G(F)_{\text{ell}}$  on the Euler characteristic  $H^*(G, b, \mu)[\rho]$  equals  $T_{b,\mu}^{J \rightarrow G}(\text{tr } \rho)_{\text{ell}}$ .*

Combining Theorem 3.5.1 with Theorem 3.2.9 shows that the trace distribution of  $H^*(G, b, \mu)[\rho]$  on  $G(F)_{\text{ell}}$  agrees with the trace distribution of

$$(-1)^d \sum_{\pi \in \Pi_\phi(J_b)} \dim \text{Hom}_{S_\phi}(\delta_{\pi,\rho}, r_\mu).$$

This implies Theorem 1.0.2.

The proof of Theorem 3.5.1 proceeds by “geometrizing” the diagram in (3.4.1). That diagram offered a way of transferring conjugation-invariant distributions from  $J_b(F)_{\text{ell}}$  to  $G(F)_{\text{ell}}$ . By “geometrizing”, we mean that we will lift this transfer of distributions to a transfer of sheaves on stacks. The groundwork for this geometrization will be laid in Section 4.

### 3.6 Proof of Theorem 3.2.9

We now give the proof of Theorem 3.2.9. We will use the notation and results of §A.1.

We are given a discrete Weil parameter  $\phi$ , a representation  $\rho \in \Pi_\phi(J_b)$  in its  $L$ -packet, and an element  $g \in G(F)_{\text{sr}}$ . We assume that  $g$  is related to an element of  $J_b(F)$ . This means there exists a triple in  $\text{Rel}_b$  of the form  $(g, j, \lambda)$ . For the moment we fix such a triple  $(g, j, \lambda)$ .

Let  $s \in S_\phi$  be a semi-simple element, and let  $\dot{s} \in S_\phi^+$  be a lift of it. Then we have the refined endoscopic datum  $\dot{\mathfrak{e}} = (H, \mathcal{H}, \dot{s}, \eta)$  defined in (A.1.1); we choose as in that section a  $z$ -pair  $\mathfrak{z} = (H_1, \eta_1)$ . Then

$$\begin{aligned}
& e(J_b) \sum_{\rho' \in \Pi_\phi(J_b)} \text{tr } \tau_{z, \mathfrak{w}, \pi'}(\dot{s}) \Theta_{\rho'}(j) \\
\stackrel{\text{(A.1.2)}}{=} & \sum_{h_1 \in H_1(F)/\text{st}} \Delta(h_1, j) S_{\Theta_{\phi^s}}(h_1) \\
\stackrel{\text{(A.1.4)}}{=} & \sum_{h_1 \in H_1(F)/\text{st}} \Delta(h_1, g) \langle \text{inv}[b](g, j), s_{h,j} \rangle S_{\Theta_{\phi^s}}(h_1) \\
= & \sum_{h_1 \in H_1(F)/\text{st}} \Delta(h_1, g) \lambda(s_{h,g}) S_{\Theta_{\phi^s}}(h_1).
\end{aligned}$$

We now multiply this expression by the kernel function  $\dim r_\mu[\lambda]$ , and then sum over all  $J_b(F)$ -conjugacy classes of elements  $j \in J_b(F)$  and all  $\lambda \in X_*(T_g)$  such that  $(g, j, \lambda)$  lies in  $\text{Rel}_b$ . We obtain

$$\begin{aligned}
& e(J_b) \sum_{(j, \lambda)} \sum_{\rho' \in \Pi_\phi(G)} \text{tr } \tau_{z, \mathfrak{w}, \rho'}(\dot{s}) \Theta_{\rho'}(j) \dim r_\mu[\lambda] \\
= & \sum_{h_1 \in H_1(F)/\text{st}} \Delta(h_1, g) S_{\Theta_{\phi^s}}(h_1) \sum_{(j, \lambda)} \lambda(s_{h,g}) \dim r_\mu[\lambda] \\
\stackrel{(*)}{=} & \sum_{h_1 \in H_1(F)/\text{st}} \Delta(h_1, g) S_{\Theta_{\phi^s}}(h_1) \text{tr } r_\mu(s_{h,j}) \\
\stackrel{(**)}{=} & \text{tr } r_\mu(s^\natural) \sum_{h_1 \in H_1(F)/\text{st}} \Delta(h_1, g) S_{\Theta_{\phi^s}}(h_1) \\
\stackrel{\text{(A.1.2)}}{=} & \text{tr } r_\mu(s^\natural) e(G) \sum_{\pi \in \Pi_\phi(G)} \text{tr } \tau_{z, \mathfrak{w}, \pi}(\dot{s}) \Theta_\pi(g).
\end{aligned}$$

We justify (\*\*): Let  $T \subset G$  be the centralizer of  $g$ . The image of  $s_{h,j}$  under any admissible embedding  $\widehat{T} \rightarrow \widehat{G}$  is conjugate to  $s^\natural$  in  $\widehat{G}$  and  $\text{tr } r_\mu$

is conjugation-invariant. Recall here that  $s^\natural \in S_\phi$  is the image of  $\dot{s}$  under (2.3.2).

We justify (\*):  $\lambda \in X_*(T)$  determines the  $J_b(F)$ -conjugacy class of  $j$ , since  $\text{inv}[b](g, j) \in B(T)$  determines it. Therefore the sum over  $(j, \lambda)$  is in reality a sum only over  $\lambda$ . There exists  $j \in J_b(F)$  with  $\kappa(\text{inv}[b](g, j))$  being the image of  $\lambda$  in  $X_*(T)_\Gamma$  if and only if the image of  $\lambda$  under  $X_*(T) \rightarrow X_*(T)_\Gamma \rightarrow \pi_1(G)_\Gamma$  equals  $\kappa(b)$ . Since the image of  $\mu$  in  $\pi_1(G)_\Gamma$  also equals  $\kappa(b)$ , the sum over  $(j, \lambda)$  is in fact the sum over  $\lambda \in X_*(T)$  having the same image as  $\mu$  in  $\pi_1(G)_\Gamma$ . In terms of the dual torus  $\hat{T}$  this is the sum over  $\lambda \in X^*(\hat{T})$  whose restriction to  $Z(\hat{G})^\Gamma$  equals that of  $\mu$ . Since for  $\lambda$  not satisfying this condition the number  $\dim r_\mu[\lambda]$  is zero, we may extend the sum to be over all  $\lambda \in X_*(T) = X^*(\hat{T})$ .

Multiply both sides of the above equation by  $\text{tr } \check{\tau}_{z, \mathfrak{w}, \rho}(\dot{s})$ . As functions of  $\dot{s} \in S_\phi^+$ , both sides then become invariant under  $Z(\hat{G})^+$  and thus become functions of the finite quotient  $\bar{S}_\phi = S_\phi^+ / Z(\hat{G})^+ = S_\phi / Z(\hat{G})^\Gamma$ . Now apply  $|\bar{S}_\phi|^{-1} \sum_{\bar{s} \in \bar{S}_\phi}$  to both sides to obtain an equality between

$$|\bar{S}_\phi|^{-1} e(J_b) \sum_{\bar{s} \in \bar{S}_\phi} \sum_{(j, \lambda)} \sum_{\rho' \in \Pi_\phi(J_b)} \text{tr } \check{\tau}_{z, \mathfrak{w}, \rho}(\dot{s}) \text{tr } \tau_{z, \mathfrak{w}, \rho'}(\dot{s}) \Theta_{\rho'}(j) \dim r_\mu[\lambda] \quad (3.6.1)$$

and

$$|\bar{S}_\phi|^{-1} e(G) \sum_{\bar{s} \in \bar{S}_\phi} \text{tr } r_\mu(s^\natural) \sum_{\pi \in \Pi_\phi(G)} \text{tr } \check{\tau}_{z, \mathfrak{w}, \rho}(\dot{s}) \text{tr } \tau_{z, \mathfrak{w}, \pi}(\dot{s}) \Theta_\pi(g), \quad (3.6.2)$$

where in both formulas  $\dot{s}$  is an arbitrary lift of  $\bar{s}$ . Executing the sum in (3.6.1) over  $\bar{s}$  in the first of the two expressions gives

$$e(J_b) \sum_{(j, \lambda)} \Theta_\rho(j) \dim r_\mu[\lambda] = e(J_b) [T_{b, \mu} \Theta_\rho](g).$$

To treat (3.6.2) note that  $\check{\tau}_{z, \mathfrak{w}, \rho} \otimes \tau_{z, \mathfrak{w}, \pi}(\dot{s}) = \check{\delta}_{\pi, \rho}(s^\natural)$ . Furthermore, the composition of the map (2.3.2) with the natural projection  $S_\phi \rightarrow S_\phi / Z(\hat{G})^\Gamma$  is equal to the natural projection  $S_\phi^+ \rightarrow S_\phi^+ / Z(\hat{G})^+ = S_\phi / Z(\hat{G})^\Gamma = \bar{S}_\phi$ . Thus  $s^\natural$  is simply a lift of  $\bar{s}$  to  $S_\phi$ . We find that (3.6.2) equals

$$e(G) |\bar{S}_\phi|^{-1} \sum_{\bar{s} \in \bar{S}_\phi} \text{tr } r_\mu(s^\natural) \text{tr } \check{\delta}_{\pi, \rho}(s^\natural) = e(G) \dim \text{Hom}_{S_\phi}(\delta_{\pi, \rho}, r_\mu).$$

We have now reduced Theorem 3.2.9 to the identity

$$e(G) e(J_b) = (-1)^{(2\rho_G, \mu)}, \quad (3.6.3)$$

where  $\rho_G$  is the sum of the positive roots. This is Lemma A.2.1.

## 4 Geometric preparations: Trace distributions

The goal of this section is to build up some machinery related to the Lefschetz fixed-point formula. Classically, the setting of the Lefschetz fixed-point formula is a compact topological space  $X$  and a continuous map  $f: X \rightarrow X$ . If the fixed points of  $f$  are discrete, then the Euler characteristic  $\mathrm{tr} f|H^*(X, \mathbf{Q})$  equals a sum of local terms  $\mathrm{loc}_x(f)$ , as  $x$  runs over the fixed point locus.

There are myriad variations on the theme. For instance  $X$  may be a proper variety over an algebraically closed field  $k$ , which comes equipped with an object  $\mathcal{F}$  of  $D_{\mathrm{\acute{e}t}}(X, \overline{\mathbf{Q}}_\ell)$ , its derived category of étale  $\overline{\mathbf{Q}}_\ell$ -sheaves. Suppose  $f: X \rightarrow X$  is an endomorphism. In order for  $f$  to induce an operator on  $H^*(X, \mathcal{F})$ , there needs to be an extra datum<sup>2</sup>, namely a morphism  $Rf_!\mathcal{F} \rightarrow \mathcal{F}$ , which expresses the equivariance of  $\mathcal{F}$  with respect to  $f$ . Then if  $p: X \rightarrow \mathrm{Spec} k$  is the structure map, the operator on cohomology is  $Rp_!\mathcal{F} = Rp_!f_!\mathcal{F} \rightarrow Rp_!\mathcal{F}$ . The Lefschetz-Verdier formula [SGA77], [Var07] expresses  $\mathrm{tr} f|H^*(X, \mathcal{F})$  in terms of data living on the fixed point locus of  $f$ . In particular, at isolated points  $x$  there are local terms  $\mathrm{loc}_x(f, \mathcal{F})$ . To apply the formula, we need to assume that  $\mathcal{F}$  satisfies a suitable finiteness hypothesis (constructible of finite Tor dimension). From there, it is a matter of applying Grothendieck’s six functor formalism.

As a further variation, suppose  $G$  is a smooth algebraic group acting on the proper variety  $X$ . Let  $\mathcal{F}$  be a  $G$ -equivariant object of the derived category of étale  $\overline{\mathbf{Q}}_\ell$ -sheaves on  $X$ . Then  $H^*(X, \mathcal{F})$  becomes a representation of  $G$  that factors through  $\pi_0(G)$ , and we might hope to have a formula for its trace in terms of fixed points. For this  $\mathcal{F}$  needs to be equipped with a descent datum to the stacky quotient  $[X/G]$ . We will work out a version of the Lefschetz formula in this setting. On one side, we have the trace character of  $G$  on  $H^*(X, \mathcal{F})$ , expressed as a function on  $[G//G]$ , the stack of conjugacy classes of  $G$ . On the other side, there will appear the analogue of the local terms  $\mathrm{loc}_x(f, \mathcal{F})$ ; this will be a function on the inertia stack  $\mathrm{In}([X/G])$ .

For our application to the Kottwitz conjecture, we need a final variation

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<sup>2</sup>This is a special case of the notion of a cohomological correspondence  $c_1^*\mathcal{F} \rightarrow c_2^!\mathcal{F}$  attached to  $(c_1, c_2): C \rightarrow X \times X$ . Here we have taken  $C = X$ ,  $c_1 = \mathrm{id}$  and  $c_2 = f$ . Note that if  $f$  is an automorphism, giving the equivariance is equivalent to giving a map  $\mathcal{F} \rightarrow f^*\mathcal{F}$ .

on the Lefschetz theme, where the objects live in the world of perfectoid spaces, diamonds, and v-stacks. Scholze [Sch17] defines for every v-stack  $\mathfrak{X}$  a triangulated category  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  (where  $\Lambda$  is a suitable coefficient ring, such as  $\mathbf{Z}/\ell^n\mathbf{Z}$ ) and establishes the six-functor formalism for a large class of morphisms between v-stacks. In particular, if  $f$  is a morphism between v-stacks which is compactifiable, representable in locally spatial diamonds, and of finite geometric transcendence degree, there are functors  $Rf_!$  and  $Rf^!$  which form an adjoint pair and obey the usual rules.

From there we will prove a general sort of Lefschetz formula, applying to sheaves on v-stacks, under the assumption that the sheaf  $\mathcal{F}$  is “strongly reflexive”.

In what follows we write  $f_*, f_!, f^!, \dots$  for morphisms between triangulated categories, where traditionally one would write  $Rf_*, Rf_!, Rf^!$ . We hope no confusion arises, as we do not really work at all in the non-derived category.

#### 4.1 The dualizing complex, and the space of distributions

Many of our definitions and results apply equally to the world of schemes and stacks as to the world of diamonds and v-stacks. Throughout, we use the symbol  $*$  to mean either  $\text{Spec } k$  for an algebraically closed field  $k$  (in the scheme setting), or  $\text{Spa } C$  for an algebraically closed perfectoid field  $C$  of residue characteristic  $p$  (in the diamond setting). All of the schemes and diamonds we mention will live over  $*$ .

Fix a coefficient ring  $\Lambda = \mathcal{O}_E/\ell^n$ , where  $\ell$  is a prime number, and  $E/\mathbf{Q}_\ell$  is a finite extension. We assume that  $\ell$  is not equal to the characteristic of  $k$  (in the scheme setting) or to  $p$  (in the diamond setting).

Let us say that a morphism  $f: X \rightarrow Y$  between schemes or diamonds is *nice* if it is locally of finite presentation (in the scheme setting), or if it is compactifiable, locally spatial, and of finite geometric transcendence degree (in the diamond setting). If  $Y = *$ , we simply say that  $X$  is nice.

Let  $X$  be a nice scheme or diamond, with structure morphism  $q: X \rightarrow *$ . We define the *dualizing complex* as

$$K_X = q^!\Lambda,$$

an object in the derived category  $D(X_{\text{ét}}, \Lambda)$ . Given an object  $\mathcal{F}$  of  $D(X_{\text{ét}}, \Lambda)$ , we define its *Verdier dual* as  $\mathbf{D}\mathcal{F} = \underline{\mathbf{R}}\text{Hom}(\mathcal{F}, K_X)$ . We record here the isomorphisms  $f^!\mathbf{D} \cong \mathbf{D}f^*$  and  $f_*\mathbf{D} \cong \mathbf{D}f_!$ , valid whenever  $f$  is nice.

**Example 4.1.1.** If  $X$  is a smooth scheme or rigid space of dimension  $d$ , then  $K_X \cong \Lambda(d)[2d]$ .

**Example 4.1.2.** Suppose we are in the diamond setting. Let  $S$  be a locally profinite topological space. We have the constant perfectoid space  $\underline{S}$ , which represents the functor  $T \mapsto C(|T|, S)$  on perfectoid spaces  $T$  over  $*$ . We may identify  $D(\underline{S}_{\text{ét}}, \Lambda)$  with the derived category of sheaves of  $\Lambda$ -modules on  $S$  with respect to the locally profinite topology, and hence by Lemma B.2.5 with the derived category of complete modules over the ring  $C(S, \Lambda)$  of continuous  $\Lambda$ -valued functions on  $S$ . Under this identification,  $K_{\underline{S}}$  is the  $\Lambda$ -linear dual of  $C_c(S, \Lambda)$ ; that is, it is the space of  $\Lambda$ -valued distributions on  $S$ , in degree 0.

It will be important to extend these definitions to stacks and v-stacks. Let us say that a morphism  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$  is *nice* if its pullback to any scheme or diamond mapping to  $\mathfrak{Y}$  is a nice morphism of schemes or diamonds.

We would like to define dualizing complexes on stacks. The trouble is that if  $q: \mathfrak{X} \rightarrow *$  is stacky, it is not clear how to define  $q_!$  and  $q^!$ . (It may be possible to set up an “enhanced six-functor formalism” as in [LZ], but we do not do this here.) The situation is better for Artin stacks or Artin v-stacks.

We fix here some definitions. An *Artin stack* is a stack  $\mathfrak{X}$  on the étale site over  $*$ , such that the diagonal map  $\Delta_{\mathfrak{X}}: \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$  is representable in algebraic spaces, and such that there exists a surjective smooth morphism  $X \rightarrow \mathfrak{X}$  from a nice scheme  $X$ . We let  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  be the derived category of sheaves of  $\Lambda$ -modules on the lisse-étale site of  $\mathfrak{X}$ . Analogously, an *Artin v-stack* is a stack  $\mathfrak{X}$  on the pro-étale site over  $*$ , such that the diagonal map  $\Delta_{\mathfrak{X}}$  is nice, and such that there exists a surjective smooth morphism  $X \rightarrow \mathfrak{X}$  from a nice diamond  $X$ . Here, *smooth* refers to the notion of  $\ell$ -cohomological smoothness between v-stacks [Sch17, §19]; strictly speaking, it depends on  $\ell$ . We let  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  be the triangulated category constructed in [Sch17]. In both cases, we call the morphism  $X \rightarrow \mathfrak{X}$  a *uniformization*. If in addition  $X$  is smooth, then  $\mathfrak{X}$  is a smooth Artin stack.

Recall from [Sch17, Proposition 23.12] that if  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$  is a smooth morphism between (v-)stacks, then  $f^!\Lambda$  is *invertible*, in the sense that it is (v-)locally isomorphic to a shift  $\Lambda[n]$ ; also, the natural transformation  $f^* \otimes f^!\Lambda \rightarrow f^!$  is an isomorphism. We write  $(f^!\Lambda)^{-1} = \underline{\text{RHom}}(f^!\Lambda, \Lambda)$  for the inverse object.

**Definition 4.1.3.** Let  $\mathfrak{X}$  be an Artin (v-)stack. A *dualizing complex* for  $\mathfrak{X}$  is an object  $K_{\mathfrak{X}}$  of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  which comes equipped with an isomorphism  $f^!K_{\mathfrak{X}} \cong K_U$  for every nice morphism  $f: U \rightarrow \mathfrak{X}$  from a nice scheme or diamond. It is required that this isomorphism is compatible with  $\mathfrak{X}$ -morphisms  $U \rightarrow V$  in the evident sense.

**Lemma 4.1.4.** *Let  $\mathfrak{X}$  be an Artin ( $v$ -)stack. The dualizing complex  $K_{\mathfrak{X}}$  is unique up to isomorphism, if it exists. It exists in the following cases:*

1. *The ( $v$ -)stack  $\mathfrak{X}$  is uniformized by a finite-type scheme or rigid-analytic space.*
2. *The ( $v$ -)stack  $\mathfrak{X}$  is a smooth Artin stack.*

*Proof.* Suppose that  $K_{\mathfrak{X}}$  is a dualizing sheaf. Let  $u: X \rightarrow \mathfrak{X}$  be a uniformization. Since  $K_X \cong u^!K_{\mathfrak{X}} \cong u^*K_{\mathfrak{X}} \otimes u^!\Lambda$ , and  $u^!\Lambda$  is invertible, we have  $u^*K_{\mathfrak{X}} \cong K_X \otimes (u^!\Lambda)^{-1}$ . Let  $\text{pr}_1, \text{pr}_2: X \times_{\mathfrak{X}} X \rightarrow X$  be the projection maps; note that each is smooth, and that  $X \times_{\mathfrak{X}} X$  is nice. Noting that invertibility is preserved under pullbacks, we have isomorphisms ( $i = 1, 2$ )

$$\begin{aligned} \text{pr}_i^*(K_X \otimes (u^!\Lambda)^{-1}) &\xrightarrow{\sim} \text{pr}_i^*K_X \otimes \text{pr}_i^*(u^!\Lambda)^{-1} \\ &\xrightarrow{\sim} \text{pr}_i^!K_X \otimes (\text{pr}_i^!u^!\Lambda)^{-1} \\ &\xrightarrow{\sim} K_{X \times_{\mathfrak{X}} X} \otimes ((u \circ \text{pr}_i)^!\Lambda)^{-1}. \end{aligned}$$

Since  $u \circ \text{pr}_1 = u \circ \text{pr}_2$ , we have an isomorphism  $\text{pr}_1^*(K_X \otimes (u^!\Lambda)^{-1}) \cong \text{pr}_2^*(K_X \otimes (u^!\Lambda)^{-1})$ . This satisfies the cocycle condition, and therefore constitutes a descent datum for  $K_X \otimes (u^!\Lambda)^{-1}$  along  $u$ .

This shows that  $K_{\mathfrak{X}}$  is unique if it exists. It also shows that  $K_{\mathfrak{X}}$  exists exactly when the above descent datum is effective. By the gluing lemma [BBD82, Theorem 3.2.4], the descent datum is effective so long as  $\text{Ext}_U^i(K_U, K_U) = H^i(\text{RHom}_U(K_U, K_U)) = 0$  for all  $i < 0$ , whenever  $U$  admits a smooth morphism to some  $n$ -fold product of  $X$  over  $\mathfrak{X}$ . Letting  $q_U: U \rightarrow *$  be the structure morphism, we have isomorphisms

$$\begin{aligned} \text{RHom}_U(K_U, K_U) &\cong \text{RHom}_U(K_U, q_U^!\Lambda) \\ &\cong \text{RHom}_{\Lambda}(q_U^!K_U, \Lambda) \\ &\cong \mathbf{D}q_U^!K_U \\ &\cong q_{U*}\mathbf{D}K_U \\ &= R\Gamma(U, \mathbf{D}\Lambda) \end{aligned}$$

There is a canonical morphism  $\Lambda \rightarrow \mathbf{D}\Lambda$ . If it is an isomorphism (which is to say, the constant sheaf is *reflexive* on  $U$ ), then  $\text{RHom}_U(K_U, K_U) \cong R\Gamma(U, \Lambda)$  has no cohomology in negative degree.

Assume that  $\mathfrak{X}$  is uniformized by a finite-type scheme or rigid-analytic space  $X$ . Then the  $n$ -fold product of  $X$  over  $\mathfrak{X}$  is of finite type, and then so is any  $U$  equipped with a smooth morphism to this  $n$ -fold product. The reflexivity of  $\Lambda$  follows from Example 4.2.4 in the scheme case and Proposition C.3.1 in the rigid analytic case.

Now suppose  $\mathfrak{X}$  is a smooth Artin (v-)stack. Let  $X \rightarrow \mathfrak{X}$  be a smooth uniformization by a smooth scheme or diamond  $X$ . Then the  $n$ -fold products of  $X$  over  $\mathfrak{X}$  are also smooth, and therefore so is any  $U$  admitting a smooth morphism to such a product, hence  $\Lambda$  is again reflexive.  $\square$

**Example 4.1.5.** Let  $G$  be a pro- $p$  group. If  $G$  acts freely on a smooth diamond  $Y$  over  $\mathrm{Spa}C$ , then  $[*/\underline{G}]$  is a smooth Artin v-stack. Indeed,  $Y/\underline{G}$  is also a smooth diamond, and then  $f: Y/\underline{G} \rightarrow [*/\underline{G}]$  is a smooth surjection. Assume we are given a  $\Lambda$ -valued Haar measure on  $G$ . Then we have an isomorphism  $K_{[*/\underline{G}]} \cong \Lambda$ , since we have one after pulling back to  $Y/\underline{G}$ , cf. [Sch17, Proposition 24.2].

We can generalize these observations to the setting where  $G$  is a locally pro- $p$  group, with pro- $p$  open subgroup  $G_0$ , such that  $G$  acts freely on a smooth diamond. Then  $[*/\underline{G}_0]$  is a smooth Artin stack as above, and since the map  $\pi: [*/\underline{G}_0] \rightarrow [*/\underline{G}]$  is an étale cover,  $[*/\underline{G}]$  is also a smooth Artin stack. Assume given a  $\Lambda$ -valued Haar measure on  $G_0$  (hence on  $G$ ); then the trivialization  $K_{[*/\underline{G}_0]} \cong \Lambda$  induces a trivialization  $K_{[*/\underline{G}]} = \pi^! K_{[*/\underline{G}_0]} \cong \pi^! \Lambda \cong \pi^* \Lambda = \Lambda$ .

**Definition 4.1.6.** Let  $\mathfrak{X}$  be an Artin (v-)stack. The *space of  $\Lambda$ -valued distributions* on  $\mathfrak{X}$  is

$$\mathrm{Dist}(\mathfrak{X}, \Lambda) = H^0(\mathfrak{X}, K_{\mathfrak{X}}).$$

(For an object  $\mathcal{F}$  of  $D_{\mathrm{ét}}(\mathfrak{X}, \Lambda)$ , the  $\Lambda$ -module  $H^0(\mathfrak{X}, \mathcal{F})$  is by definition  $\mathrm{Hom}(\Lambda, \mathcal{F})$ .)

**Example 4.1.7.** If  $\mathfrak{X}$  is a connected scheme or rigid space, smooth and proper over  $*$  of dimension  $d$ , then  $\mathrm{Dist}(\mathfrak{X}, \Lambda) = H^{2d}(\mathfrak{X}, \Lambda(d)) \cong \Lambda$ .

**Example 4.1.8.** If  $S$  is a locally profinite topological space, then there is a canonical isomorphism  $\mathrm{Dist}(\underline{S}, \Lambda) \cong \mathrm{Dist}(S, \Lambda)$ , where  $\mathrm{Dist}(S, \Lambda)$  is the module of  $\Lambda$ -valued distributions on  $S$ , i.e. the  $\Lambda$ -linear dual of the module  $C_c(S, \Lambda)$  of compactly supported continuous  $\Lambda$ -valued functions on  $S$ . Indeed,

**Example 4.1.9.** Let  $G$  be a locally pro- $p$  group which acts freely on a smooth diamond  $Y$ , as in Example 4.1.5. Thus  $[*/\underline{G}]$  is a smooth Artin v-stack. Assume we are given a  $\Lambda$ -valued Haar measure on  $G$ . Suppose that  $S$  is a locally profinite set admitting a continuous action of  $G$ . Then  $\mathfrak{X} = [\underline{S}/\underline{G}]$  is also a smooth Artin stack, since it is uniformized by  $(\underline{S} \times Y)/\underline{G}$ . We have an isomorphism

$$\mathrm{Dist}([\underline{S}/\underline{G}], \Lambda) \cong \mathrm{Dist}(S, \Lambda)^G$$

onto the space of  $G$ -invariant  $\Lambda$ -valued distributions on  $S$ .

**Example 4.1.10.** Continuing the previous example, suppose  $H \subset G$  is a closed normal subgroup which acts freely on  $S$ , so that  $\pi: S \rightarrow S/H$  is an  $H$ -torsor. Then  $[S/G] \cong [(S/H)/(G/H)]$  as stacks, and accordingly there is an isomorphism between spaces of distributions

$$\mathrm{Dist}(S, \Lambda)^G \cong \mathrm{Dist}(S/H, \Lambda)^{G/H}.$$

This isomorphism agrees with the composition of

$$\mathrm{Dist}(S, \Lambda)^G \cong (\mathrm{Dist}(S, \Lambda)^H)^{G/H}$$

with the  $G/H$ -invariants in the isomorphism  $\mathrm{Dist}(S, \Lambda)^H \rightarrow \mathrm{Dist}(S/H, \Lambda)$  of Lemma 3.4.5.

## 4.2 Reflexivity and strong reflexivity

In this subsection we define the notion of strong reflexivity for an object of  $D_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Lambda)$ . These are exactly the objects for which we can apply a Lefschetz formula.

First we recall the notion of an exterior tensor product. Suppose  $\mathfrak{X}_1$  and  $\mathfrak{X}_2$  are Artin (v-)stacks over a base  $S$ . Consider the cartesian diagram

$$\begin{array}{ccc} & \mathfrak{X}_1 \times_S \mathfrak{X}_2 & \\ p_1 \swarrow & & \searrow p_2 \\ \mathfrak{X}_1 & & \mathfrak{X}_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & S & \end{array}$$

For objects  $\mathcal{F}_i \in D_{\mathrm{\acute{e}t}}(\mathfrak{X}_i, \Lambda)$  ( $i = 1, 2$ ) we define the exterior tensor product

$$\mathcal{F}_1 \boxtimes_S \mathcal{F}_2 = p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2 \in D_{\mathrm{\acute{e}t}}(\mathfrak{X}_1 \times_S \mathfrak{X}_2, \Lambda).$$

When  $S = *$  we simply write  $\boxtimes$  for  $\boxtimes_S$ .

**Lemma 4.2.1.** *Assume that  $\pi_1$  and  $\pi_2$  are nice. There is a natural isomorphism*

$$(\pi_1 \times \pi_2)_!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \cong \pi_{1!} \mathcal{F}_1 \otimes \pi_{2!} \mathcal{F}_2.$$

*Proof.* Combining the projection formula and base change isomorphisms gives:

$$\begin{aligned} p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) &= p_{2!}(p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2) \\ &\xrightarrow{\text{proj}} p_{2!} p_1^* \mathcal{F}_1 \otimes \mathcal{F}_2 \\ &\xrightarrow{\text{BC}} \pi_2^* \pi_{1!} \mathcal{F}_1 \otimes \mathcal{F}_2, \end{aligned}$$

so that we have an isomorphism

$$p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \cong \pi_2^* \pi_{1!} \mathcal{F}_1 \otimes \mathcal{F}_2 \quad (4.2.1)$$

Now apply  $\pi_{2!}$ , and note that  $\pi_1 \times \pi_2 = \pi_2 \circ p_2$ :

$$\begin{aligned} (\pi_1 \times \pi_2)_!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) &= \pi_{2!} p_{2!}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \\ &\cong \pi_{2!}(\pi_2^* \pi_{1!} \mathcal{F}_1 \otimes \mathcal{F}_2) \\ &\xrightarrow{\text{proj}} \pi_{1!} \mathcal{F}_1 \otimes \pi_{2!} \mathcal{F}_2, \end{aligned}$$

as claimed.  $\square$

We now impose the assumption that  $\Lambda$  is a dualizing complex for the bases  $S$  (for instance if  $S = *$ , or if  $S = [*/G]$  for a smooth group object  $G$ ). In the case  $\mathcal{F}_1 = \kappa_{\mathfrak{X}_1} = \pi_1^! \Lambda$ , we have a morphism

$$\begin{aligned} p_{2!}(K_{\mathfrak{X}_1} \boxtimes_S \mathcal{F}_2) &\xrightarrow{(4.2.1)} \pi_2^* \pi_{1!} K_{\mathfrak{X}_1} \otimes \mathcal{F}_2 \\ &= \pi_2^* \pi_{1!} \pi_1^! \Lambda \otimes \mathcal{F}_2 \\ &\xrightarrow{\text{adj.}} \pi_2^* \Lambda \otimes \mathcal{F}_2 \cong \Lambda \otimes \mathcal{F}_2 \cong \mathcal{F}_2, \end{aligned}$$

which induces by adjunction a morphism

$$K_{\mathfrak{X}_1} \boxtimes_S \mathcal{F}_2 \rightarrow p_{2!} \mathcal{F}_2. \quad (4.2.2)$$

If  $\mathcal{F}_2 = K_{\mathfrak{X}_2}$ , (4.2.2) becomes

$$K_{\mathfrak{X}_1} \boxtimes_S K_{\mathfrak{X}_2} \rightarrow K_{\mathfrak{X}_1 \times_S \mathfrak{X}_2}. \quad (4.2.3)$$

The morphism in (4.2.3) exists even without the assumption that  $\pi_1$  and  $\pi_2$  are nice (but keeping an assumption that the  $K_{\mathfrak{X}_i}$  exist), by arguing on uniformizations.

We now write down two morphisms related to the notion of a double dual of an object  $\mathcal{F}$  in  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ . The first is straightforward: there is an obvious evaluation morphism

$$\text{ev}: \mathcal{F} \otimes \mathbf{D}\mathcal{F} \rightarrow K_{\mathfrak{X}},$$

which induces by adjunction a morphism  $\mathcal{F} \rightarrow \mathbf{D}\mathcal{F}$ .

The second is the morphism

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \rightarrow \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$$

coming from

$$(\mathbf{D}\mathcal{F} \otimes \mathcal{F}) \boxtimes (\mathcal{F} \otimes \mathbf{D}\mathcal{F}) \xrightarrow{\text{ev} \boxtimes \text{ev}} K_{\mathfrak{X}} \boxtimes K_{\mathfrak{X}} \xrightarrow{(4.2.3)} K_{\mathfrak{X} \times \mathfrak{X}}.$$

**Definition 4.2.2.** Let  $\mathfrak{X}$  be an Artin (v-) stack admitting a dualizing complex. An object  $\mathcal{F}$  of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  is *reflexive* if  $\mathcal{F} \rightarrow \mathbf{D}\mathcal{F}$  is an isomorphism, and *strongly reflexive* if it is reflexive and if  $\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \rightarrow \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$  is an isomorphism.

**Example 4.2.3.** We start with the case that  $X = *$  is a geometric point, so that  $D_{\text{ét}}(*, \Lambda)$  is equivalent to the derived category of  $\Lambda$ -modules, and Verdier duality is just the operation  $M \mapsto \underline{\text{RHom}}(M, \Lambda[0])$ . For an object  $M$  in this category:

1.  $M$  is reflexive if and only if each  $H^i(M)$  is finitely generated.
2.  $M$  is strongly reflexive if and only if  $M$  is a perfect complex.

See §B.1 for proofs of these claims.

**Example 4.2.4.** Let  $X \rightarrow *$  be a scheme of finite type, and let  $\mathcal{F} \in D_{\text{ét}}(X, \Lambda)$  be a complex of finite tor dimension, with constructible cohomology. Then  $\mathcal{F}$  is reflexive [Del77, Th. Finitude 4.3]. Let  $p_1, p_2: X \times X \rightarrow X$  be the projections. We have a morphism

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \rightarrow \underline{\text{RHom}}(p_1^*\mathcal{F}, p_2^!\mathcal{F}), \quad (4.2.4)$$

arising by adjunction from

$$p_1^*\mathcal{F} \otimes (\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}) = (\mathcal{F} \otimes \mathbf{D}\mathcal{F}) \boxtimes \mathcal{F} \xrightarrow{\text{ev}} K_X \boxtimes \mathcal{F} \xrightarrow{4.2.2} p_2^!\mathcal{F}.$$

The morphism in (4.2.4) is an isomorphism [SGA77, III(3.1.1)]. We get a chain of isomorphisms

$$\begin{aligned} \mathbf{D}\mathcal{F} \boxtimes \mathcal{F} &\cong \underline{\text{RHom}}(p_1^*\mathcal{F}, p_2^!\mathcal{F}) \\ &\cong \underline{\text{RHom}}(p_1^*\mathcal{F}, p_2^!\mathbf{D}\mathcal{F}) \\ &\cong \underline{\text{RHom}}(p_1^*\mathcal{F}, \mathbf{D}p_2^*\mathcal{F}) \\ &\cong \underline{\text{RHom}}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{F}, K_{X \times X}) \\ &\cong \mathbf{D}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{F}) \\ &= \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}), \end{aligned}$$

which shows that  $\mathcal{F}$  is also strongly reflexive.

**Example 4.2.5.** Let  $X$  be a scheme or nice diamond, and let  $G$  be a smooth algebraic group or smooth group diamond acting on  $X$ , so that  $[X/G]$  is an Artin (v-)stack. Let  $\mathcal{F}$  be an object of  $D_{\acute{e}t}(X, \Lambda)$  which is  $G$ -equivariant, in the sense that there is a descent datum for  $\mathcal{F}$  along  $X \rightarrow [X/G]$ . This is guaranteed if  $\mathcal{F}$  is *perverse*, and indeed  $G$ -equivariant perverse sheaves on  $X$  are in equivalence with perverse sheaves on  $[X/G]$ . See [LO09, Remark 5.5] for details. This equivalence preserves reflexivity and strong reflexivity.

**Example 4.2.6.** Suppose now that  $G$  is a locally profinite topological group, and assume that  $X = [*/G]$  is a smooth Artin v-stack, as in Example 4.1.5. Then  $D_{\acute{e}t}(\mathfrak{X}, \Lambda)$  is equivalent to the derived category of  $\Lambda$ -modules with a smooth action of  $G$ , and Verdier duality corresponds to smooth duality. (This is not quite automatic; one has to argue on a smooth chart for  $[*/G]$ .) For an object  $M$  in this category:

1.  $M$  is reflexive if and only if each  $H^i(M)$  is admissible (meaning the map to its double smooth dual is an isomorphism).
2.  $M$  is strongly reflexive if, for all open compact subgroups  $K$ , the derived  $K$ -invariants  $M^K$  are a perfect complex.

### 4.3 Inertia stacks and trace distributions

In this subsection we define the local terms appearing in our Lefschetz formula. They are centered around the concept of the inertia stack [Sta17a, §8.7].

**Definition 4.3.1.** For an Artin (v-)stack  $\mathfrak{X}$ , define the *inertia stack*  $\text{In}(\mathfrak{X})$  as the Cartesian product

$$\begin{array}{ccc} \text{In}(\mathfrak{X}) & \xrightarrow{h_{\mathfrak{X}}} & \mathfrak{X} \\ h_{\mathfrak{X}} \downarrow & & \downarrow \Delta_{\mathfrak{X}} \\ \mathfrak{X} & \xrightarrow{\Delta_{\mathfrak{X}}} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

Thus  $\mathfrak{X}$  classifies pairs  $(S, g)$ , where  $S$  is an object of  $\mathfrak{X}$  and  $g \in \text{Aut } S$ . It is also an Artin (v-)stack.

**Example 4.3.2.** Let  $\mathfrak{X} = [X/G]$  for a scheme (or diamond)  $X$  admitting an action of a group scheme (or group diamond)  $G$ . Then  $\text{In}(\mathfrak{X}) = [Y/G]$ , where

$$Y = \left\{ (x, g) \in X \times G \mid g(x) = x \right\},$$

and the action of  $G$  on  $Y$  is  $h(x, g) = (hx, hgh^{-1})$ . In particular,  $\text{In}([*/G]) = [G//G]$  is the stack of conjugacy classes of  $G$ .

**Lemma 4.3.3.** *Let  $\mathfrak{X}$  be an Artin (v-)stack which admits a dualizing sheaf. Let  $\mathcal{F}$  be a strongly reflexive object of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ . There is an isomorphism in  $D_{\text{ét}}(\mathfrak{X} \times \mathfrak{X}, \Lambda)$ :*

$$\underline{\text{RHom}}(\mathcal{F}, \mathcal{F}) \xrightarrow{\sim} \Delta_{\mathfrak{X}}^!(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}).$$

*Proof.* The claimed isomorphism is the composition

$$\begin{aligned} \underline{\text{RHom}}(\mathcal{F}, \mathcal{F}) &\cong \underline{\text{RHom}}(\mathcal{F}, \mathbf{D}\mathbf{D}\mathcal{F}) \\ &\cong \underline{\text{RHom}}(\mathcal{F}, \underline{\text{RHom}}(\mathbf{D}\mathcal{F}, K_{\mathfrak{X}})) \\ &\cong \underline{\text{RHom}}(\mathcal{F} \otimes \mathbf{D}\mathcal{F}, K_{\mathfrak{X}}) \\ &\cong \underline{\text{RHom}}(\Delta_{\mathfrak{X}}^*(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}), \Delta_{\mathfrak{X}}^! K_{\mathfrak{X} \times \mathfrak{X}}) \\ &\cong \Delta_{\mathfrak{X}}^! \underline{\text{RHom}}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}, K_{\mathfrak{X} \times \mathfrak{X}}) \\ &\cong \Delta_{\mathfrak{X}}^! \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \\ &\cong \Delta_{\mathfrak{X}}^!(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}), \end{aligned}$$

where we have used reflexivity in the first step, and strong reflexivity in the last.  $\square$

**Definition 4.3.4** (The trace distribution of a sheaf). Let  $\mathfrak{X}$  be an Artin (v-)stack that admits a dualizing complex, and let  $\mathcal{F}$  be a strongly reflexive object of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  be strongly reflexive. Let  $\text{tr}: \underline{\text{RHom}}(\mathcal{F}, \mathcal{F}) \rightarrow h_{\mathfrak{X}*} K_{\text{In}(\mathfrak{X})}$  be the composition

$$\begin{aligned} \underline{\text{RHom}}(\mathcal{F}, \mathcal{F}) &\cong \Delta_{\mathfrak{X}}^!(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \\ &\rightarrow \Delta_{\mathfrak{X}}^! \Delta_{\mathfrak{X}*}(\mathcal{F} \otimes \mathbf{D}\mathcal{F}) \\ &\xrightarrow{\text{BC}} h_{\mathfrak{X}*} h_{\mathfrak{X}}^! K_{\mathfrak{X}} \\ &= h_{\mathfrak{X}*} K_{\text{In}(\mathfrak{X})}, \end{aligned}$$

and let  $\text{tr}(\mathcal{F}) \in \text{Dist}(\text{In}(\mathfrak{X}), \Lambda)$  be the image of the identity morphism in  $\text{Hom}(\mathcal{F}, \mathcal{F}) = H^0(\mathfrak{X}, \underline{\text{RHom}}(\mathcal{F}, \mathcal{F}))$  under  $H^0(\text{tr})$ .

**Example 4.3.5.** Let  $X \rightarrow *$  be a projective scheme. Let  $f: X \rightarrow X$  be an automorphism, and let  $\mathfrak{X} = [X/f^{\mathbf{Z}}]$ . Then  $\text{In}(\mathfrak{X})$  is fibered over  $\text{In}([*/\mathbf{Z}]) \cong \mathbf{Z} \times [*/\mathbf{Z}]$ . The fiber over  $\{1\} \times [*/\mathbf{Z}]$  is  $\text{Fix}(f) \times [*/\mathbf{Z}]$ , where  $\text{Fix}(f) \subset X$  is the fixed-point locus of  $f$ .

Let  $\mathcal{F}$  be an object of  $D_{\text{ét}}(X, \Lambda)$  which is of finite tor dimension with constructible cohomology. Assume that  $\text{Ext}^i(\mathcal{F}, \mathcal{F}) = 0$  for all  $i < 0$  (for

instance,  $\mathcal{F}$  could be an honest sheaf). If we are given an isomorphism  $u: \mathcal{F} \rightarrow f^*\mathcal{F}$ , then  $\mathcal{F}$  descends to an object  $\overline{\mathcal{F}}$  of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ . Then  $\overline{\mathcal{F}}$  is strongly reflexive, and we have the element  $\text{tr } \overline{\mathcal{F}} \in \text{Dist}(\text{In}(\mathfrak{X}), \Lambda)$ . For each isolated point  $x \in \text{Fix}(f)$ , we get by restriction an element of  $\text{Dist}(x \times [*/\mathbf{Z}], \Lambda) \cong \Lambda$ . This is the local term  $\text{loc}_x(f, \mathcal{F})$ . It is denoted  $\text{LT}_x(u)$  in [Var07].

**Example 4.3.6.** Let  $G$  be a locally pro- $p$  topological group, such that  $\mathfrak{X} = [*/G]$  is a smooth Artin stack. Then a strongly reflexive object  $\mathcal{F} \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$  corresponds to a complex of smooth  $\Lambda G$ -modules  $M$ , such that  $M^K$  (derived  $K$ -invariants) is a perfect complex for each compact open subgroup  $K \subset G$ . After choosing a Haar measure on  $G$ , we have (after example 4.1.9) an isomorphism  $\text{Dist}(\text{In}(\mathfrak{X}), \Lambda) \cong \text{Dist}(G, \Lambda)^G$ . With respect to this isomorphism,  $\text{tr}(\mathcal{F})$  is the trace distribution of  $G$  on  $M$ . Explicitly: if  $f \in C_c(G, \Lambda)$  is a compactly supported smooth function, then it is bi- $K$ -invariant for a compact open subgroup  $K \subset G$ , and then  $\text{tr}(\mathcal{F})(f)$  equals the trace of integration against  $f$  on  $M^K$  (that is, we represent  $M^K$  by a bounded complex of finite projective  $\Lambda$ -modules and take the alternating sum of traces of this endomorphism acting on each term of that complex).

#### 4.4 Compatibility of the trace distribution with proper pushforward and smooth pullback

In this subsection we show that formation of the trace distribution  $\text{tr } \mathcal{F}$  is compatible with proper pushforward and smooth pullback under morphisms  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ .

**Theorem 4.4.1.** *Let  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$  be a morphism between Artin stacks which is representable (respectively, a morphism between Artin  $v$ -stacks which is representable in nice diamonds).*

1. *If  $f$  is proper, there exists a natural morphism in  $D_{\text{ét}}(\text{In}(\mathfrak{Y}), \Lambda)$*

$$f_{\natural}: \text{In}(f)_* K_{\text{In}(\mathfrak{X})} \rightarrow K_{\text{In}(\mathfrak{Y})},$$

*which induces on global sections a pushforward map*

$$f_{\natural}: \text{Dist}(\text{In}(\mathfrak{X}), \Lambda) \rightarrow \text{Dist}(\text{In}(\mathfrak{Y}), \Lambda).$$

*If  $\mathcal{F} \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$  is strongly reflexive, then so is  $f_{\natural}\mathcal{F}$ , and then*

$$f_{\natural}(\text{tr } \mathcal{F}) = \text{tr } f_{\natural}\mathcal{F}.$$



of  $x$  lifting  $u$ ; thus the fiber product is exactly  $\text{In}(\mathfrak{X})$ , with  $b$  taking  $(x, u')$  to  $(x, u)$ .

*For Part (1) of the theorem:* We suppose that  $f$  is proper, so that  $f_* = f_!$ . If  $\mathcal{F}$  is reflexive, then so is  $f_*\mathcal{F}$ , because

$$\mathbf{D}\mathbf{D}f_*\mathcal{F} = \mathbf{D}\mathbf{D}f_!\mathcal{F} \cong \mathbf{D}f_*\mathbf{D}\mathcal{F} \cong \mathbf{D}f_*\mathbf{D}\mathcal{F} \cong f_!\mathbf{D}\mathbf{D}\mathcal{F} \cong f_!\mathcal{F} = f_*\mathcal{F}.$$

If  $\mathcal{F}$  is strongly reflexive, then so is  $f_*\mathcal{F}$ , because (using  $f_* = f_!$  and  $(f \times f)_* = (f \times f)_!$ )

$$\begin{aligned} \mathbf{D}f_*\mathcal{F} \boxtimes f_*\mathcal{F} &\cong f_*\mathbf{D}\mathcal{F} \boxtimes f_*\mathcal{F} \\ &\cong (f \times f)_*(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}) \\ &\cong (f \times f)_*\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \\ &\cong \mathbf{D}(f \times f)_*(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \\ &\cong \mathbf{D}(f_*\mathcal{F} \boxtimes f_*\mathbf{D}\mathcal{F}) \\ &\cong \mathbf{D}(f_*\mathcal{F} \boxtimes \mathbf{D}f_*\mathcal{F}). \end{aligned}$$

We claim that  $\text{In}(f): \text{In}(\mathfrak{X}) \rightarrow \text{In}(\mathfrak{Y})$  is proper. This will follow from the properness of the maps  $a$  and  $b$  in (4.4.1). The map  $a$  is proper because it is a base change of  $f$ . The map  $b$  is proper because it is a base change of  $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ , and  $\Delta_{\mathfrak{X}/\mathfrak{Y}}$  is proper because  $f$ , being a proper map, is separated.

The desired map  $f_!$  is the composite

$$\text{In}(f)_*K_{\text{In}(\mathfrak{X})} = \text{In}(f)_!K_{\text{In}(\mathfrak{X})} = \text{In}(f)_!\text{In}(f)^!K_{\text{In}(\mathfrak{Y})} \rightarrow K_{\text{In}(\mathfrak{Y})} :$$

thus roughly speaking it is “fiberwise integration.”

We now trace through the construction of the trace distribution for  $\mathcal{F}$

and  $f_*\mathcal{F}$ :

$$\begin{array}{ccc}
f_*\underline{\mathrm{RHom}}(\mathcal{F}, \mathcal{F}) & \xrightarrow{(1)} & \underline{\mathrm{RHom}}(f_*\mathcal{F}, f_*\mathcal{F}) \\
\cong \downarrow & & \downarrow \cong \\
f_*\underline{\mathrm{RHom}}(\mathcal{F} \otimes \mathbf{D}\mathcal{F}, K_{\mathfrak{X}}) & \xrightarrow{(2)} & \underline{\mathrm{RHom}}(f_*\mathcal{F} \otimes \mathbf{D}f_*\mathcal{F}, K_{\mathfrak{Y}}) \\
\cong \downarrow & & \downarrow \cong \\
f_*\Delta_{\mathfrak{X}}^!(\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}) & \xrightarrow{(3)} & \Delta_{\mathfrak{Y}}^!(\mathbf{D}f_*\mathcal{F} \boxtimes f_*\mathcal{F}) \\
\downarrow & & \downarrow \\
f_*\Delta_{\mathfrak{X}}^!\Delta_{\mathfrak{X}*}K_{\mathfrak{X}} & \xrightarrow{(4)} & \Delta_{\mathfrak{Y}}^!\Delta_{\mathfrak{Y}*}K_{\mathfrak{Y}} \\
\cong \downarrow & & \downarrow \cong \\
f_*h_{\mathfrak{X}*}K_{\mathrm{In}(\mathfrak{X})} & \xrightarrow{(5)} & h_{\mathfrak{Y}*}K_{\mathrm{In}(\mathfrak{Y})}.
\end{array} \tag{4.4.2}$$

The horizontal maps are obtained by six functor manipulations, making use of the facts that  $f_* = f_!$  and  $\mathrm{In}(f)_* = \mathrm{In}(f)!$  since these maps are proper. For instance, the map labeled (1) is

$$\begin{aligned}
f_*\underline{\mathrm{RHom}}(\mathcal{F}, \mathcal{F}) &\rightarrow f_*\underline{\mathrm{RHom}}(\mathcal{F}, f^!f_!\mathcal{F}) \\
&\cong \underline{\mathrm{RHom}}(f_!\mathcal{F}, f_!\mathcal{F}) \\
&\cong \underline{\mathrm{RHom}}(f_*\mathcal{F}, f_*\mathcal{F}).
\end{aligned}$$

The map labeled (5) arises this way: the isomorphism  $K_{\mathrm{In}(\mathfrak{X})} \xrightarrow{\sim} \mathrm{In}(f)^!K_{\mathrm{In}(\mathfrak{Y})}$  is adjoint to a map

$$f^{\natural}: \mathrm{In}(f)_*K_{\mathrm{In}(\mathfrak{X})} = \mathrm{In}(f)_{!}K_{\mathrm{In}(\mathfrak{X})} \rightarrow K_{\mathrm{In}(\mathfrak{Y})}, \tag{4.4.3}$$

which in turn induces a map

$$f_*h_{\mathfrak{X}*}K_{\mathrm{In}(\mathfrak{X})} = h_{\mathfrak{Y}*}\mathrm{In}(f)_*K_{\mathrm{In}(\mathfrak{X})} \xrightarrow{f^{\natural}} h_{\mathfrak{Y}*}K_{\mathrm{In}(\mathfrak{Y})}.$$

The commutativity of each square in (4.4.2) is an appeal to the naturality of the six functor operations, especially base change. We omit the details. Tracing through the image of the identity map on  $\mathcal{F}$  in (4.4.2) gives the desired equality  $f_{\natural} \mathrm{tr} \mathcal{F} = \mathrm{tr} f_*\mathcal{F}$ .

*For Part (2) of the theorem:* Assume that  $f$  is smooth. This means that  $f^!\Lambda$  is invertible in  $D_{\mathrm{et}}(\mathfrak{X}, \Lambda)$  and that we have a natural isomorphism

$f^! \cong f^! \Lambda \otimes f^*$ . If  $\mathcal{F}$  is reflexive, then so is  $f^* \mathcal{F}$ , because

$$\begin{aligned}
\mathbf{D} \mathbf{D} f^* \mathcal{F} &\cong \mathbf{D} f^! \mathbf{D} \mathcal{F} \cong \mathbf{D}(f^! \Lambda \otimes f^* \mathbf{D} \mathcal{F}) \\
&\cong \mathbf{D} f^* \mathbf{D} \mathcal{F} \otimes f^! \Lambda^{-1} \\
&\cong f^! \mathbf{D} \mathbf{D} \mathcal{F} \otimes f^! \Lambda^{-1} \\
&\cong f^! \mathcal{F} \otimes f^! \Lambda^{-1} \cong f^* \mathcal{F}.
\end{aligned}$$

If  $\mathcal{F}$  is strongly reflexive, then

$$\begin{aligned}
\mathbf{D} f^* \mathcal{F} \boxtimes f^* \mathcal{F} &\cong f^! \mathbf{D} \mathcal{F} \boxtimes f^* \mathcal{F} \\
&\cong (f^! \Lambda \otimes f^* \mathbf{D} \mathcal{F}) \boxtimes f^* \mathcal{F} \\
&\cong (f \times f)^*(\mathbf{D} \mathcal{F} \boxtimes \mathcal{F}) \otimes (f^! \Lambda \boxtimes \Lambda) \\
&\cong (f \times f)^* \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D} \mathcal{F}) \otimes (f^! \Lambda \boxtimes \Lambda) \\
&\cong (f \times f)^! \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D} \mathcal{F}) \otimes (\Lambda \boxtimes f^! \Lambda^{-1}) \\
&\cong \mathbf{D}(f^* \mathcal{F} \boxtimes f^* \mathbf{D} \mathcal{F}) \otimes (\Lambda \boxtimes f^! \Lambda^{-1}) \\
&\cong \mathbf{D}(f^* \mathcal{F} \boxtimes (f^! \Lambda \otimes f^* \mathbf{D} \mathcal{F})) \\
&\cong \mathbf{D}(f^* \mathcal{F} \boxtimes f^! \mathbf{D} \mathcal{F}) \\
&\cong \mathbf{D}(f^* \mathcal{F} \boxtimes \mathbf{D} f^* \mathcal{F}),
\end{aligned}$$

so that  $f^* \mathcal{F}$  is strongly reflexive as well.

We use the isomorphism  $f^! \cong f^! \Lambda \otimes f^*$  repeatedly to trace through the constructions in Lemma 4.3.3 and Definition 4.3.4, as applied to  $\mathcal{F}$  and  $f^* \mathcal{F}$ :

$$\begin{array}{ccc}
f^* \underline{\mathbf{R} \mathbf{H} \mathbf{om}}(\mathcal{F}, \mathcal{F}) & \xrightarrow[\cong]{(1)} & \underline{\mathbf{R} \mathbf{H} \mathbf{om}}(f^* \mathcal{F}, f^* \mathcal{F}) & (4.4.4) \\
\cong \downarrow & & \downarrow \cong & \\
f^* \underline{\mathbf{R} \mathbf{H} \mathbf{om}}(\mathcal{F} \otimes \mathbf{D} \mathcal{F}, K_{\mathfrak{y}}) & \xrightarrow[\cong]{(2)} & \underline{\mathbf{R} \mathbf{H} \mathbf{om}}(f^* \mathcal{F} \otimes f^* \mathbf{D} \mathcal{F}, K_{\mathfrak{x}}) & \\
\cong \downarrow & & \downarrow \cong & \\
f^* \Delta_{\mathfrak{y}}^! (\mathbf{D} \mathcal{F} \boxtimes \mathcal{F}) & \xrightarrow[\cong]{(3)} & \Delta_{\mathfrak{x}}^! (\mathbf{D} f^* \mathcal{F} \boxtimes f^* \mathcal{F}) & \\
\downarrow & & \downarrow & \\
f^* \Delta_{\mathfrak{y}}^! \Delta_{\mathfrak{y}*} K_{\mathfrak{y}} & \xrightarrow{(4)} & \Delta_{\mathfrak{x}}^! \Delta_{\mathfrak{x}*} K_{\mathfrak{x}} & \\
\cong \downarrow & & \downarrow \cong & \\
f^* h_{\mathfrak{y}*} K_{\mathbf{In}(\mathfrak{y})} & \xrightarrow{(5)} & h_{\mathfrak{x}*} K_{\mathbf{In}(\mathfrak{x})}. &
\end{array}$$

For instance, the map labeled (1) is [Sch17, Proposition 23.17]:

$$\begin{aligned}
f^* \underline{\mathrm{RHom}}(\mathcal{F}, \mathcal{F}) &\cong f^! \Lambda^{-1} \otimes f^! \underline{\mathrm{RHom}}(\mathcal{F}, \mathcal{F}) \\
&\cong f^! \Lambda^{-1} \otimes \underline{\mathrm{RHom}}(f^* \mathcal{F}, f^! \mathcal{F}) \\
&\cong \underline{\mathrm{RHom}}(f^* \mathcal{F}, f^! \Lambda^{-1} \otimes f^! \mathcal{F}) \\
&\cong \underline{\mathrm{RHom}}(f^* \mathcal{F}, f^* \mathcal{F}).
\end{aligned}$$

The morphism labeled (5) is more involved. Referring to (4.4.1), we construct the composite

$$\begin{aligned}
a^* K_{\mathrm{In}(\mathfrak{Y})} &= a^* h_{\mathfrak{Y}}^! K_{\mathfrak{Y}} \\
&\xrightarrow{\mathrm{BC}} c^! f^* K_{\mathfrak{Y}} \\
&= d^! p_1^! f^* K_{\mathfrak{Y}} \\
&\xrightarrow{\mathrm{BC}} d^! p_2^* f^! K_{\mathfrak{Y}} \\
&= d^! p_2^* \mathrm{id}_* f^! K_{\mathfrak{Y}} \\
&= d^! p_2^* p_{2*} \Delta_{\mathfrak{X}/\mathfrak{Y}*} f^! K_{\mathfrak{Y}} \\
&\rightarrow d^! \Delta_{\mathfrak{X}/\mathfrak{Y}*} f^! K_{\mathfrak{Y}} \\
&\xrightarrow{\mathrm{BC}} b_* h_{\mathfrak{X}}^! f^! K_{\mathfrak{Y}} \\
&= b_* K_{\mathrm{In}(\mathfrak{X})},
\end{aligned}$$

where we have used a base change isomorphism for all three cartesian squares in (4.4.1). This induces by adjunction a morphism  $b^* a^* K_{\mathrm{In}(\mathfrak{Y})} = \mathrm{In}(f)^* K_{\mathrm{In}(\mathfrak{Y})} \rightarrow K_{\mathrm{In}(\mathfrak{X})}$ . Finally, from this we derive the map

$$f^* h_{\mathfrak{Y}*} K_{\mathrm{In}(\mathfrak{Y})} \xrightarrow{\mathrm{BC}} h_{\mathfrak{X}*} \mathrm{In}(f)^* K_{\mathrm{In}(\mathfrak{Y})} \rightarrow h_{\mathfrak{X}*} K_{\mathrm{In}(\mathfrak{X})},$$

which is (5).

The map in (5) is adjoint to a map  $h_{\mathfrak{Y}*} K_{\mathrm{In}(\mathfrak{Y})} \rightarrow f_* h_{\mathfrak{X}*} K_{\mathrm{In}(\mathfrak{X})}$ , which on global sections is our map  $f^{\natural}$ . Once again, the commutativity of the squares in (4.4.4) results from the functoriality of the six functor operations (especially base change). Applied to the identity map on  $\mathcal{F}$ , it implies that  $f^{\natural}(\mathrm{tr} \mathcal{F}) = \mathrm{tr} f^* \mathcal{F}$ .  $\square$

Evidently the morphism  $f^{\natural}$  (for  $f$  smooth) is rather subtle. However, there is a situation in which it is quite easy to describe.

**Lemma 4.4.2.** *Suppose we are in the situation of Theorem 4.4.1(2). Let  $V \subset \mathrm{In}(\mathfrak{Y})$  be an open substack such that  $\mathrm{In}(f)$  is étale over  $V$ . Let  $U \subset \mathrm{In}(\mathfrak{X})$  be the preimage of  $V$ . Then  $f^{\natural}$  is the isomorphism:*

$$\mathrm{In}(f)^* K_V = \mathrm{In}(f)^! K_V = K_U.$$

**Example 4.4.3.** Let  $X$  be a smooth scheme or diamond, and let  $u: X \rightarrow X$  be an automorphism. Thus we have a smooth morphism  $f: [X/u^{\mathbf{Z}}] \rightarrow [*/\mathbf{Z}]$ . Let  $x \in X$  be an isolated fixed point of  $u$ . The condition that  $\mathrm{In}(f)$  be étale at  $(x, u)$  is equivalent to the condition that the diagonal  $\Delta_X$  and the graph  $\Gamma_u$  meet transversely in  $X \times X$ . Under this condition, Lemma 4.4.2 states that  $\mathrm{loc}_x(u, \Lambda) = 1$ .

#### 4.5 Application: a Lefschetz formula for group actions

At this point we can reap some of the benefits of our machinery, to compute the action of an algebraic group or group diamond  $G$  on the cohomology of a proper scheme or diamond  $X$ .

Let us assume that  $[*/G]$  is an Artin (v-)stack, and that  $X \rightarrow *$  is a proper scheme or diamond. Let  $\mathfrak{X} = [X/G]$ , and let  $f: \mathfrak{X} \rightarrow [*/G]$  be the descent of the structure morphism on  $X$ . Suppose  $\mathcal{F} \in D_{\acute{\mathrm{e}}\mathrm{t}}(\mathfrak{X}, \Lambda)$  is strongly reflexive. Since  $f$  is proper, Theorem 4.4.1 applies to give

$$\mathrm{tr} f_* \mathcal{F} = f_{\natural} \mathrm{tr} \mathcal{F}.$$

The left side is an element of  $\mathrm{Dist}([G//G], \Lambda)$ ; it gives the trace distribution of  $G$  on the cohomology of  $X$ . On the right side,  $\mathrm{tr} \mathcal{F}$  is a distribution on  $\mathrm{In}(\mathfrak{X})$ . Recall from Example 4.3.2 that  $\mathrm{In}(\mathfrak{X}) = [Y/G]$ , where  $Y$  classifies pairs  $(x, g)$  with  $x \in X$ ,  $g \in G$  and  $g(x) = x$ . The equality between the sides expresses the fact that the trace distribution of  $G$  acting on the cohomology of  $X$  can be computed in terms of objects living on the inertia stack  $\mathrm{In}[X/G]$ .

#### 4.6 Universal local acyclicity, and compatibility of the trace distribution with fiber products

We present here the notion of universal local acyclicity in the diamond setting, introduced in [FS].

**Definition 4.6.1.** Let  $f: X \rightarrow Y$  be a nice morphism of locally spatial diamonds, and let  $\mathcal{F}$  be an object of  $D_{\acute{\mathrm{e}}\mathrm{t}}(X, \Lambda)$ . We call  $\mathcal{F}$  *locally acyclic* (LA) with respect to  $f$  if the following conditions hold:

1. The object  $\mathcal{F}$  is overconvergent; that is, for any specialization of geometric points  $x \rightsquigarrow y$  of  $X$ , the associated map on stalks  $\mathcal{F}_y \rightarrow \mathcal{F}_x$  is an isomorphism.
2. For every étale morphism  $U \rightarrow X$  for which  $f|_U: U \rightarrow Y$  is qcqs, the object  $f|_U! \mathcal{F}$  of  $D_{\acute{\mathrm{e}}\mathrm{t}}(Y, \Lambda)$  is constructible.

Now suppose  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$  is a nice morphism of v-stacks, and  $\mathcal{F}$  is an object of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ . We call  $\mathcal{F}$  *universally locally acyclic* (ULA) with respect to  $f$  if for all morphisms  $Y \rightarrow \mathfrak{Y}$  from a locally spatial diamond, with pullback  $f': X = \mathfrak{X} \times_{\mathfrak{Y}} Y \rightarrow Y$ , the pullback of  $\mathcal{F}$  to  $X$  is locally acyclic with respect to  $f'$ .

**Remark 4.6.2.** This definition of ULA looks different than its schematic analogue. We say that a bounded complex  $\mathcal{F}$  of constructible sheaves on  $X_{\text{ét}}$  is LA with respect to a finitely presented morphism of schemes  $f: X \rightarrow Y$  if for all geometric points  $x \rightarrow X$  and  $y \rightarrow Y$  with  $x$  mapping to  $y$ , and all geometric points  $z \rightarrow Y$  specializing to  $y$ , the restriction map

$$R\Gamma(X_x, \mathcal{F}) \rightarrow R\Gamma(X_x \times_{Y_y} Y_z, \mathcal{F}) \quad (4.6.1)$$

is an isomorphism. Informally, this means that the fiber of  $\mathcal{F}$  at a closed point is determined by its fibers at nearby points. (In particular, any  $\mathcal{F}$  is ULA when  $Y$  is a point.) This concept is important for some approaches to geometric Satake [Ric14], where it is used to show that the category of equivariant perverse sheaves on the affine Grassmannian is symmetric monoidal.

In the setting of  $p$ -adic geometry, the part of the ULA condition demanding that (4.6.1) be an isomorphism is captured by the overconvergence property. But the notion of constructible sheaf is too restrictive for our purposes. For instance, a skyscraper sheaf supported at a point in rigid-analytic  $\mathbf{P}^1$  is not constructible, because the complement of a point is not quasi-compact. Instead of demanding that  $\mathcal{F}$  itself be constructible in Definition 4.6.1, we instead place a constructibility condition on its cohomology; this is condition (2).

We list here some properties of ULA objects, valid in both the scheme and diamond settings.

1. If  $\mathcal{F}$  is locally constant on  $\mathfrak{X}$  and  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$  is smooth, then  $\mathcal{F}$  is ULA with respect to  $f$ .
2. If  $\mathcal{F}$  is ULA with respect to  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ , then so is  $\mathbf{D}\mathcal{F}$  (Verdier dual relative to  $\mathfrak{Y}$ ).
3. If  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are ULA with respect to  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ , then  $\mathcal{F}_1 \boxtimes_{\mathfrak{Y}} \mathcal{F}_2$  is ULA with respect to  $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \rightarrow \mathfrak{Y}$ .
4. The ULA condition is smooth-local on  $\mathfrak{X}$ .

We gather here two consequences of the ULA property, and discuss the relation to the Lefschetz trace formula.

**Theorem 4.6.3** ([FS]). *Suppose  $\mathcal{F}$  is ULA with respect to a nice morphism of  $v$ -stacks  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ .*

1. *For every cartesian diagram of  $v$ -stacks*

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ g' \downarrow & & \downarrow g \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}, \end{array}$$

*the natural map*

$$(g')^* \underline{\mathrm{RHom}}(\mathcal{F}, f^! \Lambda) \rightarrow \underline{\mathrm{RHom}}((g')^* \mathcal{F}, (f')^! \Lambda)$$

*is an isomorphism.*

2. *For every object  $\mathcal{G}$  of  $D_{\acute{\mathrm{e}}\mathrm{t}}(\mathfrak{Y}, \Lambda)$ , the natural map*

$$\underline{\mathrm{RHom}}(\mathcal{F}, f^! \Lambda) \otimes f^* \mathcal{G} \rightarrow \underline{\mathrm{RHom}}(\mathcal{F}, f^! \mathcal{G})$$

*is an isomorphism.*

Consider now a cartesian diagram of Artin ( $v$ -)stacks

$$\begin{array}{ccc} & \mathfrak{X}_1 \times_S \mathfrak{X}_2 & \\ p_1 \swarrow & & \searrow p_2 \\ \mathfrak{X}_1 & & \mathfrak{X}_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & S & \end{array} \tag{4.6.2}$$

in which  $\pi_1$  and  $\pi_2$  are nice.

**Lemma 4.6.4.** *For  $i = 1, 2$ , let  $\mathcal{F}_i$  be an object of  $D_{\acute{\mathrm{e}}\mathrm{t}}(\mathfrak{X}_i, \Lambda)$ . Assume that  $\mathcal{F}_1$  is ULA with respect to  $\pi_1$ , and that  $K_S = \Lambda$ . Then the natural map*

$$\mathbf{D}\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_2 \rightarrow \mathbf{D}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2)$$

*is an isomorphism.*

*Proof.* The inverse map is

$$\begin{aligned}
\mathbf{D}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) &\cong \underline{\mathbf{R}\mathrm{Hom}}(p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2, K_{\mathfrak{X}_1 \times_S \mathfrak{X}_2}) \\
&\cong \underline{\mathbf{R}\mathrm{Hom}}(p_1^* \mathcal{F}_1, \underline{\mathbf{R}\mathrm{Hom}}(p_2^* \mathcal{F}_2, K_{\mathfrak{X}_1 \times_S \mathfrak{X}_2})) \\
&\cong \underline{\mathbf{R}\mathrm{Hom}}(p_1^* \mathcal{F}_2, p_2^! \mathbf{D}\mathcal{F}_2) \\
&\cong \underline{\mathbf{R}\mathrm{Hom}}(p_1^* \mathcal{F}_1, p_2^! \Lambda) \otimes p_2^* \mathbf{D}\mathcal{F}_2 \\
&\cong p_1^* \underline{\mathbf{R}\mathrm{Hom}}(\mathcal{F}_1, \pi_1^! \Lambda) \otimes p_2^* \mathbf{D}\mathcal{F}_2 \\
&\cong p_1^* \mathbf{D}\mathcal{F}_1 \otimes p_2^* \mathbf{D}\mathcal{F}_2 \\
&= \mathbf{D}\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_2
\end{aligned}$$

Here we used Theorem 4.6.3(2) applied to  $p_1^* \mathcal{F}_1$  for the fourth isomorphism, and Theorem 4.6.3(1) for the fifth isomorphism.  $\square$

**Lemma 4.6.5.** *Let  $\pi: \mathfrak{X} \rightarrow S$  be a nice morphism of  $v$ -stacks, with  $K_S = \Lambda$ . If an object  $\mathcal{F}$  of  $D_{\text{ét}}(\mathfrak{X}, \Lambda)$  is reflexive and ULA with respect to  $\pi$ , then  $\mathcal{F}$  is strongly reflexive.*

*Proof.* Lemma 4.6.4 shows that  $\mathbf{D}(\mathcal{F} \boxtimes_S \mathbf{D}\mathcal{F}) \cong \mathbf{D}\mathcal{F} \boxtimes_S \mathbf{D}\mathbf{D}\mathcal{F}$ , which by reflexivity is isomorphic to  $\mathbf{D}\mathcal{F} \boxtimes_S \mathcal{F}$ .  $\square$

We return now to the general setting of (4.6.2). Passing to inertia stacks in that diagram, we arrive at an isomorphism

$$\mathrm{In}(\mathfrak{X}_1 \times_S \mathfrak{X}_2) \cong \mathrm{In}(\mathfrak{X}_1) \times_S \mathrm{In}(\mathfrak{X}_2).$$

Keep the assumption that  $K_S = \Lambda$ . The Künneth morphism (4.2.3) induces a  $\Lambda$ -bilinear map

$$\mathrm{Dist}(\mathrm{In}(\mathfrak{X}_1), \Lambda) \times \mathrm{Dist}(\mathrm{In}(\mathfrak{X}_2), \Lambda) \rightarrow \mathrm{Dist}(\mathrm{In}(\mathfrak{X}_1 \times_S \mathfrak{X}_2), \Lambda)$$

which we will denote simply by  $(\mu_1, \mu_2) \mapsto \mu_1 \boxtimes \mu_2$ .

**Proposition 4.6.6.** *For  $i = 1, 2$ , let  $\mathcal{F}_i$  be a strongly reflexive object of  $D_{\text{ét}}(\mathfrak{X}_i, \Lambda)$ . Assume that  $\mathcal{F}_1$  is ULA with respect to  $\pi_1$ . Then  $\mathcal{F}_1 \boxtimes_S \mathcal{F}_2$  is strongly reflexive, and*

$$\mathrm{tr}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) = \mathrm{tr}(\mathcal{F}_1) \boxtimes \mathrm{tr}(\mathcal{F}_2).$$

*Proof.* By Lemma 4.6.4, the natural map  $\mathbf{D}\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_2 \rightarrow \mathbf{D}(\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_2)$  is an isomorphism, and similarly for  $\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathbf{D}\mathcal{F}_2 \rightarrow \mathbf{D}(\mathbf{D}\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_2)$ , from

which we deduce (using reflexivity of  $\mathcal{F}_1$  and  $\mathcal{F}_2$ ) that  $\mathcal{F}_1 \boxtimes_S \mathcal{F}_2$  is reflexive. Strong reflexivity of  $\mathcal{F}_1 \boxtimes_S \mathcal{F}_2$  also follows, since

$$\begin{aligned}
\mathbf{D}(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \boxtimes_S (\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) &\cong (\mathbf{D}\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_2) \boxtimes_S (\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \\
&\cong (\mathbf{D}\mathcal{F}_1 \boxtimes_S \mathcal{F}_1) \boxtimes_S (\mathbf{D}\mathcal{F}_2 \boxtimes_S \mathcal{F}_2) \\
&\cong \mathbf{D}(\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_1) \boxtimes_S \mathbf{D}(\mathcal{F}_2 \boxtimes_S \mathbf{D}\mathcal{F}_2) \\
&\cong \mathbf{D}((\mathcal{F}_1 \boxtimes_S \mathbf{D}\mathcal{F}_1) \boxtimes_S (\mathcal{F}_2 \boxtimes_S \mathbf{D}\mathcal{F}_2)) \\
&\cong \mathbf{D}((\mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \boxtimes_S (\mathcal{F}_1 \boxtimes_S \mathcal{F}_2))
\end{aligned}$$

(In the second-to-last isomorphism, we applied Lemma 4.6.4 to  $\mathbf{D}\mathcal{F}_1 \boxtimes_S \mathcal{F}_2$ , which is ULA with respect to  $\mathfrak{X}_1 \times_S \mathfrak{X}_2 \rightarrow S$ .)

The claim about trace distributions is a tedious diagram chase.  $\square$

## 5 The affine Grassmannian

### 5.1 Lefschetz for flag varieties

Let  $G$  be a connected reductive algebraic group over an algebraically closed field, or else the rigid space attached to a connected reductive algebraic group over an algebraically closed perfectoid field.

Let  $P \subset G$  be a parabolic subgroup. The flag variety  $X = G/P$  is proper and smooth, and admits an action of  $G$ . Presently we will use our machinery to calculate the Euler characteristic of  $X$ . Before doing this, let us give an informal sketch of how this should work. The action of  $G$  on  $H^*(X, \Lambda)$  should be trivial, since  $G$  is connected. Thus  $\dim H^*(X, \Lambda)$  equals the trace of  $g$  on  $H^*(X, \Lambda)$  for any  $g \in G$ . Let us choose  $g$  to be a strongly regular element of a maximal torus  $T \subset P$ . Then the fixed point locus  $X^g$  is a finite set of points, in correspondence with  $W/W_M$ , where  $W$  is the Weyl group of  $G$  associated to  $T$ , and  $W_M$  is the Weyl group of the Levi quotient  $M$  of  $P$ . Furthermore, for each  $x \in X^g$  the local term  $\text{loc}_x(g, \Lambda)$  equals one, because the graph of  $g$  and the diagonal meet transversely in  $X \times X$  at  $(x, x)$ . Thus  $\dim H^*(X, \Lambda) = \#W/W_M$ . This confirms what is already well-known about the cohomology of the flag variety: there is a basis vector for each Weyl translate of  $P$ , corresponding to the Schubert cells, and these all live in even degree.

## 5.2 The stack $[X/G]$

The quotient  $\mathfrak{X} = [X/G]$  is a proper smooth Artin stack. We have an isomorphism

$$\mathrm{In}(\mathfrak{X}) \cong [P//P].$$

Indeed, a point of  $\mathrm{In}(\mathfrak{X})$  is a conjugacy class of pairs  $(x, g)$  with  $x \in X$  and  $g \in G$  and  $g(x) = x$ . If  $x = hP$ , this means that  $h^{-1}gh \in P$ . The isomorphism above sends  $(x, g)$  to  $h^{-1}gh$ . Thus  $\mathrm{In}(\mathfrak{X})$  is a smooth Artin stack of dimension 0.

The morphism  $f: \mathfrak{X} \rightarrow [*/G]$  is proper and smooth (because  $X$  is proper and smooth). Therefore by Theorem 4.4.1 there are morphisms  $f_{\natural}$  and  $f^{\natural}$  which push forward and pull back distributions. We have already seen that  $f_{\natural}$  is the natural ‘‘integration over fibers’’ map, so let us consider  $f^{\natural}$ . Both  $\mathrm{In}(\mathfrak{X})$  and  $\mathrm{In}[*/G]$  are connected smooth Artin stacks of dimension 0, so their dualizing complexes can be identified with the constant sheaf  $\Lambda$ . Thus  $f^{\natural}$  is an element of

$$\mathrm{Hom}_{D_{\acute{e}t}(\mathrm{In} \mathfrak{X}, \Lambda)}(\Lambda, \Lambda) = H^0(\mathrm{In} \mathfrak{X}, \Lambda) = \Lambda.$$

We claim that this element is 1. It suffices to check this over any open subset of  $\mathrm{In}[*/G] = [G//G]$ . Let  $[G//G]_{\mathrm{sr}}$  denote the open substack of strongly regular elements, and let  $\mathrm{In}(\mathfrak{X})_{\mathrm{sr}}$  be its preimage in  $\mathrm{In}(\mathfrak{X})$ . If  $T \subset P$  is a maximal torus, then  $[G//G]_{\mathrm{sr}} \cong [T_{\mathrm{sr}}//N_G(T)]$  and  $\mathrm{In}(\mathfrak{X})_{\mathrm{sr}} \cong [T_{\mathrm{sr}}//N_G(T) \cap P]$ . The restricted morphism  $\mathrm{In}(f): \mathrm{In}(\mathfrak{X})_{\mathrm{sr}} \rightarrow [G//G]_{\mathrm{sr}}$  is finite étale, with fibers  $W/W_M$ . Therefore by Lemma 4.4.2, the restriction of  $f^{\natural}$  over  $\mathrm{In}(\mathfrak{X})_{\mathrm{sr}}$  is the identity map, and the claim follows.

We can use these observations to calculate the Euler characteristic of  $X$ . The trace distribution of the constant sheaf  $\Lambda$  on  $[*/G]$  is  $1 \in \Lambda \cong \mathrm{Dist}([G//G], \Lambda)$ . By Theorem 4.4.1(2),

$$\mathrm{tr} \Lambda_{\mathfrak{X}} = \mathrm{tr} f^* \Lambda_{[*/G]} = f^{\natural} \mathrm{tr} \Lambda = 1$$

as well. By Theorem 4.4.1(1) we have

$$\mathrm{tr} f_* \Lambda_{\mathfrak{X}} = f_{\natural} \mathrm{tr} \Lambda_{\mathfrak{X}} = f_{\natural}(1).$$

This last quantity is  $\#W/W_M$ , since this is the degree of  $\mathrm{In}(f)$  on the strongly regular locus, and  $f_{\natural}$  is fiberwise summation on that locus. On the other hand,  $\mathrm{tr} f_* \Lambda_{\mathfrak{X}}$  is the dimension of the Euler characteristic of  $X$ . Therefore the Euler characteristic of  $X$  is  $\#W/W_M$ .

### 5.3 Some remarks on the affine Grassmannian

Once again,  $G$  is a connected reductive group, either over an algebraically closed field  $k$ , or else over a  $p$ -adic field  $F$  (in which case let  $C/F$  be an algebraically closed perfectoid field). Let  $T \subset G$  be a maximal torus,

Let  $\mathrm{Gr}_G = LG/L^+G$  denote either the classical affine Grassmannian  $G(F((t)))/G(F[[t]])$ , or else the  $B_{\mathrm{dR}}$ -Grassmannian  $G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$  [SW14]. Then  $\mathrm{Gr}_G$  is an ind-object in the category of schemes or locally spatial diamonds, which admits an action of  $L^+G$ . For a dominant cocharacter  $\mu \in X_*(T)$ , we have a bounded version  $\mathrm{Gr}_{G, \leq \mu}$ , which is a scheme or locally spatial diamond. When  $\mu$  is minuscule,  $\mathrm{Gr}_{G, \leq \mu}$  is isomorphic to the flag variety  $G/P_\mu$ .

**Example 5.3.1.** When  $G = T$  is a torus,  $\mathrm{Gr}_T$  is discrete, with points corresponding to cocharacters in  $X_*(T)$ .

We would like to generalize our calculations on flag varieties to a statement about the cohomology of  $\mathrm{Gr}_G$ . This involves the geometric Satake equivalence [MV07], which has been extended to the  $B_{\mathrm{dR}}$  case [FS]. This is an equivalence between tensor categories:  $L^+G$ -equivariant perverse sheaves on  $[\mathrm{Gr}_G/L^+G]$  on the one hand, and algebraic representations of the dual group  $\widehat{G}$ . Given a dominant cocharacter  $\mu \in X_*(T)$ , let  $r_\mu$  be the irreducible representation of  $\widehat{G}$  of highest weight  $\mu$ . Then the corresponding perverse sheaf on  $[\mathrm{Gr}_G/L^+G]$  is the intersection complex  $\mathrm{IC}_\mu$ , which is supported on  $\mathrm{Gr}_{G, \leq \mu}$ . Under this correspondence,  $H^*(\mathrm{Gr}_G, \mathrm{IC}_\mu)$  is supported in one degree, and has dimension  $\dim r_\mu$ .

Proceeding as in (5.1), we can try to apply the Lefschetz formula to the action of  $L^+G$  on  $H^*(\mathrm{Gr}_G, \mathrm{IC}_\mu)$ . We first need to investigate the nature of fixed points of elements of  $L^+G$  on  $\mathrm{Gr}_G$ .

**Proposition 5.3.2.** *Let  $g \in L^+T \subset L^+G$  be strongly regular, and let  $\mathrm{Gr}_G^g$  be the locus in  $\mathrm{Gr}_G$  fixed by  $g$ . The inclusion  $\mathrm{Gr}_G^g \rightarrow \mathrm{Gr}_G$  factors through an isomorphism  $\mathrm{Gr}_G^g \cong \mathrm{Gr}_T$ . Thus there is a bijection  $\lambda \mapsto L_\lambda$  between  $X_*(T)$  and points of  $\mathrm{Gr}_G^g$ .*

*Proof.* Let  $K = F((t))$  and  $K^+ = F[[t]]$ , or  $K = B_{\mathrm{dR}}$  and  $K^+ = B_{\mathrm{dR}}^+$ .

Let  $\mathcal{B}$  be the (reduced) Bruhat-Tits building of the split reductive group  $G_K$  over the discretely valued field  $K$ . Thus  $\mathcal{B}$  is a locally finite simplicial complex admitting an action of  $LG$ . We will identify the  $LG$ -set  $\mathrm{Gr}_G$  with a piece of this building.

By [BT84, 5.1.40] there exists a hyperspecial point  $\bar{o} \in \mathcal{B}$  corresponding to  $L^+G$ . The point  $\bar{o}$  can be characterized by [BT84, 4.6.29] as the unique

fixed point of  $L^+G$ . Let  $\mathcal{B}^{\text{ext}}$  be the extended Bruhat-Tits building of  $G_K$ . Recall that  $\mathcal{B}^{\text{ext}} = \mathcal{B} \times X_*(A_G)_{\mathbf{R}}$ , where  $A_G$  is the connected center of  $G$ . The group  $LG$  acts on  $X_*(A_G)_{\mathbf{R}}$  via the isomorphism  $X_*(A_G)_{\mathbf{R}} \rightarrow X_*(A'_G)_{\mathbf{R}}$ , where  $A'_G$  is the maximal abelian quotient of  $G$ . Let  $o = (\bar{o}, z)$  be any point in  $\mathcal{B}^{\text{ext}}$  lying over  $\bar{o}$ . Then  $L^+G$  can be characterized as the full stabilizer of  $o$  in  $G(\mathcal{B}_{\text{dR}})$ : It is clear that  $L^+G$  stabilizes  $o$ , and the reverse inclusion follows from the Cartan decomposition  $LG = L^+G \cdot X_*(T) \cdot L^+G$  (which relies on  $\bar{o}$  being hyperspecial) and the fact that  $X_*(T)$  acts on the apartment of  $T$  in  $\mathcal{B}^{\text{ext}}$  by translations. It follows that the action of  $LG$  on  $\mathcal{B}^{\text{ext}}$  provides an  $LG$ -equivariant bijection from  $\text{Gr}_G$  to the orbit of  $LG$  through  $o$ .

Now suppose  $x \in \text{Gr}_G$  is fixed by a strongly regular element  $g \in L^+T_{\text{sr}}$ . Then its image in  $\mathcal{B}^{\text{ext}}$  is a  $g$ -fixed point belonging to the orbit of  $o$ , and we can write  $x = ho$  for some  $h \in LG$ . For every root  $\alpha: T \rightarrow \mathbf{G}_m$ , the element  $\alpha(g)$  does not lie in the kernel of  $L^+\mathbf{G}_m \rightarrow \mathbf{G}_m$ . According to [Tit79, 3.6.1] the image of  $x$  in  $\mathcal{B}$  belongs to the apartment  $\mathcal{A}$  of  $T$ . At the same time,  $g \in L^+G$  also fixes  $\bar{o}$ , so for the same reason  $\bar{o} \in \mathcal{A}$ . Thus  $\bar{o}$  belongs to both apartments  $\mathcal{A}$  and  $h^{-1}\mathcal{A}$ . Since  $L^+G$  acts transitively on the apartments containing  $\bar{o}$  [BT84, 4.6.28], we can multiply  $h$  on the right by an element of  $L^+G$  to ensure that  $h^{-1}\mathcal{A} = \mathcal{A}$ . By [BT72, 7.4.10] we then have  $h \in L^+N(T, G)$ . Since  $\bar{o}$  is hyperspecial, every Weyl reflection is realized in  $L^+G$  and hence we may again modify  $h$  on the right to achieve  $h \in LT$ . We see now that  $x = ho$  is fixed by all of  $LT$  and that furthermore the coset  $x = hL^+G$  is the image of the coset  $hL^+T$ .  $\square$

Once again, suppose  $g \in L^+T_{\text{sr}}$ . Given  $\lambda \in X_*(T)$ , with corresponding fixed point  $L_\lambda$ , we can ask about the local terms  $\text{loc}_{L_\lambda}(g, \text{IC}_\mu)$ . By Lefschetz, we have

$$\sum_{\lambda \in X_*(T)} \text{loc}_{L_\lambda}(g, \text{IC}_\mu) = \dim H^*(\text{Gr}_G, \text{IC}_\mu) = \dim r_\mu.$$

A reasonable guess is that

$$\text{loc}_{L_\lambda}(g, \text{IC}_\mu) = \dim r_\mu[\lambda], \quad (5.3.1)$$

with  $r_\mu[\lambda]$  being the  $\lambda$ -weight space of  $r_\mu$ .

We can formalize this discussion in terms of stacks, as follows. We have a morphism of inertia stacks  $\text{In}[\text{Gr}_G/L^+G] \rightarrow \text{In}[* / L^+G]$ . Let  $\text{In}[\text{Gr}_G/L^+G]_{\text{sr}}$  be the open substack of  $\text{In}[\text{Gr}_G/L^+G]$  lying over  $\text{In}[* / L^+G]_{\text{sr}}$ . Then

$$\text{In}[\text{Gr}_G/L^+G]_{\text{sr}} \rightarrow \text{In}[* / L^+G]_{\text{sr}}$$

is étale and surjective, with fibers  $X_*(T)$ . Explicitly,

$$\begin{aligned} \mathrm{In}[\mathrm{Gr}_G/L^+G]_{\mathrm{sr}} &= \left[ \left\{ (x, g) \mid g \in L^+G_{\mathrm{sr}}, x \in \mathrm{Gr}_G^g \right\} / L^+G \right] \\ &\cong \left[ \left\{ (x, g) \mid g \in L^+T_{\mathrm{sr}}, x \in \mathrm{Gr}_G^g \right\} / L^+N_G(T) \right] \\ &\cong [(X_*(T) \times L^+T_{\mathrm{sr}}) / L^+N_G(T)]. \end{aligned}$$

The connected components of this stack are in bijection with  $X_*(T)/W$ , and each one is isomorphic to the 0-dimensional smooth Artin stack  $[L^+T_{\mathrm{sr}}/L^+N_G(T)]$ , whose dualizing sheaf is trivial. Therefore

$$\mathrm{Dist}(\mathrm{In}[\mathrm{Gr}_G/L^+G]_{\mathrm{sr}}, \Lambda) \cong H^0(X_*(T), \Lambda)^W \quad (5.3.2)$$

is the module of  $\Lambda$ -valued  $W$ -invariant functions on  $X_*(T)$ . For a strongly reflexive object  $\mathcal{F}$  of  $D_{\mathrm{ét}}([\mathrm{Gr}_G/L^+G], \Lambda)$ , let  $\mathrm{tr}_{\mathrm{sr}} \mathcal{F} \in H^0(X_*(T), \Lambda)$  be the corresponding  $W$ -invariant function.

We have the intersection complex  $\mathrm{IC}_\mu$ , an  $L^+G$ -equivariant perverse sheaf on  $\mathrm{Gr}_G$ . By Example 4.2.5,  $\mathrm{IC}_\mu$  descends to a perverse sheaf on the ind-Artin (v-)stack  $[\mathrm{Gr}_G/L^+G]$ , which we continue to call  $\mathrm{IC}_\mu$ .

**Theorem 5.3.3.** *The object  $\mathrm{IC}_\mu$  of  $D_{\mathrm{ét}}([\mathrm{Gr}_{G, \leq \mu}/L^+G], \Lambda)$  is Verdier self-dual and ULA with respect to the structure morphism  $[\mathrm{Gr}_{G, \leq \mu}/L^+G] \rightarrow [*/L^+G]$ . Therefore by Lemma 4.6.5 it is strongly reflexive. With respect to the isomorphism in (5.3.2), its trace distribution  $\mathrm{tr}_{\mathrm{sr}} \mathrm{IC}_\mu$  is the function  $\lambda \mapsto \dim r_\mu[\lambda]$ .*

**Remark 5.3.4.** Since the ULA condition is smooth local on the source, the first claim of the theorem is equivalent to the claim that  $\mathrm{IC}_\mu$  is ULA with respect to  $\mathrm{Gr}_{G, \leq \mu} \rightarrow *$ . This claim is trivial in the scheme setting (as any bounded constructible complex is ULA over a base which is a point), but nontrivial in the  $p$ -adic setting. The claim about trace distributions appears equally non-trivial in both settings, and our proof will treat both settings the same.

The proof of the theorem will occupy us for the next two subsections.

## 5.4 Minuscule and quasi-minuscule cocharacters

We work here with a fixed triple  $(G, B, T)$ , with  $T \subset G$  a maximal torus and  $B \subset G$  a Borel subgroup containing  $T$ . Let  $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \mathbf{Z}$  be the pairing between roots and coroots.

Let  $\Phi \subset X^*(T)$  be the set of roots of  $G$ , and let  $\Phi^+ \subset \Phi$  be the set of positive roots. Let  $\Phi^\vee \subset X_*(T)$  be the set of roots in the dual root system, and let  $(\Phi^\vee)^+$  be the set of positive coroots. There is a bijection  $\alpha \mapsto \alpha^\vee$  between  $\Phi$  and  $\Phi^\vee$ ; we have  $\langle \alpha, \alpha^\vee \rangle = 2$ .

The cone of  $X_*(T)^+$  of dominant cocharacters is

$$X_*(T)^+ = \left\{ \mu \in X_*(T) \mid \langle \alpha, \mu \rangle \geq 0 \ \forall \alpha \in \Phi^+ \right\}$$

The set  $X_*(T)$  is partially ordered under the relation defined as follows:  $\mu \leq \mu'$  if and only if  $\mu' - \mu \in X_*(T)$  lies in the monoid generated by  $(\Phi^\vee)^+$ .

For a dominant cocharacter  $\mu \in X_*(T)^+$ , let  $r_\mu$  denote the irreducible representation of  $\widehat{G}$  (with  $\Lambda$  coefficients) of highest weight  $\mu$ . For each  $\lambda \in X_*(T) = X^*(\widehat{T})$  we have the weight space  $r_\mu[\lambda]$ . The set of weights of  $r_\mu$  is

$$\Omega(\mu) = \left\{ \lambda \in X_*(T) \mid \forall w \in W, w\lambda \leq \mu \right\}.$$

Here  $W$  is the Weyl group of  $T$ .

Let  $M$  be the set of elements of  $X_*(T)^+ \setminus \{0\}$  that are minimal for the order on  $X_*(T)^+ \setminus \{0\}$  given by  $\leq$ .

**Lemma 5.4.1** ([NP01, Lemme 1.1]). *Let  $\mu \in M$ . Exactly one of the following is true:*

1. *For all  $\alpha \in \Phi$  we have  $\langle \alpha, \mu \rangle \in \{0, \pm 1\}$ . Then  $\mu$  is also minimal in  $X_*(T)^+$  and the set of weights of  $r_\mu$  is  $\Omega(\mu) = W\mu$ , the  $W$ -orbit of  $\mu$ .*
2. *There exists a unique root  $\gamma \in \Phi$  with  $\langle \gamma, \mu \rangle \geq 2$ . We have  $\mu = \gamma^\vee$  and  $\Omega(\mu) = W\mu \cup \{0\}$ .*

The cocharacter  $\mu$  is called *minuscule* or *quasi-minuscule*, respectively. We will first prove Theorem 5.3.3 in each of these cases, and then show how to deduce the general case using the convolution product.

Assume first that  $\mu$  is minuscule. Then  $\text{Gr}_{\leq \mu} = \text{Gr}_\mu$  is isomorphic to the flag variety  $G/P$ , where  $P$  is the parabolic subgroup associated to the set of roots  $\alpha \in \Phi$  with  $\langle \alpha, \mu \rangle \leq 0$  [NP01, Lemme 6.2]. Since  $G/P$  is smooth, and we have  $\text{IC}_\mu = \Lambda$  on  $G/P$  up to an even shift. Therefore  $\text{IC}_\mu$  is ULA with respect to the structure morphism.

The  $T$ -fixed points in  $G/P$  are in bijection with  $W\mu$ . In §5.1 we saw that the local term of  $\Lambda$  at each  $\lambda \in W\mu$  is 1. The multiplicity of each such  $\lambda$  in  $r_\mu$  is also 1.

Now assume that  $\mu$  is quasi-minuscule. A resolution of  $\mathrm{Gr}_{\leq\mu}$  is described in [NP01, §7]. The space  $\mathrm{Gr}_{G,\leq\mu}$  is the union of  $\mathrm{Gr}_{G,\mu}$  and a unique singular point  $e_0$ , corresponding to the class of the identity in  $L^+G$ . Let  $\gamma \in \Phi$  be the root from Lemma 5.4.1(2), and let  $\mathfrak{g}_\gamma$  be the weight space of  $\gamma$  in  $\mathfrak{g}$ . This space carries a natural action of  $P$ . Let  $\mathbb{L}_\gamma = G \times^P \mathfrak{g}_\gamma$ , a line bundle on  $G/P$ . Let us write  $s_0: G/P \rightarrow \mathbb{L}_\gamma$  for the zero section. This has a natural compactification  $\mathbb{L}_\gamma \hookrightarrow \mathbb{P}_\gamma$ , where  $\mathbb{P}_\gamma \rightarrow G/P$  is a  $\mathbb{P}^1$ -bundle over  $G/P$ ; this is obtained by adding a section at infinity. Thus there is an infinity section  $s_\infty: G/P \rightarrow \mathbb{P}_\gamma$ . There is an  $L^+G$ -equivariant isomorphism  $\mathbb{L}_\gamma \xrightarrow{\sim} \mathrm{Gr}_{G,\mu}$  which extends to a  $L^+G$ -equivariant morphism  $\pi: \mathbb{P}_\gamma \rightarrow \mathrm{Gr}_{G,\leq\mu}$ . This is a resolution of  $\mathrm{Gr}_{\leq\mu}$ ; the preimage of the singular point  $e_0$  is  $s_\infty(G/P)$ .

Up to a shift and twist, the derived pushforward of  $\Lambda$  through  $\pi$  is  $\mathrm{IC}_\mu \oplus \mathcal{C}$ , where  $\mathcal{C}$  is supported on  $e_0$  [NP01, Corollaire 8.2]. Since the ULA property is preserved under proper pushforwards and direct summands,  $\mathrm{IC}_\mu$  is ULA.

Let  $g \in L^+T_{\mathrm{sr}}$ . The fixed points of  $g$  on  $\mathrm{Gr}_{\leq\mu}$  are the points  $L_\lambda$ , where  $\lambda \in W\mu \cup \{0\}$ . We wish to calculate  $\mathrm{loc}_{L_\lambda}(g, \mathrm{IC}_\mu)$  for each  $\lambda$ .

First suppose  $\lambda = w\mu$  for an element  $w \in W$ . Then  $L_\lambda \in \mathrm{Gr}_\mu$  is non-singular, and  $\mathrm{IC}_\mu$  is isomorphic to the constant sheaf on it. With respect to the isomorphism  $\mathrm{Gr}_{G,\mu} \cong \mathbb{L}_\gamma$ , the point  $L_\lambda$  corresponds to  $s_0(we)$ , where  $e \in G/P$  is the identity coset. In a neighborhood of this point, there is a trivialization  $\mathbb{L}_\gamma \cong G/P \times \mathfrak{g}_\gamma$ , with respect to which the action of  $g$  corresponds to the action of  $(g, \gamma(g))$ . The local term of this action at the origin can be computed on each factor; it is 1 for the first factor (by the calculation we have already done) and 1 for the second factor (since  $\gamma(g) \neq 1$ , in turn since  $g$  is regular). Thus  $\mathrm{loc}_{L_\lambda}(g, \mathrm{IC}_\mu) = 1 = \dim r_\mu[\lambda]$  for each  $\lambda \in W\mu$ , which confirms Theorem 5.3.3 for those  $\lambda$ .

To deduce the result for  $\lambda = 0$ , we apply the global Lefschetz formula (valid because  $\mathrm{Gr}_{G,\leq\mu}$  is proper), together with the global result that  $\dim H^*(\mathrm{Gr}_{G,\leq\mu}, \mathrm{IC}_\mu) = \dim r_\mu$ , to deduce that  $\mathrm{loc}_{e_0}(g, \mathrm{IC}_\mu) = \dim r_\mu[0]$ .

## 5.5 The convolution product

Having proved Theorem 5.3.3 for minuscule and quasi-minuscule cocharacters, we will prove the general case by means of the convolution product. We review the definition of the convolution product, in terms of sheaves on the stack  $[\mathrm{Gr}_G / L^+G] = [L^+G \backslash LG / L^+G]$ . Let  $L^+G \backslash LG \times_{L^+G} LG / L^+G$  be the stacky quotient of  $L^+G \backslash LG \times LG / L^+G$  by the action of  $L^+G$  defined

by  $(x, y) \mapsto (xg^{-1}, gy)$ . Consider the diagram of stacks

$$\begin{array}{ccc} L^+G \backslash LG \times_{L^+G} LG/L^+G & \xrightarrow{\pi_2} & [L^+G \backslash LG/L^+G] \\ \pi_1 \downarrow & & \\ [L^+G \backslash LG/L^+G] \times [L^+G \backslash LG/L^+G] & & \end{array} \quad (5.5.1)$$

where  $\pi_1(x, y) = (xL^+G, L^+Gy)$  and  $\pi_2(x, y) = xy$ . Or, to abbreviate notation:

$$\begin{array}{ccc} \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G & \xrightarrow{\pi_2} & [\mathrm{Gr}_G / L^+G] \\ \pi_1 \downarrow & & \\ [\mathrm{Gr}_G / L^+G] \times [\mathrm{Gr}_G / L^+G] & & \end{array} \quad (5.5.2)$$

The convolution product is defined for objects  $\mathcal{F}_1, \mathcal{F}_2$  in  $D_{\text{ét}}([L^+G \backslash LG/L^+G])$ . It is

$$\mathcal{F}_1 * \mathcal{F}_2 = \pi_{2*} \pi_1^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

If  $\mathcal{F}_i$  is supported on  $\mathrm{Gr}_{G, \leq \mu_i}$  for  $i = 1, 2$ , then  $\mathcal{F}_1 * \mathcal{F}_2$  is supported on  $\mathrm{Gr}_{G, \leq \mu_1 + \mu_2}$ .

**Lemma 5.5.1.** *For  $i = 1, 2$ , let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be objects in  $D_{\text{ét}}([\mathrm{Gr}_{G, \leq \mu_i} / L^+G], \Lambda)$  which are ULA with respect to  $[\mathrm{Gr}_{G, \leq \mu_i} / L^+G] \rightarrow [*/L^+G]$ . Then  $\mathcal{F}_1 * \mathcal{F}_2$  is also ULA, and*

$$\mathrm{tr}_{\mathrm{sr}}(\mathcal{F}_1 * \mathcal{F}_2) = \mathrm{tr}_{\mathrm{sr}}(\mathcal{F}_1) * \mathrm{tr}_{\mathrm{sr}}(\mathcal{F}_2),$$

where the  $*$  on the right means convolution of functions on  $X_*(T)$ .

*Proof.* The morphism  $\pi_1$  is ind-smooth (being an  $L^+G$ -torsor), and the morphism  $\pi_2$  is ind-proper (its fibers are  $\mathrm{Gr}_G$ ). Therefore Theorem 4.4.1 applies: the convolution product preserves strong reflexivity, and we have

$$\mathrm{tr}(\mathcal{F}_1 * \mathcal{F}_2) = \pi_{2\sharp} \pi_1^\natural \mathrm{tr}(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

We therefore want to compute  $\pi_{2\sharp} \pi_1^\natural$  as a transfer of distributions from  $\mathrm{In}[\mathrm{Gr}_G / L^+G]^2$  to  $\mathrm{In}[\mathrm{Gr}_G / L^+G]$ , at least over the strongly regular locus in  $\mathrm{In}[*/L^+G]$ . The preimage of this locus in  $\mathrm{In}[\mathrm{Gr}_G / L^+G]^2$  is contained in  $(\mathrm{In}[\mathrm{Gr}_G / L^+G]_{\mathrm{sr}})^2$ .

Taking inertia stacks in (5.5.2) and restricting to strongly regular loci gives a diagram

$$\begin{array}{ccc} \mathrm{In}(\mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G)_{\mathrm{sr}} & \xrightarrow{\mathrm{In}(\pi_2)} & \mathrm{In}(\mathrm{Gr}_G / L^+G)_{\mathrm{sr}} \\ \mathrm{In}(\pi_1) \downarrow & & \\ \mathrm{In}(\mathrm{Gr}_G / L^+G)_{\mathrm{sr}} \times \mathrm{In}(\mathrm{Gr}_G / L^+G)_{\mathrm{sr}} & & \end{array}$$

in which  $\mathrm{In}(\mathrm{Gr}_G \widetilde{\times} \mathrm{Gr}_G)_{\mathrm{sr}}$  is defined as the preimage of  $\mathrm{In}(\mathrm{Gr}_G / L^+G)_{\mathrm{sr}} \times \mathrm{In}(\mathrm{Gr}_G / L^+G)_{\mathrm{sr}}$  under  $\mathrm{In}(\pi_2)$ . The objects above can be computed explicitly; they are

$$\begin{array}{ccc} [(X_*(T) \times X_*(T) \times L^+T_{\mathrm{sr}}) / L^+N_G(T)] & \longrightarrow & [(X_*(T) \times L^+T_{\mathrm{sr}}) / L^+T] \\ \downarrow & & \\ [(X_*(T) \times L^+T_{\mathrm{sr}}) / L^+N_G(T)] & & \end{array}$$

where the vertical arrow is the projection map, and the horizontal arrow is induced from the addition law  $X_*(T) \times X_*(T) \rightarrow X_*(T)$ . In particular,  $\mathrm{In}(\pi_1)$  is étale on this locus, so Lemma 4.4.2 applies. Applying  $\mathrm{Dist}(\cdot, \Lambda)$  to the above diagram gives

$$\begin{array}{ccc} H^0(X_*(T) \times X_*(T), \Lambda)^W & \xrightarrow{\pi_{2\mathfrak{h}}} & H^0(X_*(T), \Lambda)^W \\ \pi_1^{\mathfrak{h}} \uparrow & & \\ H^0(X_*(T), \Lambda)^W \times H^0(X_*(T), \Lambda)^W & & \end{array}$$

where  $\pi_1^{\mathfrak{h}}$  is the identity map on functions on  $X_*(T) \times X_*(T)$ , and  $\pi_{2\mathfrak{h}}$  is the convolution map. Thus the composite  $\pi_{2\mathfrak{h}}\pi_1^{\mathfrak{h}}$  is the convolution map as claimed.  $\square$

Lemma 5.5.1 extends in an obvious way to arbitrary convolution products of the form  $\mathcal{F}_1 * \cdots * \mathcal{F}_n$ . Therefore if  $\mu_1, \dots, \mu_n$  are minuscule or quasi-minuscule cocharacters, the convolution  $\mathrm{IC}_{\mu_1} * \cdots * \mathrm{IC}_{\mu_n}$  is ULA on  $[\mathrm{Gr}_G / L^+G]$ , and

$$\mathrm{tr}_{\mathrm{sr}}(\mathrm{IC}_{\mu_1} * \cdots * \mathrm{IC}_{\mu_n})(\lambda) = \dim(r_{\mu_1} \otimes \cdots \otimes r_{\mu_n})[\lambda] \quad (5.5.3)$$

for every  $\lambda \in X_*(T)$ .

Now suppose  $\mu \in X_*(T)_+$  is an arbitrary dominant cocharacter. By [NP01, Proposition 9.6],  $\mathrm{IC}_{\mu}$  is a direct summand of  $\mathrm{IC}_{\mu_1} * \cdots * \mathrm{IC}_{\mu_n}$ , where

each  $\mu_i$  is minuscule or quasi-minuscule. Therefore  $\mathrm{IC}_\mu$  is ULA. Furthermore,  $r_\mu$  can be expressed as a virtual sum of representations of the form  $r_{\mu_1} \otimes \cdots \otimes r_{\mu_n}$ ; this is implicit in the proof of [NP01, Lemma 10.3]. Applying (5.5.3) to this virtual sum shows that  $(\mathrm{tr}_{\mathrm{sr}} \mathrm{IC}_\mu)(\lambda) = \dim r_\mu[\lambda]$ . This concludes the proof of Theorem 5.3.3.

## 6 Application to the Hecke stacks

In this section we work exclusively in the diamond context. Let  $F/\mathbf{Q}_p$  be a finite extension, with residue field  $k$  having cardinality  $q$ .

Let  $G/F$  be a connected reductive group. If  $S = \mathrm{Spa} C$  is a geometric point (meaning that  $C$  is an algebraically closed perfectoid field containing  $k$ ), there is a bijection [Far15]

$$b \mapsto \mathcal{E}^b$$

between Kottwitz' set  $B(G)$  and isomorphism classes of  $G$ -bundles on  $X_C$ . Therefore a moduli stack of  $G$ -bundles would be a geometric version of  $B(G)$ .

**Definition 6.0.1.**  $\mathrm{Bun}_G$  is the v-stack over  $\mathrm{Spa} \bar{k}$  which assigns to a perfectoid space  $S$  the groupoid of  $G$ -bundles on  $X_S$ . Given a class  $b \in B(G)$ , let  $i_b: \mathrm{Bun}_G^b \rightarrow \mathrm{Bun}_G$  be the substack classifying  $G$ -bundles which are isomorphic to  $\mathcal{E}^b$  at every geometric point.

Then  $\mathrm{Bun}_G^b \cong [\mathrm{Spa} \bar{k} / \mathcal{J}_b]$ , where  $\mathcal{J}_b = \underline{\mathrm{Aut}} \mathcal{E}_b$  is a group diamond. If  $b$  is basic, then  $i_b$  is an open immersion, and  $\mathcal{J}_b = \underline{J}_b(F)$ .

**Theorem 6.0.2** ([FS]).  *$\mathrm{Bun}_G$  is a smooth Artin v-stack of dimension 0.*

Therefore by Lemma 4.1.4,  $\mathrm{Bun}_G$  admits a dualizing complex, and a notion of Verdier duality.

**Theorem 6.0.3** ([FS]). *Let  $b \in B(G)$  be a basic class, and let  $\rho$  be an irreducible admissible representation of  $J_b(F)$  with coefficients in  $\Lambda$ . Let  $\mathcal{L}_\rho$  be the corresponding object in  $D_{\acute{\mathrm{e}}\mathrm{t}}(\mathrm{Bun}_G^b, \Lambda)$ . Then  $i_{b*} \mathcal{L}_\rho$  is a strongly reflexive object of  $D_{\acute{\mathrm{e}}\mathrm{t}}(\mathrm{Bun}_G, \Lambda)$ .*

## 6.1 The Hecke correspondences on $\text{Bun}_G$

**Definition 6.1.1.** Let  $\mu$  be a dominant cocharacter of  $G$ , with reflex field  $E$ . The Hecke correspondence associated to  $\mu$  is a diagram

$$\begin{array}{ccc} & \text{Hecke}_{\leq \mu} & \\ h_1 \swarrow & & \searrow h_2 \\ \text{Bun}_G & & \text{Bun}_G \times \text{Spd } \check{E}. \end{array}$$

For a perfectoid space  $S$  over  $\bar{k}$ , the  $S$ -points of  $\text{Hecke}_{\leq \mu}$  are quadruples  $(\mathcal{E}_1, \mathcal{E}_2, S^\sharp, f)$ , where  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are  $G$ -bundles on  $X_S$ ,  $S^\sharp$  is an untilt of  $S$  over  $\check{E}$ , and

$$f: \mathcal{E}_2|_{X_S \setminus D_{S^\sharp}} \cong \mathcal{E}_1|_{X_S \setminus D_{S^\sharp}}$$

is an isomorphism. We require that  $f$  be fiberwise bounded by  $\mu$ .

Let  $S = \text{Spa}(R, R^+)$  be affinoid. The fiber of  $h_2$  over a pair  $(\mathcal{E}_2, S^\sharp)$  is the set of modifications of  $\mathcal{E}_2$  at  $D_{S^\sharp}$ . If we trivialize  $\mathcal{E}_2$  in a neighborhood of  $D_{S^\sharp}$ , then this set of modifications gets identified with

$$\text{Gr}_{G, \leq \mu}(S^\sharp) \subset G(B_{\text{dR}}(R^\sharp))/G(B_{\text{dR}}^+(R^\sharp)).$$

Thus  $h_2$  is a “ $\text{Gr}_{G, \leq \mu}$ -fibration”. This observation allows us to define an intersection complex  $\text{IC}_\mu \in D_{\text{ét}}(\text{Hecke}_{\leq \mu}, \Lambda)$ .

Define the Hecke operator as follows:

$$\begin{aligned} T_\mu: D_{\text{ét}}(\text{Bun}_G, \Lambda) &\rightarrow D_{\text{ét}}(\text{Bun}_G \times \text{Spd } \check{E}, \Lambda) \\ \mathcal{F} &\mapsto h_{2!}(h_1^* \mathcal{F} \otimes \text{IC}_\mu). \end{aligned}$$

The geometrization program outlined in [Far] conjectures the existence of an object  $\mathcal{F}_\phi \in D_{\text{ét}}(\text{Bun}_G, \Lambda)$  for each Langlands parameter  $\phi: W_F \rightarrow {}^L G$ . This is supposed to be a Hecke eigensheaf, in the sense that

$$T_\mu(\mathcal{F}_\phi) \cong \mathcal{F}_\phi \boxtimes (r_\mu \circ \phi_E)$$

for each dominant cocharacter  $\mu$ , where  $r_\mu \circ \phi_E$  was defined in the introduction.

## 6.2 Relation to local shtuka spaces

Fix now a basic class  $b \in B(G)$ . We explain here the relation between Hecke stacks  $\text{Hecke}_{\leq \mu}$  and the local shtuka spaces  $\text{Sht}_{G, b, \mu}$ .

At this point it will be convenient to restrict the stack  $\text{Hecke}_{\leq\mu}$  from perfectoid spaces over  $\bar{k}$  to perfectoid spaces over  $C$ , where  $C/F$  is an algebraically closed perfectoid field. We write  $\text{Hecke}_{\leq\mu,C}$  for this restriction. Thus if  $S/C$  is a perfectoid space, the curve  $X_{S^b}$  comes with a distinguished divisor  $\infty$  (corresponding to the untild  $S$  of  $S^b$ ), and then  $\text{Hecke}_{\leq\mu,C}(S)$  classifies triples  $(\mathcal{E}_1, \mathcal{E}_2, f)$ , where the  $\mathcal{E}_i$  are  $G$ -bundles on  $X_{S^b}$ , and  $f$  is an isomorphism between  $\mathcal{E}_1$  and  $\mathcal{E}_2$  away from  $\infty$ , which is bounded by  $\mu$  fiberwise on  $S$ . We write  $h_1, h_2: \text{Hecke}_{\leq\mu,C} \rightarrow \text{Bun}_{G,C}$  for the projections onto  $\mathcal{E}_1$  and  $\mathcal{E}_2$ .

We have a commutative diagram of stacks

$$\begin{array}{ccccc}
& & & \xrightarrow{h_2^a} & \\
& & & \text{Hecke}_{\leq\mu,C}^{b,1} & \xrightarrow{i_b} & \text{Hecke}_{\leq\mu,C}^{-,1} & \xrightarrow{h_2} & \text{Bun}_{G,C}^1 & \\
& & & \downarrow i_1 & & \downarrow i_1 & & \downarrow i_1 & \\
& & & \text{Hecke}_{\leq\mu,C}^{b,-} & \xrightarrow{i_b} & \text{Hecke}_{\leq\mu,C} & \xrightarrow{h_2} & \text{Bun}_{G,C} & \\
& & & \downarrow h_1 & & \downarrow h_1 & & & \\
& & & \text{Bun}_{G,C}^b & \xrightarrow{i_b} & \text{Bun}_{G,C} & & & 
\end{array} \tag{6.2.1}$$

in which all squares are cartesian, the morphisms labeled  $i_1$  and  $i_b$  are open immersions, and the morphisms labeled  $h_1$  and  $h_2$  are proper. We can identify the leftmost column rather explicitly, via the diagram

$$\begin{array}{ccc}
[\text{Gr}_{G,\leq\mu}^{b,1} / \underline{J_b(F)}] & \xrightarrow{\cong} & \text{Hecke}_{\leq\mu,C}^{b,1} \\
\downarrow & & \downarrow i_1 \\
[\text{Gr}_{G,\leq\mu}^b / \underline{J_b(F)}] & \xrightarrow{\cong} & \text{Hecke}_{\leq\mu,C}^{b,-} \\
\downarrow & & \downarrow h_1 \\
[* / \underline{J_b(F)}] & \xrightarrow{\cong} & \text{Bun}_{G,C}^b.
\end{array}$$

Explanation:  $\text{Gr}_{G,\leq\mu}^b$  assigns to a perfectoid  $C$ -algebra  $(R, R^+)$  the set of pairs  $(\mathcal{E}, \gamma)$ , where  $\mathcal{E}$  is a  $G$ -bundle, and  $\gamma: \mathcal{E}|_{X_R \setminus D_R} \cong \mathcal{E}^b|_{X_R \setminus D_R}$  is an isomorphism, which is bounded by  $\mu$  at  $D_R$  pointwise on  $\text{Spa}(R, R^+)$ . If we choose a trivialization of  $\mathcal{E}^b$  at  $D_R$ , then we obtain an isomorphism  $\text{Gr}_{G,\leq\mu}^b \cong \text{Gr}_{G,\leq\mu}$  onto the  $B_{\text{dR}}$ -Grassmannian of §5.3.

Within  $\mathrm{Gr}_{G,\leq\mu}^b$ , we have the open *admissible locus*  $\mathrm{Gr}_{G,\leq\mu}^{b,1}$  consisting of pairs  $(\mathcal{E}, \gamma)$ , where  $\mathcal{E}$  is everywhere isomorphic to  $\mathcal{E}^1$ .

Similarly, the top row of (6.2.1) can be identified as:

$$\begin{array}{ccccc} [\mathrm{Gr}_{G,\leq\mu}^{1,b}/\underline{G(F)}] & \longrightarrow & [\mathrm{Gr}_{G,\leq\mu}^1/\underline{G(F)}] & \longrightarrow & [*/\underline{G(F)}] \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \mathrm{Hecke}_{\leq\mu,C}^{b,1} & \xrightarrow{i_b} & \mathrm{Hecke}_{\leq\mu,C}^{-,1} & \xrightarrow{h_2} & \mathrm{Bun}_{G,C}^1. \end{array}$$

Here,  $\mathrm{Gr}_{G,\leq\mu}^1$  assigns to a perfectoid  $C$ -algebra  $(R, R^+)$  the set of pairs  $(\mathcal{E}, \gamma)$ , where  $\mathcal{E}$  is a  $G$ -bundle on  $X_R$ , and  $\gamma: \mathcal{E}^1|_{X_R \setminus D_R} \cong \mathcal{E}|_{X_R \setminus D_R}$  is an isomorphism, subject to the following boundedness condition. If we choose a trivialization  $i: \mathcal{E} \rightarrow \mathcal{E}^1$  over a formal neighborhood of  $D_R$ , then  $i \circ \gamma$  is a well-defined element  $g \in [L^+G \setminus LG]$ . We require that  $g^{-1} \in \mathrm{Gr}_{G,\leq\mu}$ .

Within  $\mathrm{Gr}_{G,\leq\mu}^1$  we have the open locus  $\mathrm{Gr}_{G,\leq\mu}^{1,b}$ , consisting of those pairs  $(\mathcal{E}, \gamma)$ , where  $\mathcal{E}$  is everywhere isomorphic to  $\mathcal{E}^b$ .

We have identified  $\mathrm{Hecke}_{\leq\mu}^{b,1}$  as a quotient of an admissible locus in two ways:

$$\mathrm{Hecke}_{\leq\mu,C}^{b,1} \cong [\mathrm{Gr}_{G,\leq\mu}^{b,1}/\underline{J_b(F)}] \cong [\mathrm{Gr}_{G,\leq\mu}^{1,b}/\underline{G(F)}].$$

Therefore there exists is a diamond  $\mathrm{Sht}_{G,b,\mu}$  over  $C$  admitting an action of  $\underline{G(F)} \times \underline{J_b(F)}$ , and a diagram

$$\begin{array}{ccc} & \mathrm{Sht}_{G,b,\mu} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathrm{Gr}_{G,\leq\mu}^{b,1} & & \mathrm{Gr}_{G,\leq\mu}^{1,b} \end{array}$$

in which  $\pi_1$  is a  $\underline{J_b(F)}$ -equivariant  $\underline{G(F)}$ -torsor, and  $\pi_2$  is a  $\underline{G(F)}$ -equivariant  $\underline{J_b(F)}$ -torsor. This is the space of (infinite-level) mixed-characteristic shtukas as defined by Scholze [SW14, Corollary 23.2.2], and whose definition was recalled in §2.4 (We have suppressed the subscript  $\infty$ , because we will have no use for the moduli spaces at finite level.) Let  $\rho$  be an irreducible admissible representation with coefficients in  $\Lambda = \mathcal{O}_L$ , where  $L/\mathbf{Q}_\ell$  is a finite extension. It corresponds to a reflexive object  $\mathcal{L}_\rho$  of  $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda)$ . Recall the cohomology complex  $H^*(G, b, \mu)[\rho]$  from Definition 2.4.3:

$$H^*(G, b, \mu)[\rho] = \varinjlim_{K \subset G(F)} R\mathrm{Hom}_{J_b(F)}(R\Gamma_c(\mathrm{Sht}_{G,b,\mu}/\underline{K}, \mathrm{IC}_\mu), \rho),$$

which lives in the derived category of smooth representations of  $G(F)$  with coefficients in  $\Lambda$ .

We abuse notation by writing “ $\mathrm{IC}_\mu$ ” for many objects. For instance,  $\mathrm{IC}_\mu$  refers to an object living on  $\mathrm{Gr}_{G,\leq\mu}^1$ , but also to its descent to  $[\mathrm{Gr}_{G,\leq\mu}^1/\underline{K}]$ , and also for the pullback of the latter to  $\mathrm{Sht}_{G,b,\mu}/\underline{K}$ . This abuse of notation also appears in the proof of the following proposition.

**Proposition 6.2.1.** *The object  $i_1^*T_\mu i_{b*}\mathcal{L}_\rho \in D_{\acute{e}t}(\mathrm{Bun}_{G,C}^1, \Lambda)$  corresponds to the  $\Lambda G(F)$ -representation  $H^*(G, b, \mu)[\rho]$ .*

*Proof.* The representation of  $G(F)$  corresponding to  $i_1^*T_\mu i_{b*}\mathcal{L}_\rho$  is the colimit of  $R\Gamma([*/K], i_1^*T_\mu i_{b*}\mathcal{L}_\rho)$  as  $K \subset G(F)$  runs through open compact subgroups. For each such  $K$ , we have a diagram of v-stacks

$$\begin{array}{ccccc}
[\mathrm{Gr}_{G,\leq\mu}^{1,b}/\underline{K}] & \xrightarrow{i'_b} & [\mathrm{Gr}_{G,\leq\mu}^1/\underline{K}] & \xrightarrow{h_{2K}} & [*/\underline{K}] \\
\downarrow q' & & \downarrow i'_{1K} & & \downarrow i_{1K} \\
& & \mathrm{Hecke}_{\leq\mu,C} & \xrightarrow{h_2} & \mathrm{Bun}_{G,C} \\
& & \downarrow h_1 & & \\
[*/\underline{J}_b(F)] & \xrightarrow{i_b} & \mathrm{Bun}_{G,C} & & 
\end{array}$$

with both rectangles cartesian. (Here  $i_K$  is the composition of  $[*/\underline{K}] \rightarrow [*/\underline{G}(F)] \cong \mathrm{Bun}_G^1$  with  $i_1$ .) In the following calculation, we use the property that  $\mathrm{IC}_\mu \in D_{\acute{e}t}(\mathrm{Hecke}_{\leq\mu,C}, \Lambda)$  is Verdier self-dual and ULA with respect to  $h_1$ , so that for any object  $\mathcal{F}$  of  $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ , we have

$$\begin{aligned}
T_\mu \mathcal{F} &\cong h_{2!}(\mathrm{IC}_\mu \otimes h_1^* \mathcal{F}) \\
&\cong h_{2!}(\mathbf{D} \mathrm{IC}_\mu \otimes h_1^* \mathcal{F}) \\
&\cong h_{2!}(\underline{\mathrm{RHom}}_{\mathrm{Hecke}_{\leq\mu,C}}(\mathrm{IC}_\mu, h_1^! \Lambda) \otimes h_1^* \mathcal{F}) \\
&\cong h_{2!} \underline{\mathrm{RHom}}_{\mathrm{Hecke}_{\leq\mu,C}}(\mathrm{IC}_\mu, h_1^! \mathcal{F})
\end{aligned}$$

The derived sections of  $i_1^*T_\mu i_{b*}\mathcal{L}_\rho$  over  $[*/\underline{K}]$  are

$$\begin{aligned}
R\Gamma([*/\underline{K}], i_K^*T_\mu i_{b*}\mathcal{L}_\rho) &\cong R\Gamma([*/\underline{K}], i_K^*h_{2!} \underline{\mathrm{RHom}}_{\mathrm{Hecke}_{\leq\mu,C}}(\mathrm{IC}_\mu, h_1^! i_{b*}\mathcal{L}_\rho)) \\
&\cong R\Gamma([*/\underline{K}], h_{2K}!(i'_{1K})^* \underline{\mathrm{RHom}}_{\mathrm{Hecke}_{\leq\mu,C}}(\mathrm{IC}_\mu, h_1^! i_{b*}\mathcal{L}_\rho)).
\end{aligned}$$

Since  $h_{2K}$  is proper and  $i'_{1K}$  is étale, we may rewrite this as

$$\begin{aligned}
R\Gamma([*/\underline{K}], i_K^* T_\mu i_{b*} \mathcal{L}_\rho) &\cong R\Gamma([*/\underline{K}], h_{2K*} i'_{1K} \underline{\mathrm{RHom}}_{\mathrm{Hecke}_{\leq \mu, C}}(\mathrm{IC}_\mu, h_1^! i_{b*} \mathcal{L}_\rho)) \\
&\cong R\Gamma([\mathrm{Gr}_{G, \leq \mu}^1 / \underline{K}], \underline{\mathrm{RHom}}_{[\mathrm{Gr}_{G, \leq \mu}^1 / \underline{K}]}(\mathrm{IC}_\mu, q^! i_{b*} \mathcal{L}_\rho)) \\
&\cong \mathrm{RHom}_{[\mathrm{Gr}_{G, \leq \mu}^1 / \underline{K}]}(\mathrm{IC}_\mu, q^! i_{b*} \mathcal{L}_\rho) \\
&\cong \mathrm{RHom}_{\mathrm{Bun}_G}(q^! \mathrm{IC}_\mu, i_{b*} \mathcal{L}_\rho) \\
&\cong \mathrm{RHom}_{[* / \underline{J}_b(F)]}(i_b^* q^! \mathrm{IC}_\mu, \mathcal{L}_\rho) \\
&\cong \mathrm{RHom}_{[* / \underline{J}_b(F)]}(q^! \mathrm{IC}_\mu, \mathcal{L}_\rho) \\
&\cong \underline{\mathrm{RHom}}_{\underline{J}_b(F)}(R\Gamma_c(\mathrm{Sht}_{G, b, \mu} / K, \mathrm{IC}_\mu), \rho)
\end{aligned}$$

where in the last step we used the cartesian diagram

$$\begin{array}{ccc}
\mathrm{Sht}_{G, b, \mu} / K & \longrightarrow & [\mathrm{Gr}_{G, \leq \mu}^{1, b} / \underline{K}] \\
\downarrow & & \downarrow q' \\
* & \longrightarrow & [* / \underline{J}_b(F)]
\end{array}$$

Taking the direct limit over  $K$  gives the result.  $\square$

### 6.3 The inertia stack of the Hecke stack: local structure

Consider the v-stack

$$\mathrm{Hecke}_{\leq \mu, C}^{\mathrm{loc}} = [\mathrm{Gr}_{\leq \mu} / L^+ G] = [L^+ G \backslash LG / L^+ G].$$

For a perfectoid  $C$ -algebra  $R$ , this classifies pairs of  $G$ -bundles  $\mathcal{E}_1, \mathcal{E}_2$  over  $B_{\mathrm{dR}}^+(R)$  together with an isomorphism over  $B_{\mathrm{dR}}(R)$  which is bounded by  $\mu$  fiberwise on  $\mathrm{Spa} R$ . Whereas,  $\mathrm{Bun}_{G, C}^{\mathrm{loc}} = [* / L^+ G]$  classifies  $G$ -bundles over  $B_{\mathrm{dR}}^+(R)$ . We have a cartesian diagram of v-stacks over  $* = \mathrm{Spa} C$ :

$$\begin{array}{ccc}
\mathrm{Hecke}_{\leq \mu, C} & \xrightarrow{h_i} & \mathrm{Bun}_{G, C} \\
\downarrow & & \downarrow \\
\mathrm{Hecke}_{\leq \mu, C}^{\mathrm{loc}} & \xrightarrow{h_i^{\mathrm{loc}}} & \mathrm{Bun}_{G, C}^{\mathrm{loc}}
\end{array} \tag{6.3.1}$$

for  $i = 1, 2$ , in which the vertical maps are smooth. We write  $\mathrm{IC}_\mu$  for both the object in  $D_{\mathrm{ét}}(\mathrm{Hecke}_{\leq \mu, C}^{\mathrm{loc}}, \Lambda)$  and for its pullback to  $\mathrm{Hecke}_{\leq \mu, C}$ .

We will now apply our results on the behavior of the trace distribution under exterior tensor products and proper pushforwards (in Proposition 4.6.6 and Theorem 4.4.1, respectively) to compare  $\text{tr } \mathcal{F}$  and  $\text{tr } T_\mu \mathcal{F}$ . For this we will have to determine the structure of the inertia stack  $\text{In}(\text{Hecke}_{\leq \mu, C})$ , or at least a certain part of it (the strongly regular locus).

Taking inertia stacks in (6.3.1) gives a cartesian diagram

$$\begin{array}{ccc} \text{In}(\text{Hecke}_{\leq \mu, C}) & \xrightarrow{\text{In}(h_1)} & \text{In}(\text{Bun}_{G, C}) \\ \downarrow & & \downarrow \\ \text{In}([\text{Gr}/L^+G]) & \xrightarrow{\text{In}(h)} & \text{In}([\ast/L^+G]) \end{array}$$

Now let  $b \in B(G)$  be basic. Within  $\text{In}(\text{Bun}_{G, C})$  we have the open subsets

$$\begin{aligned} \text{In}(\text{Bun}_{G, C}^1) &\cong [G(F) // G(F)] \\ \text{In}(\text{Bun}_{G, C}^b) &\cong [\underline{J_b}(F) // \underline{J_b}(F)] \end{aligned}$$

and, within these, we have the open subsets

$$\begin{aligned} \text{In}(\text{Bun}_{G, C}^1)_{\text{sr}} &\cong [G(F)_{\text{sr}} // G(F)] \\ \text{In}(\text{Bun}_{G, C}^b)_{\text{sr}} &\cong [\underline{J_b}(F)_{\text{sr}} // \underline{J_b}(F)] \end{aligned}$$

(recall that sr means strongly regular).

Let  $\text{In}(\text{Hecke}_{\leq \mu, C}^{-, 1})_{\text{sr}}$  be the pullback of  $\text{In}(\text{Bun}_{G, C}^1)_{\text{sr}}$  through  $\text{In}(h_2)$ . Similarly, let  $\text{In}(\text{Hecke}_{\leq \mu, C}^{b, -})_{\text{sr}}$  be the pullback of  $\text{In}(\text{Bun}_{G, C}^b)_{\text{sr}}$  through  $\text{In}(h_1)$ . Given our prior description of  $\text{In}(\text{Hecke}_{\leq \mu, C}^{\text{loc}})_{\text{sr}}$  from §5.3, we have

$$\begin{aligned} \text{In}(\text{Hecke}_{\leq \mu, C}^{-, 1})_{\text{sr}} &\cong \left[ \underline{I \text{Gr}}_{\leq \mu, \text{sr}}^1 / G(F) \right] \\ \text{In}(\text{Hecke}_{\leq \mu, C}^{b, -})_{\text{sr}} &\cong \left[ \underline{I \text{Gr}}_{\leq \mu, \text{sr}}^b / \underline{J_b}(F) \right], \end{aligned}$$

where

$$\begin{aligned} \underline{I \text{Gr}}_{\leq \mu, \text{sr}}^1 &= \left\{ (g, \lambda) \mid g \in G(F)_{\text{sr}}, \lambda \in X_*(T_g)_{\leq \mu} \right\} \\ \underline{I \text{Gr}}_{\leq \mu, \text{sr}}^b &= \left\{ (g', \lambda') \mid g' \in J_b(F)_{\text{sr}}, \lambda' \in X_*(T_{g'})_{\leq \mu} \right\}. \end{aligned}$$

As usual we define  $T_g = \text{Cent}(g, G)$  and  $T_{g'} = \text{Cent}(g', J_b)$ .

Recall the Hecke operator  $T_\mu(\mathcal{F}) = h_{2!}(h_1^*\mathcal{F} \otimes \mathrm{IC}_\mu)$ , defined for objects  $\mathcal{F}$  of  $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ . Ultimately we will calculate its effect on trace distributions, at least when restricted to the appropriate loci in  $\mathrm{In}(\mathrm{Bun}_G)$ . But first we need to know that  $T_\mu$  preserves strong reflexivity.

**Lemma 6.3.1.** *Let  $\mathcal{F}$  be a strongly reflexive object in  $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ . Then  $T_\mu\mathcal{F}$  is also strongly reflexive.*

*Proof.* By Proposition 4.6.6, the object  $h_1^*\mathcal{F} \otimes \mathrm{IC}_\mu = \mathcal{F} \boxtimes_{[*]/L+G} \mathrm{IC}_\mu$  is strongly reflexive, and then  $T_\mu\mathcal{F} = h_{2!}(h_1^*\mathcal{F} \otimes \mathrm{IC}_\mu)$  is strongly reflexive by Theorem 4.4.1.  $\square$

Proposition 4.6.6 shows that

$$\mathrm{tr}(\mathcal{F} \boxtimes \mathrm{IC}_\mu) = \mathrm{tr} \mathcal{F} \otimes \mathrm{tr} \mathrm{IC}_\mu$$

as elements of  $\mathrm{Dist}(\mathrm{In}(\mathrm{Hecke}_{\leq\mu, C}), \Lambda)$ . When we restrict this to  $\mathrm{In}(\mathrm{Hecke}_{\leq\mu, C}^{b, -})_{\mathrm{sr}}$  and unravel definitions, we obtain the following result. We let  $D_{\acute{e}t}(\mathfrak{X}, \Lambda)^{\mathrm{st.ref.}}$  denote the full subcategory of strongly reflexive objects of  $D_{\acute{e}t}(\mathfrak{X}, \Lambda)$ .

**Lemma 6.3.2.** *The following diagram commutes:*

$$\begin{array}{ccc} D_{\acute{e}t}(\mathrm{Bun}_{G, C}, \Lambda)^{\mathrm{st.ref.}} & \xrightarrow{\boxtimes \mathrm{IC}_\mu} & D_{\acute{e}t}(\mathrm{Hecke}_{\leq\mu, C}, \Lambda)^{\mathrm{st.ref.}} \\ \mathrm{tr} \downarrow & & \downarrow \mathrm{tr} \\ \mathrm{Dist}(\mathrm{In}(\mathrm{Bun}_G), \Lambda) & \xrightarrow{\otimes \mathrm{tr}(\mathrm{IC}_\mu)} & \mathrm{Dist}(\mathrm{In}(\mathrm{Hecke}_{\leq\mu, C}), \Lambda) \\ \mathrm{res} \downarrow & & \downarrow \mathrm{res} \\ \mathrm{Dist}(\mathrm{In}(\mathrm{Bun}_G^b)_{\mathrm{sr}}, \Lambda) & \xrightarrow{\otimes \mathrm{tr}(\mathrm{IC}_\mu)} & \mathrm{Dist}(\mathrm{In}(\mathrm{Hecke}_{\leq\mu, C}^{b, -})_{\mathrm{sr}}, \Lambda) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Dist}(J_b(F)_{\mathrm{sr}}, \Lambda)^{J_b(F)} & \xrightarrow{K_\mu \cdot \pi_1^*} & \mathrm{Dist}(I \mathrm{Gr}_{\leq\mu, \mathrm{sr}}^b, \Lambda)^{J_b(F)} \end{array}$$

Here, the top arrow is  $\mathcal{F} \mapsto h_1^*(\mathcal{F}) \otimes \mathrm{IC}_\mu = \mathcal{F} \boxtimes_{\mathrm{Bun}_{G, C}^{\mathrm{loc}}} \mathrm{IC}_\mu$ , and the bottom arrow is dual to the composition

$$C_c(I \mathrm{Gr}_{\leq\mu, \mathrm{sr}}^b, \Lambda)_{J_b(F)} \xrightarrow{\cdot K_\mu} C_c(I \mathrm{Gr}_{\leq\mu, \mathrm{sr}}^b, \Lambda)_{J_b(F)} \xrightarrow{\pi_{1*}} C_c(J_b(F)_{\mathrm{sr}}, \Lambda)_{J_b(F)},$$

where  $K_\mu$  is the function  $(g, \lambda) \mapsto \dim r_\mu[\lambda]$ , and  $\pi_{1*}$  is the ‘‘sum-over-fibers’’ map induced from  $\pi_1 : I \mathrm{Gr}_{\leq\mu, \mathrm{sr}}^b \rightarrow J_b(F)_{\mathrm{sr}}$ .

We need a corresponding result for the map  $h_2: \text{Hecke}_{\leq \mu, C} \rightarrow \text{Bun}_{G, C}$ . This result is an application of the compatibility of the trace distribution with proper pushforwards, Theorem 4.4.1.

**Lemma 6.3.3.** *The following diagram commutes:*

$$\begin{array}{ccc}
D_{\text{ét}}(\text{Hecke}_{\leq \mu, C}, \Lambda)^{\text{st.ref.}} & \xrightarrow{h_{2!}} & D_{\text{ét}}(\text{Bun}_{G, C}, \Lambda)^{\text{st.ref.}} \\
\text{tr} \downarrow & & \downarrow \text{tr} \\
\text{Dist}(\text{In}(\text{Hecke}_{\leq \mu, C}, \Lambda)) & \xrightarrow{h_{2!}} & \text{Dist}(\text{In}(\text{Bun}_{G, C}), \Lambda) \\
\text{res} \downarrow & & \downarrow \text{res} \\
\text{Dist}(\text{In}(\text{Hecke}_{\leq \mu, C}^{-, 1}_{\text{sr}}, \Lambda)) & \xrightarrow{h_{2!}} & \text{Dist}(\text{In}(\text{Bun}_{G, C}^1)_{\text{sr}}, \Lambda) \\
\cong \downarrow & & \downarrow \cong \\
\text{Dist}(I \text{Gr}_{\leq, \text{sr}}^1, \Lambda)^{G(F)} & \xrightarrow{\pi_{2*}} & \text{Dist}(G(F), \Lambda)^{G(F)}
\end{array}$$

where the bottom arrow is dual to the pullback map

$$\pi_2^*: C_c(G(F)_{\text{sr}}, \Lambda)_{G(F)} \rightarrow C_c(I \text{Gr}_{\leq \mu, \text{sr}}^1, \Lambda)_{G(F)}.$$

#### 6.4 The inertia stack of the Hecke stack: global structure

We now apply our analysis to diagrams in which  $h_1$  and  $h_2$  appear simultaneously, to wit

$$\begin{array}{ccc}
& \text{Hecke}_{\leq \mu, C}^{b, 1} & \\
h_1^a \swarrow & & \searrow h_2^a \\
\text{Bun}_{G, C}^b & & \text{Bun}_{G, C}^1
\end{array} \tag{6.4.1}$$

Recall the local shtuka space  $\text{Sht}_{G, b, \mu}$ . In terms of  $\text{Sht}_{G, b, \mu}$ , the diagram in (6.4.1) becomes

$$\begin{array}{ccc}
& [\text{Sht}_{G, b, \mu} / (G(F) \times J_b(F))] & \\
\cong \swarrow & & \searrow \cong \\
[\text{Gr}_{G, \leq \mu}^{b, 1} / J_b(F)] & & [\text{Gr}_{G, \leq \mu}^{1, b} / G(F)] \\
\downarrow & & \downarrow \\
[* / J_b(F)] & & [* / G(F)],
\end{array} \tag{6.4.2}$$

where the diagonal isomorphisms are induced from the  $\underline{G(F)}$ -torsor  $\text{Sht}_{G,b,\mu} \rightarrow \text{Gr}_{G,\leq\mu}^{b,1}$  and the  $\underline{J_b(F)}$ -torsor  $\text{Sht}_{G,b,\mu} \rightarrow \text{Gr}_{G,\leq\mu}^{1,b}$ , respectively.

We intend to pass to inertia stacks in (6.4.2), and so we must understand the nature of the fixed points of the action of  $\underline{G(F)} \times \underline{J_b(F)}$  on  $\text{Sht}_{G,b,\mu}$ . It will help to introduce some notation. Suppose  $\mathcal{E}$  is a  $G$ -bundle on  $X_{C^b}$  equipped with a trivialization at  $\infty = D_C$ . Let  $T \subset G$  be a maximal torus. We have seen in Proposition 5.3.2 that there is a bijection  $\lambda \mapsto L_\lambda$  between  $X_*(T)$  and the set of  $T$ -fixed points of  $\text{Gr}_G$ . Given  $\lambda \in X_*(T)$ , we let  $\mathcal{E}[\lambda]$  be the modification of  $\mathcal{E}$  corresponding to  $L_\lambda$ .

**Lemma 6.4.1.** *Let  $\mathcal{E}$  be a  $G$ -bundle on  $X_C$ , let  $T \subset G$  be a maximal torus, let  $\lambda \in X_*(T)$  be a cocharacter, and let  $\widehat{\lambda} \in X^*(\widehat{T})$  be the corresponding character. In the group  $X^*(Z(\widehat{G})^\Gamma)$  we have*

$$\kappa(\mathcal{E}[\lambda]) = \kappa(\mathcal{E}) - \widehat{\lambda}|_{Z(\widehat{G})^\Gamma}.$$

*Proof.* The proof is a devissage argument based on the functoriality of the Kottwitz map  $\kappa$  and the following easily established fact: If  $f: G \rightarrow H$  is a homomorphism of reductive groups, write  $f_*$  for the functor  $\mathcal{E} \mapsto H \times_{G,f} \mathcal{E}$  carrying  $G$ -bundles to  $H$ -bundles. Then for a  $G$ -bundle  $\mathcal{E}$  we have

$$f_*(\mathcal{E}[\lambda]) \cong (f_*\mathcal{E})[f \circ \lambda]. \quad (6.4.3)$$

**Step 0:**  $G = \mathbf{G}_m$ . In this case  $\kappa: B(\mathbf{G}_m) \rightarrow \mathbf{Z}$  is an isomorphism, which agrees with the degree map on vector bundles. The claim reduces to the fact that modifying a line bundle by the cocharacter  $t \mapsto t^n$  reduces its degree by  $n$ .

**Step 1:**  $G = T = \text{Res}_{E/F} \mathbf{G}_m$  for a finite extension  $E/F$ . In this case  $X^*(\widehat{T}) = X_*(T)$  is the group ring  $\mathbf{Z}[\Gamma_{E/F}]$ . The norm maps  $N: T \rightarrow \mathbf{G}_m$  and  $N: \mathbf{Z}[\Gamma_{E/F}] \rightarrow \mathbf{Z}$  fit into the commutative diagram

$$\begin{array}{ccc} B(T) & \xrightarrow{N} & B(\mathbf{G}_m) \\ \kappa \downarrow & & \downarrow \kappa \\ X^*(\widehat{T}^\Gamma) & \xrightarrow{N} & \mathbf{Z} \end{array}$$

and all four maps are isomorphisms. The claim follows from (6.4.3) and Step 0.

**Step 2:  $G = T$  is a torus.** Let  $E/F$  be the splitting field of  $T$  and  $M$  a free  $\mathbf{Z}[\Gamma_{E/F}]$ -module together with a  $\Gamma$ -equivariant surjection  $M \rightarrow X_*(T)$ . If  $S$  is the torus with  $X_*(S) = M$  then  $S$  is a product of tori of the form  $\text{Res}_{E/F} \mathbf{G}_m$  and we have a surjection  $S \rightarrow T$  with connected kernel, and hence [Kot85, §1.9] a surjection  $B(S) \rightarrow B(T)$ , as well as a surjection  $X_*(S)_\Gamma \rightarrow X_*(T)_\Gamma$ . We have  $\mathcal{E} \cong \mathcal{E}_b$  for some  $b \in B(T)$ ; let  $b_S \in B(S)$  be a lift of  $b$  and let  $\lambda_S \in X_*(S)$  be a lift of  $\lambda$ . The claim follows from (6.4.3) and Step 1 applied to  $S$ .

**Step 3:  $G_{\text{der}}$  is simply connected.** According to [Kot97, §7.5] the map  $\kappa$  is given by  $B(G) \rightarrow B(D) \rightarrow X^*(\widehat{D}^\Gamma) = X^*(Z(\widehat{G})^\Gamma)$ , where  $D = G/G_{\text{der}}$ , and the claim follows from (6.4.3) and Step 2 applied to  $D$ .

**Step 4: General  $G$ .** Let  $1 \rightarrow K \rightarrow \widetilde{G} \rightarrow G \rightarrow 1$  be a  $z$ -extension. Again we have a surjection  $B(\widetilde{G}) \rightarrow B(G)$  as well as surjections  $X_*(\widetilde{T}) \rightarrow X_*(T)$  for any maximal torus  $\widetilde{T} \subset \widetilde{G}$  with image  $T \subset G$ . This allows us to lift both  $b$  and  $\lambda$  to elements  $\widetilde{b} \in \widetilde{G}(\check{F})$  and  $\widetilde{\lambda} : \mathbf{G}_m \rightarrow \widetilde{G}$ . The claim follows from (6.4.3) and Step 3 applied to  $\widetilde{G}$ .  $\square$

**Proposition 6.4.2.** *Suppose a pair  $(g, g') \in G(F)_{\text{sr}} \times J_b(F)_{\text{sr}}$  fixes a point  $x \in \text{Sht}_{G,b,\mu}(C)$ . Let  $T = \text{Cent}(g, G)$  and  $T' = \text{Cent}(g', J_b)$ . Then  $\pi_1(x) \in (\text{Gr}_{G, \leq \mu}^{b,1})^{g'}$  and  $\pi_2(x) \in (\text{Gr}_{G, \leq \mu}^{1,b})^g$  correspond to cocharacters  $\lambda' \in X_*(T')$  and  $\lambda \in X_*(T)$ , respectively.*

*There exists  $y \in G(\check{F})$  such that  $\text{ad } y$  is an  $F$ -rational isomorphism  $T \rightarrow T'$ , which carries  $g$  to  $g'$  and  $\lambda$  onto  $\lambda'$ . The invariant  $\text{inv}[b](g, g') \in B(T) \cong X_*(T)_\Gamma$  agrees with the image of  $\lambda$  under  $X_*(T) \rightarrow X_*(T)_\Gamma$ .*

*Proof.* The point  $x$  corresponds to an isomorphism  $\gamma : \mathcal{E}^1 \rightarrow \mathcal{E}^b[\lambda']$ . Let  $g' \in T'(F)_{\text{sr}}$ . The action of  $g'$  on  $\mathcal{E}^b$  extends to an action on  $\mathcal{E}^b[\lambda']$ , and may therefore be pulled back via  $\gamma$  to an automorphism  $g \in \text{Aut } \mathcal{E}^1 = G(F)$ . Since  $g$  and  $g'$  become conjugate over the ring  $B_e = H^0(X_C \setminus \{\infty\}, \mathcal{O}_{X_C})$ , they are conjugate over  $\overline{F}$  and (by Lemma 3.2.1 they are even conjugate over  $\check{F}$ ). Let  $y \in G(\check{F})$  be an element such that  $(\text{ad } y)(g) = g'$ . Then  $\text{ad } y$  is a  $\check{F}$ -rational isomorphism  $T \rightarrow T'$  which carries  $g$  onto  $g'$ . In fact, since there is only one such isomorphism, we can conclude that  $\text{ad } y$  is an  $F$ -rational isomorphism  $T \rightarrow T'$ . Let  $\lambda = (\text{ad } y^{-1})(\lambda') \in X_*(T)$ .

Let  $b_0 = y^{-1}by^\sigma$ . Then (cf. Definition 3.2.2) we have  $b_0 \in T(F)$ . The element  $y$  induces isomorphisms  $y : \mathcal{E}^{b_0} \rightarrow \mathcal{E}^b$  and  $y : \mathcal{E}^{b_0}[\lambda] \rightarrow \mathcal{E}^b[\lambda']$ . The isomorphism  $y^{-1}\gamma : \mathcal{E}^1 \rightarrow \mathcal{E}^{b_0}[\lambda]$  descends to an isomorphism of  $T$ -bundles. Therefore by Lemma 6.4.1 the identity  $\kappa(\mathcal{E}^{b_0}) = \lambda$  holds in  $B(T)$ . But also  $\kappa(\mathcal{E}^{b_0})$  is the class of  $[b_0]$  in  $B(T)$ , which is  $\text{inv}[b](g, g')$  by definition.  $\square$

Proposition 6.4.2 shows that if  $(g, g')$  fixes a point of  $\text{Sht}_{G,b,\mu}$  and  $g'$  is strongly regular, then so is  $g$ , and then they are related. However the converse may fail: if a pair of related strongly regular elements  $(g, g')$  is given, it is not necessarily true that  $(g, g')$  fixes a point of  $\text{Sht}_{G,b,\mu}$ . Indeed, a necessary condition for that is that the action of  $g'$  on  $\text{Gr}_{G,\leq\mu}^b$  has a fixed point in the admissible locus, and this is not automatic.

This result is always true, however, if  $g$  (or equivalently,  $g'$ ) lies in the elliptic locus.

**Theorem 6.4.3.** *Let  $g \in J_b(F)_{\text{ell}}$ . Then fixed points of  $g$  in  $\text{Gr}_{G,\leq\mu}^b$  lie in the admissible locus  $\text{Gr}_{G,\leq\mu}^{b,1}$ . Similarly, if  $g \in G(F)_{\text{ell}}$ , then fixed points of  $g$  in  $\text{Gr}_{G,\leq\mu}^1$  lie in the locus  $\text{Gr}_{G,\leq\mu}^{1,b}$ .*

*Proof.* Let  $g \in J_b(F)_{\text{ell}}$ , and let  $T' = \text{Cent}(g, J_b)$  be the elliptic maximal torus containing  $g$ . Suppose we are given a  $g$ -fixed point  $x \in \text{Gr}_{G,\leq\mu}(C)$ . Then  $x$  corresponds to a cocharacter  $\lambda \in X_*(T')$ , which in turn corresponds to a modification  $\mathcal{E}^b[\lambda]$  of the  $G$ -bundle  $\mathcal{E}^b$ . We wish to show that  $\mathcal{E}^b[\lambda]$  is the trivial  $G$ -bundle. First we will show that it is semistable.

Let  $b' \in G(\bar{F})$  be an element whose class in  $B(G)$  corresponds to the isomorphism class of  $\mathcal{E}_b[\lambda]$ . We wish to show that  $b'$  is basic. We have the algebraic group  $J_{b'}/F$ . A priori,  $J_{b'}(F)$  is an inner form of a Levi subgroup  $M^*$  of  $G^*$ , where  $G^*$  is the quasi-split inner form of  $G$ . Showing that  $[b']$  is basic is equivalent to showing that  $M^* = G^*$ .

We have an isomorphism  $\gamma: \mathcal{E}_b \cong \mathcal{E}_{b'}[\lambda]$ . The action of  $g \in T'(F)$  on  $\mathcal{E}_{b'}$  extends to an action on  $\mathcal{E}_{b'}[\lambda]$ , which can be pulled back via  $\gamma$  to obtain an automorphism  $g' \in \mathcal{J}_{b'}(C) = \text{Aut } \mathcal{E}_{b'}$ . Let  $\bar{g}'$  be the image of  $g'$  under the projection  $\mathcal{J}_{b'}(C) \rightarrow J_{b'}(F)$ .

We now choose trivializations of  $\mathcal{E}_b$  and  $\mathcal{E}_{b'}$  over  $\text{Spec } B_{\text{dR}}^+(C)$ . In doing so, we obtain embeddings of  $J_b(F) = \text{Aut } \mathcal{E}_b$  and  $\mathcal{J}_{b'}(C) = \text{Aut } \mathcal{E}_{b'}$  into  $G(B_{\text{dR}}^+(C))$ ; we denote both of these by  $h \mapsto h_\infty$ . We also have the isomorphism  $\gamma_\infty$  between  $\mathcal{E}_b$  and  $\mathcal{E}_{b'}$  over  $\text{Spec } B_{\text{dR}}(C)$ ; we may identify  $\gamma_\infty$  with an element of  $G(B_{\text{dR}}(C))$ , and then  $g'_\infty = \gamma_\infty g_\infty \gamma_\infty^{-1}$  holds in  $G(B_{\text{dR}}(C))$ .

The element  $\bar{g}'_\infty$  is conjugate to  $g'_\infty$ , so  $\bar{g}'_\infty$  is conjugate to  $g_\infty$  in  $G(B_{\text{dR}})$ . Since  $g$  and  $\bar{g}'$  are both regular semisimple  $\bar{F}$ -points of  $G$ , being conjugate in  $G(B_{\text{dR}})$  is the same as being conjugate in  $G(\bar{F})$ . Their centralizers, being  $F$ -rational tori, are thus isomorphic over  $F$ . Thus  $J_{b'}$  contains a maximal torus that is elliptic for  $G$ . Elliptic maximal tori transfer across inner forms [Kot86, §10], which means that the Levi subgroup  $M^* \subset G^*$  of which  $J_{b'}$  is an inner form contains a maximal torus that is elliptic for  $G^*$ . Therefore  $M^* = G^*$ .

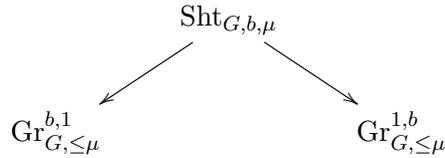
We have shown that  $\mathcal{E}_b[\lambda] \cong \mathcal{E}_{b'}$  is semistable, implying that  $\text{Aut } \mathcal{E}_{b'} = J_{b'}(F)$  and that  $g' \in J_{b'}(F)$ . Lemma 6.4.1 shows that  $\kappa([b']) = \kappa([b]) - \lambda = 0$ . Since  $b'$  is basic, we have  $[b'] = [1]$  by [Kot85, Proposition 5.6].  $\square$

We are now ready to define the locally profinite set  $\text{Sht}_{b,\mu,\text{ell}}$  from §3.4.

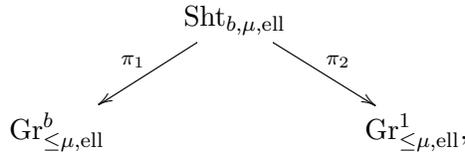
**Definition 6.4.4.** Let  $\text{Sht}_{b,\mu,\text{ell}} \subset \text{Sht}_{G,b,\mu}(C)$  be the set of elements which are fixed by a pair  $(g, g') \in G(F) \times J_b(F)$ , where  $g \in G(F)_{\text{ell}}$  (equivalently,  $g' \in J_b(F)_{\text{ell}}$ ). Let  $I\text{Sht}_{b,\mu,\text{ell}}$  denote the set of triples  $(g, g', x)$ , where  $(g, g') \in G(F) \times J_b(F)$  is a pair of elliptic elements which fixes  $x \in \text{Sht}_{G,b,\mu}(C)$ .

With this definition, we can verify Theorem 3.4.2. Recall (Definition 3.4.1) that  $\text{Gr}_{\leq \mu, \text{ell}}^1$  is the set of pairs  $(T, \lambda)$ , where  $T \subset G$  is an elliptic rational torus, and  $\lambda \in X_*(T)_{\leq \mu}$ . This may be identified with the set of  $C$ -points of  $\text{Gr}_{\leq \mu}^1$  which are fixed by an elliptic rational torus. Theorem 6.4.3 shows that all such elliptic fixed points are admissible. Thus,  $\text{Gr}_{\leq \mu, \text{ell}}^1$  may be identified with the set of elements of  $\text{Gr}_{G, \leq \mu}^{1,b}(C)$  which are fixed by an element of  $G(F)_{\text{ell}}$ . Similarly,  $\text{Gr}_{\leq \mu, \text{ell}}^b$  may be identified with the set of elements of  $\text{Gr}_{G, \leq \mu}^{b,1}(C)$  which are fixed by an element of  $J_b(F)_{\text{ell}}$ .

Now recall the diagram of  $(G(F) \times J_b(F))$ -sets



where the arrow on the left is a  $G(F)$ -torsor, and the arrow on the right is a  $J_b(F)$ -torsor. Passing to loci fixed by pairs  $(g, g') \in G(F)_{\text{ell}} \times J_b(F)_{\text{ell}}$ , we obtain a diagram



in which  $\pi_1$  is a  $G(F)$ -torsor and  $\pi_2$  is a  $J_b(F)$ -torsor. We now verify condition (3) of Theorem 3.4.2. Let  $x \in \text{Sht}_{b,\mu,\text{ell}}$ , and let  $\pi_2(x) = (T_x, \lambda_x)$  and  $\pi_1(x) = (T'_x, \lambda'_x)$ . Then  $x$  corresponds to an isomorphism of  $G$ -bundles  $\gamma: \mathcal{E}^1 \rightarrow \mathcal{E}^b[\lambda_x]$ . By Proposition 6.4.2, there exists  $y \in G(\check{F})$  which conjugates  $(T_x, \lambda_x)$  onto  $(T'_x, \lambda'_x)$ . Conjugation by  $y$  induces an isomorphism of

$F$ -rational tori  $\iota_x: T_x \xrightarrow{\sim} T'_x$ . Then for all  $t \in T_x(F)$  we have a commutative diagram of  $G$ -bundles

$$\begin{array}{ccc} \mathcal{E}^1 & \xrightarrow{\gamma} & \mathcal{E}^b[\lambda'_x] \\ g \downarrow & & \downarrow \iota_x(g) \\ \mathcal{E}^1 & \xrightarrow{\gamma} & \mathcal{E}^b[\lambda'_x], \end{array}$$

which means exactly that  $(t, \iota_x(t)) \cdot x = x$ .

## 6.5 Calculation of the Hecke correspondence on distributions

Recall the Hecke operator  $T_\mu: D_{\text{ét}}(\text{Bun}_G, \Lambda) \rightarrow D_{\text{ét}}(\text{Bun}_G, \Lambda)$ , defined by  $T_\mu(\mathcal{F}) = h_{2!}(h_1^* \mathcal{F} \otimes \text{IC}_\mu)$ . By Corollary 6.3.1,  $T_\mu$  preserves strongly reflexive objects.

**Theorem 6.5.1.** *The following diagram commutes:*

$$\begin{array}{ccc} D_{\text{ét}}(\text{Bun}_G, \Lambda)^{\text{st.ref.}} & \xrightarrow{T_\mu} & D_{\text{ét}}(\text{Bun}_G, \Lambda)^{\text{st.ref.}} \\ \text{tr} \downarrow & & \downarrow \\ \text{Dist}(\text{In}(\text{Bun}_G), \Lambda) & & \text{Dist}(\text{In}(\text{Bun}_G), \Lambda) \\ \text{res} \downarrow & & \downarrow \text{res} \\ \text{Dist}(\text{In}(\text{Bun}_G^b)_{\text{ell}}, \Lambda) & & \text{Dist}(\text{In}(\text{Bun}_G^1)_{\text{ell}}, \Lambda) \\ \cong \downarrow & & \downarrow \cong \\ \text{Dist}(J_b(F)_{\text{ell}}, \Lambda)^{J_b(F)} & \xrightarrow{T_{b,\mu}^{J \rightarrow G}} & \text{Dist}(G(F)_{\text{ell}}, \Lambda)^{G(F)}. \end{array}$$

Here,  $T_{b,\mu}^{J \rightarrow G}$  is the transfer of distributions from Definition 3.4.7.

*Proof.* We embed the diagram in the theorem into a larger one:

$$\begin{array}{ccccc}
& & \xrightarrow{T_\mu} & & \\
D_{\text{ét}}(\text{Bun}_G, \Lambda)^{\text{st.ref.}} & \xrightarrow{\boxtimes \text{IC}_\mu} & D_{\text{ét}}(\text{Hecke}_{\leq \mu, C}, \Lambda)^{\text{st.ref.}} & \xrightarrow{h_{2!}} & D_{\text{ét}}(\text{Bun}_{G, C}, \Lambda)^{\text{st.ref.}} \\
\text{tr} \downarrow & & \text{tr} \downarrow & & \text{tr} \downarrow \\
\text{Dist}(\text{In}(\text{Bun}_G), \Lambda) & \xrightarrow{\otimes \text{tr}(\text{IC}_\mu)} & \text{Dist}(\text{In}(\text{Hecke}_{\leq \mu, C}), \Lambda) & \xrightarrow{h_{2!}} & \text{Dist}(\text{In}(\text{Bun}_{G, C}), \Lambda) \\
\text{res} \downarrow & & \text{res} \downarrow & & \text{res} \downarrow \\
\text{Dist}(\text{In}(\text{Bun}_{G, C}^b)_{\text{ell}}, \Lambda) & \longrightarrow & \text{Dist}(\text{In}(\text{Hecke}^{b, 1})_{\text{ell}}, \Lambda) & \longrightarrow & \text{Dist}(\text{In}(\text{Bun}_{G, C}^1)_{\text{ell}}, \Lambda) \\
\cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
\text{Dist}(J_b(F)_{\text{ell}}, \Lambda)^{J_b(F)} & \xrightarrow{K_{\mu, \pi_1^*}} & \text{Dist}(I \text{Gr}_{\leq \mu, \text{ell}}^b, \Lambda)^{J_b(F)} & & \cong \downarrow \\
& & * \downarrow & & \downarrow \\
& & \text{Dist}(I \text{Gr}_{\leq \mu, \text{ell}}^1, \Lambda)^{G(F)} & \xrightarrow{\pi_{2*}} & \text{Dist}(G(F)_{\text{ell}}, \Lambda)^{G(F)}
\end{array}$$

The left and right halves of this diagram commute by Lemma 6.3.2 and Lemma 6.3.3, respectively. (Or rather, one has to restrict these diagrams from the strongly loci to the elliptic loci.) The map labeled  $*$  is the composite

$$\begin{aligned}
\text{Dist}(I \text{Gr}_{\leq \mu, \text{ell}}^b, \Lambda)^{J_b(F)} &\xrightarrow{\sim} \text{Dist}(I \text{Sht}_{G, b, \mu, \text{ell}}, \Lambda)^{G(F) \times J_b(F)} \\
&\xrightarrow{\sim} \text{Dist}(I \text{Gr}_{\leq \mu, \text{ell}}^1, \Lambda)^{G(F)}
\end{aligned}$$

where each isomorphism is an application of Lemma 3.4.5. Tracing through definitions, we see that the map  $\text{Dist}(J_b(F)_{\text{ell}}, \Lambda)^{J_b(F)} \rightarrow \text{Dist}(G(F)_{\text{ell}}, \Lambda)^{G(F)}$  in the above diagram is our transfer of distributions  $T_{b, \mu}^{J \rightarrow G}$ .  $\square$

We can now prove Theorem 3.5.1. Let  $\phi$  be a discrete Langlands parameter with coefficients in  $E$ , a finite extension of  $\mathbf{Q}_\ell$ . Let  $\phi \in \Pi_\phi(J_b)$ , so that  $\phi$  is an irreducible admissible representation of  $J_b(F)$  with coefficients in  $E$ .

We have assumed that  $\rho$  admits a  $J_b(F)$ -invariant  $\mathcal{O}_E$ -lattice. Let  $\rho_n$  be the reduction of this lattice modulo  $\ell^n$ , so that  $\rho_n$  is an admissible representation of  $J_b(F)$  on a free  $\mathcal{O}_E/\ell^n$ -module. This corresponds to an object  $\mathcal{L}_{\rho_n}$  of  $D_{\text{ét}}(\text{Bun}_G^b, \mathcal{O}_E/\ell^n)$ .

We apply Theorem 6.5.1 to the strongly reflexive object  $i_{b*}\mathcal{L}_{\rho_n}$ . We have the strongly reflexive object  $i_{b*}\mathcal{L}_{\rho_n}$  of  $D_{\text{ét}}(\text{Bun}_{G, C}^b, \mathcal{O}_E/\ell^n)$ . On the one hand, the object  $i_1^*T_\mu i_{b*}\mathcal{L}_{\rho_n}$  of  $D_{\text{ét}}(\text{Bun}_{G, C}^1, \Lambda)$  corresponds to  $H^*(G, b, \mu)[\rho_n]$  by Proposition 6.2.1. On the other hand, Theorem 6.5.1 shows that the trace

distribution of  $G(F)_{\text{ell}}$  on  $i_1^* T_\mu i_{b*} \mathcal{L}_{\rho_n}$  is  $T_{b,\mu}^{J \rightarrow G}(\Theta_{\rho_n} |_{J_b(F)_{\text{ell}}})$ . Taking the limit as  $n \rightarrow \infty$ , we complete the proof of Theorem 3.5.1.

## A Endoscopy

### A.1 Endoscopic character relations

We recall here the endoscopic character identities, which are part of the refined local Langlands correspondence, following the formulation of [Kal16b, §5.4], also recalled in [Kal16a, §4.2]. They will be an important ingredient in the proof of our main result.

We summarize the notation established so far.

- $F/\mathbf{Q}_p$  is a finite extension.
- $G$  is a connected reductive group defined over  $F$ .
- $G^*$  is a quasi-split connected reductive group defined over  $F$ .
- $\Psi$  is a  $G^*$ -conjugacy class of inner twists  $\psi: G^* \rightarrow G$ .
- $\bar{z}_\sigma = \psi^{-1} \sigma(\psi) \in G_{\text{ad}}^*$ , so that  $\bar{z} \in Z^1(F, G_{\text{ad}}^*)$ .
- $z \in Z^1(u \rightarrow W, Z(G^*) \rightarrow G^*)$  is a lift of  $\bar{z}$ .
- $b \in G(F^{\text{nr}})$  is a decent basic element.
- $J_b$  is the corresponding inner form of  $G$ .
- $\xi: G_{F^{\text{nr}}} \rightarrow J_{b, F^{\text{nr}}}$  is the identity map.
- $z_b \in Z^1(u \rightarrow W, Z(G) \rightarrow G)$  is the image of  $b$  under (2.3.1).
- $\mathfrak{w}$  is a Whittaker datum for  $G^*$ .
- $\phi: W_F \rightarrow {}^L G$  is a discrete parameter.
- $S_\phi = \text{Cent}(\phi, \widehat{G})$ .
- $S_\phi^+$  is the group defined in Definition 2.3.1.

Associated to  $\psi$  are the  $L$ -packets  $\Pi_\phi(G)$  and  $\Pi_\phi(J_b)$  and the bijections

$$\Pi_\phi(G) \rightarrow \text{Irr}(\pi_0(S_\phi^+), \lambda_z), \quad \Pi_\phi(J_b) \rightarrow \text{Irr}(\pi_0(S_\phi^+), \lambda_z + \lambda_{z_b})$$

denoted by  $\pi \mapsto \tau_{z, \mathfrak{w}, \pi}$  and  $\rho \mapsto \tau_{z, \mathfrak{w}, \rho}$ .

We now choose a semi-simple element  $s \in S_\phi$  and an element  $\dot{s} \in S_\phi^+$  which lifts  $s$ . Let  $e(G)$  and  $e(J_b)$  be the Kottwitz signs of the groups  $G$  and  $J_b$ , as defined in [Kot83]. Consider the virtual characters

$$e(G) \sum_{\pi \in \Pi_\phi(G)} \text{tr } \tau_{z, \mathfrak{w}, \pi}(\dot{s}) \cdot \Theta_\pi \quad \text{and} \quad e(J_b) \sum_{\rho \in \Pi_\phi(J_b)} \text{tr } \tau_{z, \mathfrak{w}, \rho}(\dot{s}) \cdot \Theta_\rho.$$

The endoscopic character identities are equations which relate these two virtual characters to virtual characters on an endoscopic group  $H_1$  of  $G$  and  $J_b$ . From the pair  $(\phi, \dot{s})$  one obtains a refined elliptic endoscopic datum

$$\dot{\mathbf{e}} = (H, \mathcal{H}, \dot{s}, \eta) \tag{A.1.1}$$

in the sense of [Kal16b, §5.3] as follows. Let  $\widehat{H} = \text{Cent}(s, \widehat{G})^\circ$ . The image of  $\phi$  is contained in  $\text{Cent}(s, \widehat{G})$ , which in turns acts by conjugation on its connected component  $\widehat{H}$ . This gives a homomorphism  $W_F \rightarrow \text{Aut}(\widehat{H})$ . Letting  $\Psi_0(\widehat{H})$  be the based root datum of  $\widehat{H}$  [Kot84b, §1.1] and  $\Psi_0^\vee(\widehat{H})$  its dual, we obtain the homomorphism

$$W_F \rightarrow \text{Aut}(\widehat{H}) \rightarrow \text{Out}(\widehat{H}) = \text{Aut}(\Psi_0(\widehat{H})) = \text{Aut}(\Psi_0^\vee(\widehat{H})).$$

Since the target is finite, this homomorphism extends to  $\Gamma_F$  and we obtain a based root datum with Galois action, hence a quasi-split connected reductive group  $H$  defined over  $F$ . Its dual group is by construction equal to  $\widehat{H}$ . We let  $\mathcal{H} = \widehat{H} \cdot \phi(W_F)$ , noting that the right factor normalizes the left so their product  $\mathcal{H}$  is a subgroup of  ${}^L G$ . Finally, we let  $\eta : \mathcal{H} \rightarrow {}^L G$  be the natural inclusion. Note that by construction  $\phi$  takes image in  $\mathcal{H}$ , i.e. it factors through  $\eta$ .

We can realize the  $L$ -group of  $H$  as  ${}^L H = \widehat{H} \rtimes W_F$ , but we caution the reader that  $W_F$  does not act on  $\widehat{H}$  via the map  $W_F \rightarrow \text{Aut}(\widehat{H})$  given by  $\phi$  as above. Rather, we have to modify this action to ensure that it preserves a pinning of  $\widehat{H}$ . More precisely, after fixing an arbitrary pinning of  $\widehat{H}$  we obtain a splitting  $\text{Out}(\widehat{H}) \rightarrow \text{Aut}(\widehat{H})$  and the action of  $W_F$  on  $\widehat{H}$  we use to form  ${}^L H$  is given by composing the above map  $W_F \rightarrow \text{Out}(\widehat{H})$  with this splitting.

Both  ${}^L H$  and  $\mathcal{H}$  are thus extensions of  $W_F$  by  $\widehat{H}$ , but they need not be isomorphic. If they are, we fix arbitrarily an isomorphism  $\eta_1 : \mathcal{H} \rightarrow {}^L H$  of extensions. Then  $\phi^s = \eta_1 \circ \phi$  is a supercuspidal parameter for  $H$ .

In the general case we need to introduce a  $z$ -pair  $\mathfrak{z} = (H_1, \eta_1)$  as in [KS99, §2]. It consists of a  $z$ -extension  $H_1 \rightarrow H$  (recall this means that

$H_1$  has a simply connected derived subgroup and the kernel of  $H_1 \rightarrow H$  is an induced torus) and  $\eta_1 : \mathcal{H} \rightarrow {}^L H_1$  is an  $L$ -embedding that extends the natural embedding  $\widehat{H} \rightarrow \widehat{H}_1$ . As is shown in [KS99, §2.2], such a  $z$ -pair always exists. Again we set  $\phi^s = \eta_1 \circ \phi$  and obtain a supercuspidal parameter for  $H_1$ . In the situation where an isomorphism  $\eta_1 : \mathcal{H} \rightarrow {}^L H$  does exist, we will allow ourselves to take  $H = H_1$  and so regard  $\mathfrak{z} = (H, \eta_1)$  as a  $z$ -pair, even though in general  $H$  will not have a simply connected derived subgroup.

The virtual character on  $H_1$  that the above virtual characters on  $G$  and  $J_b$  are to be related to is

$$S\Theta_{\phi^s} := \sum_{\pi^s \in \Pi_{\phi^s}(H_1)} \dim(\tau_{\pi^s}) \Theta_{\pi^s}.$$

Here  $\pi^s \mapsto \tau_{\pi^s}$  is a bijection  $\Pi_{\phi^s}(H_1) \rightarrow \text{Irr}(\pi_0(\text{Cent}(\phi^s, \widehat{H}_1)/Z(\widehat{H}_1)^\Gamma))$  determined by an arbitrary choice of Whittaker datum for  $H_1$ . The argument in the proof of Lemma 2.3.3 shows the independence of  $\dim(\tau_{\pi^s})$  of the choice of a Whittaker datum for  $H_1$ .

The relationship between the virtual characters on  $G$ ,  $J_b$ , and  $H_1$ , is expressed in terms of the Langlands-Shelstad transfer factor  $\Delta'_{\text{abs}}[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\psi, z)]$  for the pair of groups  $(H_1, G)$  and the corresponding Langlands-Shelstad transfer factor  $\Delta'_{\text{abs}}[\dot{\mathfrak{e}}, \mathfrak{z}, \mathfrak{w}, (\xi \circ \psi, \psi^{-1}(z_b) \cdot z)]$  for the pair of groups  $(H_1, J_b)$ , both of which are defined by [Kal16b, (5.10)]. We will abbreviate both of them to just  $\Delta$ . It is a simple consequence of the Weyl integration formula that the character relation [Kal16b, (5.11)] can be restated in terms of character functions (rather than character distributions) as

$$e(G) \sum_{\pi \in \Pi_{\phi}(G)} \text{tr } \tau_{z, \mathfrak{w}, \pi}(\dot{s}) \Theta_{\pi}(g) = \sum_{h_1 \in H_1(F)/\text{st.}} \Delta(h_1, g) S\Theta_{\phi^s}(h_1) \quad (\text{A.1.2})$$

for any strongly regular semi-simple element  $g \in G(F)$ . The sum on the right runs over stable conjugacy classes of strongly regular semi-simple elements of  $H_1(F)$ . We also have the analogous identity for  $J_b$ :

$$e(J_b) \sum_{\rho \in \Pi_{\phi}(J_b)} \text{tr } \tau_{z, \mathfrak{w}, \rho}(\dot{s}) \Theta_{\rho}(j) = \sum_{h_1 \in H_1(F)/\text{st.}} \Delta(h_1, j) S\Theta_{\phi^s}(h_1). \quad (\text{A.1.3})$$

We are only interested in the right hand sides of these two equations as a bridge between their left-hand sides. Essential for this bridge is a certain compatibility between the transfer factors appearing on both right-hand sides:

**Lemma A.1.1.**

$$\Delta(h_1, j) = \Delta(h_1, g) \cdot \langle \text{inv}[b](g, j), s_{h,g} \rangle. \quad (\text{A.1.4})$$

We need to explain the second factor. Given maximal tori  $T_H \subset H$  and  $T \subset G$ , there is a notion of an admissible isomorphism  $T_H \rightarrow T$ , for which we refer the reader to [Kal16a, §1.3]. Two strongly regular semi-simple elements  $h \in H(\mathbf{Q}_p)$  and  $g \in G(\mathbf{Q}_p)$  are called *related* if there exists an admissible isomorphism  $T_h \rightarrow T_g$  between their centralizers mapping  $h$  to  $g$ . If such an isomorphism exists, it is unique, and in particular defined over  $F$ , and shall be called  $\varphi_{h,g}$ . An element  $h_1 \in H_1(F)$  is called related to  $g \in G(F)$  if and only if its image  $h \in H(F)$  is so. Since  $g$  and  $j$  are stably conjugate, an element  $h_1 \in H_1(F)$  is related to  $g$  if and only if it is related to  $j$ . If that is not the case, both  $\Delta(h_1, j)$  and  $\Delta(h_1, g)$  are zero and (A.1.4) is trivially true. Thus assume that  $h_1$  is related to both  $g$  and  $j$ . Let  $s^\natural \in S_\phi$  be the image of  $\dot{s}$  under (2.3.2). Note that  $s^\natural \in s \cdot Z(\widehat{G})^{\circ, \Gamma}$  and hence the preimage of  $s^\natural$  under  $\eta$  belongs to  $Z(\widehat{H})^\Gamma$ , which in turns embeds naturally into  $\widehat{T}_h^\Gamma$ . Using the admissible isomorphism  $\varphi_{h,g}$  we transport  $s^\natural$  into  $\widehat{T}_g^\Gamma$  and denote it by  $s_{h,g}$ . It is then paired with  $\text{inv}[b](g, j)$  via the isomorphism  $B(T_g) \cong X^*(\widehat{T}_g^\Gamma)$  of [Kot85, §2.4].

*Proof.* For every finite subgroup  $Z \subset Z(G) \subset T_g$  one obtains from  $\varphi_{h,g}$  an isomorphism  $T_h/\varphi_{h,g}^{-1}(Z) \rightarrow T_g/Z$ . Using the subgroups  $Z_n$  from the previous subsection we form the quotients  $T_{h,n} = T_h/\varphi_{h,g}^{-1}(Z_n)$  and  $T_{g,n} = T_g/Z_n$ . From  $\varphi_{h,g}$  we obtain an isomorphism

$$\widehat{T}_h \rightarrow \widehat{T}_g$$

between the limits over  $n$  of the tori dual to  $T_{h,n}$  and  $T_{g,n}$ . Let  $\dot{s}_{h,g} \in [\widehat{T}_g]^+$  be the image of  $\dot{s}$  under this isomorphism. Let  $\text{inv}[z_b](g, j) \in H^1(u \rightarrow W, Z(G) \rightarrow T_g)$  be the invariant defined in [Kal16b, §5.1]. If we replace  $\langle \text{inv}[b](g, j), s_{h,g} \rangle$  by  $\langle \text{inv}[z_b](g, j), \dot{s}_{h,g} \rangle$  then the lemma follows immediately from the defining formula [Kal16b, (5.10)] of the transfer factors. The lemma follows from the equality  $\langle \text{inv}[b](g, j), s_{h,g} \rangle = \langle \text{inv}[z_b](g, j), \dot{s}_{h,g} \rangle$  proved in [Kal18, §4.2].  $\square$

## A.2 The Kottwitz sign

We will give a formula for the Kottwitz sign  $e(G)$  in terms of the dual group  $\widehat{G}$ . Fix a quasi-split inner form  $G^*$  and an inner twisting  $\psi: G^* \rightarrow G$ . Let  $h \in H^1(\Gamma, G_{\text{ad}}^*)$  be the class of  $\sigma \mapsto \psi^{-1}\sigma(\psi)$ . Via the Kottwitz

homomorphism [Kot86, Theorem 1.2] the class  $h$  corresponds to a character  $\nu \in X^*(Z(\widehat{G}_{\text{sc}})^\Gamma)$ .

Choose an arbitrary Borel pair  $(\widehat{T}_{\text{sc}}, \widehat{B}_{\text{sc}})$  of  $\widehat{G}_{\text{sc}}$  and let  $2\rho \in X_*(\widehat{T}_{\text{sc}})$  be the sum of the  $\widehat{B}_{\text{sc}}$ -positive coroots. The restriction map  $X^*(\widehat{T}_{\text{sc}}) \rightarrow X^*(Z(\widehat{G}_{\text{sc}}))$  is surjective and we can lift  $\nu$  to  $\dot{\nu} \in X^*(\widehat{T}_{\text{sc}})$  and form  $\langle 2\rho, \dot{\nu} \rangle \in \mathbf{Z}$ . A different lift  $\dot{\nu}$  would differ by an element of  $X^*(\widehat{T}_{\text{ad}})$ , and since  $\rho \in X_*(\widehat{T}_{\text{ad}})$  we see that the image of  $\langle 2\rho, \dot{\nu} \rangle$  in  $\mathbf{Z}/2\mathbf{Z}$  is independent of the choice of lift  $\dot{\nu}$ . We thus write  $\langle 2\rho, \nu \rangle \in \mathbf{Z}/2\mathbf{Z}$ . Since any two Borel pairs in  $\widehat{G}_{\text{sc}}$  are conjugate  $\langle 2\rho, \nu \rangle$  does not depend on the choice of  $(\widehat{T}_{\text{sc}}, \widehat{B}_{\text{sc}})$ .

**Lemma A.2.1.**

$$e(G) = (-1)^{\langle 2\rho, \nu \rangle}.$$

*Proof.* We fix  $\Gamma$ -invariant Borel pairs  $(T_{\text{ad}}, B_{\text{ad}})$  in  $G_{\text{ad}}^*$  and  $(\widehat{T}_{\text{sc}}, \widehat{B}_{\text{sc}})$  in  $\widehat{G}_{\text{sc}}$ . Then we have the identification  $X^*(T_{\text{ad}}) = X_*(\widehat{T}_{\text{sc}})$ . Let  $(T_{\text{sc}}, B_{\text{sc}})$  be the preimage in  $G_{\text{sc}}^*$  of  $(T_{\text{ad}}, B_{\text{ad}})$ .

By definition the Kottwitz sign is the image of  $h$  under

$$H^1(\Gamma, G_{\text{ad}}^*) \xrightarrow{\delta} H^2(\Gamma, Z(G_{\text{sc}}^*)) \xrightarrow{\rho} H^2(\Gamma, \{\pm 1\}) \longrightarrow \{\pm 1\},$$

where  $\rho \in X^*(T_{\text{sc}})$  is half the sum of the  $B_{\text{sc}}$ -positive roots and its restriction to  $Z(G_{\text{sc}}^*)$  is independent of the choice of  $(T_{\text{ad}}, B_{\text{ad}})$ . By functoriality of the Tate-Nakayama pairing this is the same as pairing  $\delta h \in H^2(\Gamma, Z(G_{\text{sc}}^*))$  with  $\rho \in H^0(\Gamma, X^*(Z(G_{\text{sc}}^*)))$ . The canonical pairing  $X^*(T_{\text{ad}}) \otimes X^*(\widehat{T}_{\text{sc}}) \rightarrow \mathbf{Z}$  induces the perfect pairing  $X^*(T_{\text{sc}})/X^*(T_{\text{ad}}) \otimes X^*(\widehat{T}_{\text{sc}})/X^*(\widehat{T}_{\text{ad}}) \rightarrow \mathbf{Q}/\mathbf{Z}$  and hence the isomorphism  $X^*(Z(G_{\text{sc}}^*)) \rightarrow \text{Hom}_{\mathbf{Z}}(X^*(Z(\widehat{G}_{\text{sc}})), \mathbf{Q}/\mathbf{Z}) = Z(\widehat{G}_{\text{sc}})$ , where the last equality uses the exponential map. Under this isomorphism  $\rho \in X^*(Z(G_{\text{sc}}^*))^\Gamma$  maps to the element  $(-1)^{2\rho} \in Z(\widehat{G}_{\text{sc}})^\Gamma$  obtained by mapping  $(-1) \in \mathbf{C}^\times$  under  $2\rho \in X^*(T_{\text{ad}}) = X_*(\widehat{T}_{\text{sc}})$ . The lemma now follows from [Kot86, Lemma 1.8].  $\square$

We now prove (3.6.3). Recall that  $G^*$  is a quasi-split inner form of  $G$ . Let  $\mu_1, \mu_2 \in X^*(Z(\widehat{G}_{\text{sc}})^\Gamma)$  be the elements corresponding to the inner twists  $G^* \rightarrow G$  and  $G^* \rightarrow J_b$  by Kottwitz's homomorphism [Kot86, Theorem 1.2]. By Lemma A.2.1 we have  $e(J_b)e(G) = (-1)^{\langle 2\rho, \mu_2 - \mu_1 \rangle}$ . But since  $J_b$  is obtained from  $G$  by twisting by  $b$ , the difference  $\mu_2 - \mu_1$  is equal to the image of  $\kappa(b) \in X^*(Z(\widehat{G})^\Gamma)$  under the map  $X^*(Z(\widehat{G})^\Gamma) \rightarrow X^*(Z(\widehat{G}_{\text{sc}})^\Gamma)$  dual to the natural map  $Z(\widehat{G}_{\text{sc}}) \rightarrow Z(\widehat{G})$ . Since  $b \in B(G, \mu)$  we see that  $\mu_2 - \mu_1 = \mu$  and conclude that

$$e(J_b)e(G) = (-1)^{\langle 2\rho, \mu \rangle}.$$

## B Elementary Lemmas

### B.1 Homological algebra

**Lemma B.1.1.** *Let  $R$  be a discrete valuation ring with maximal ideal  $\mathfrak{m}$ . Let  $\kappa = R/\mathfrak{m}$  be the residue field, and let  $\Lambda = R/\mathfrak{m}^k$  for some  $k > 0$ . For a  $\Lambda$ -module  $M$  we have the dual module  $M^* = \text{Hom}_\Lambda(M, \Lambda)$  and the natural morphisms  $M \rightarrow M^{**}$  and  $(M^* \otimes M) \rightarrow (M \otimes M^*)^*$ .*

*The morphism  $M \rightarrow M^{**}$  is an isomorphism if and only if  $M$  is finitely generated.*

*Proof.* For the “if” direction of the first point we note that the structure theorem for  $R$ -modules implies that a finitely generated  $\Lambda$ -module is a direct sum of finitely many cyclic  $\Lambda$ -modules, and each cyclic  $\Lambda$ -module is isomorphic to its own double dual.

Conversely assume that  $M \rightarrow M^{**}$  is an isomorphism. We induct on  $k$ . If  $k = 1$ , then  $\Lambda$  is a field, and this is well-known. For general  $k$  we consider  $N = M/\mathfrak{m}M$ . The ring  $\Lambda$  is an Artinian serial ring, and hence it is injective as a module over itself. Thus the dualization functor is exact, and we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}M & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & (\mathfrak{m}M)^{**} & \longrightarrow & M^{**} & \longrightarrow & N^{**} & \longrightarrow & 0, \end{array} \quad (\text{B.1.1})$$

which shows that the right-most vertical map is surjective and the left-most vertical map is injective.

We have an isomorphism of  $\Lambda$ -modules  $\mathfrak{m}^{m-1}\Lambda \rightarrow \kappa$ , from which we obtain

$$N^* = \text{Hom}_\Lambda(N, \Lambda) = \text{Hom}_\Lambda(N, \mathfrak{m}^{m-1}\Lambda) \cong \text{Hom}_\kappa(N, \kappa).$$

Thus  $N^{**}$  is also the double dual of  $N$  in the category of  $\kappa$ -vector spaces, and it is easy to check that the right-most vertical map in (B.1.1) is the canonical map in that category. Thus, this map is an isomorphism, and  $N$  is finitely generated as a  $\kappa$ -vector space.

By the Snake Lemma, the left-most vertical arrow in (B.1.1) is an isomorphism. We can apply the inductive hypothesis to the  $(\Lambda/\mathfrak{m}^{m-1})$ -module  $\mathfrak{m}M$  and conclude that it is finitely generated. Thus so is  $M$ . □

**Lemma B.1.2.** *Let  $\Lambda$  be an arbitrary ring, and let  $D(\Lambda)$  be the derived category of  $\Lambda$ -modules. For an object  $M$  of  $D(\Lambda)$ , let  $\mathbf{D}M = \mathbf{R}\mathrm{Hom}(M, \Lambda[0])$ .*

1. *Assume that  $\Lambda$  is self-injective. The natural morphism  $M \rightarrow \mathbf{D}\mathbf{D}M$  is an isomorphism if and only if each  $H^i(M)$  is finitely generated.*
2. *The following are equivalent:*
  - (a) *The natural maps  $M \rightarrow \mathbf{D}\mathbf{D}M$  and  $\mathbf{D}M \otimes M \rightarrow \mathbf{D}(M \otimes \mathbf{D}M)$  are isomorphisms.*
  - (b) *The natural map  $M \otimes \mathbf{D}M \rightarrow \mathbf{R}\mathrm{Hom}(M, M)$  is an isomorphism.*
  - (c)  *$M$  is strongly dualizable; that is, for any object  $N$ ,  $N \otimes \mathbf{D}M \rightarrow \mathbf{R}\mathrm{Hom}(M, N)$  is an isomorphism.*
  - (d)  *$M$  is a compact object; that is, the functor  $N \mapsto \mathbf{R}\mathrm{Hom}(M, N)$  commutes with colimits.*
  - (e)  *$M$  is a perfect complex; that is,  $M$  is isomorphic to a bounded complex of finitely generated projective  $\Lambda$ -modules.*

*(Throughout, the  $\otimes$  means derived tensor product.)*

*Proof.* For the first statement, the injectivity of  $\Lambda$  implies that  $H^i(\mathbf{D}M) \cong H^{-i}(M)^*$ , so that  $H^i(\mathbf{D}\mathbf{D}M) \cong H^i(M)^{**}$ . Therefore  $M \rightarrow \mathbf{D}\mathbf{D}M$  is an isomorphism if and only if each  $H^i(M) \rightarrow H^i(M)^{**}$  is an isomorphism. By Lemma B.1.1, this is equivalent to each  $H^i(M)$  being finitely generated.

We now turn to the second statement. For (a)  $\implies$  (b), assume that  $M \rightarrow \mathbf{D}\mathbf{D}M$  and  $\mathbf{D}M \otimes M \rightarrow \mathbf{D}(M \otimes \mathbf{D}M)$  are isomorphisms. Then  $\mathbf{R}\mathrm{Hom}(M, M) \cong \mathbf{R}\mathrm{Hom}(M, \mathbf{D}\mathbf{D}M) \cong \mathbf{R}\mathrm{Hom}(M \otimes \mathbf{D}M, \Lambda) \cong \mathbf{D}(M \otimes \mathbf{D}M) \cong \mathbf{D}M \otimes M$ .

For (b)  $\implies$  (c), the identity map on  $M$  induces a morphism  $\varepsilon: \Lambda[0] \rightarrow \mathbf{R}\mathrm{Hom}(M, M) \xrightarrow{\sim} M \otimes \mathbf{D}M$  (the coevaluation map). The required inverse to  $N \otimes \mathbf{D}M \rightarrow \mathbf{R}\mathrm{Hom}(M, N)$  is

$$\mathbf{R}\mathrm{Hom}(M, N) \xrightarrow{\mathrm{id} \otimes \varepsilon} \mathbf{R}\mathrm{Hom}(M, N) \otimes M \otimes \mathbf{D}M \rightarrow N \otimes \mathbf{D}M.$$

For (c)  $\implies$  (d), we use the fact that  $\otimes$  commutes with colimits.

For (d)  $\implies$  (e), we use the fact that compact objects of  $D(\Lambda)$  are perfect [Sta17b, Tag 07LT].

Finally, for (e) implies (a), we can write  $M$  as a bounded complex of finitely generated projective  $\Lambda$ -modules. Then duals and derived tensor products can be computed on the level of chain complexes. We are reduced to showing, for finitely generated projective  $\Lambda$ -modules  $A$  and  $B$ , that  $A \rightarrow$

$A^{**}$  and  $A^* \otimes B \rightarrow (A \otimes B^*)^*$  are isomorphisms. After localizing on  $\Lambda$ , we may assume that  $A$  and  $B$  are free of finite rank (since duals commute over direct sums), where these statements are easy to check.  $\square$

We thank David Hansen and Bhargav Bhatt for their help with the above proof.

## B.2 Sheaves on locally profinite sets

Let  $S$  be a locally profinite set and  $\Lambda$  a discrete ring. We have the ring  $\mathcal{C}^\infty(S, \Lambda)$  of locally constant functions on  $S$  and the ring  $\mathcal{C}_c^\infty(S, \Lambda)$  of locally constant compactly supported functions on  $S$ . For each compact open subset  $U \subset S$  let  $\mathbf{1}_U$  denote the characteristic function. Then  $\mathcal{C}^\infty(U, \Lambda)$  is a principal ideal of both  $\mathcal{C}_c^\infty(S, \Lambda)$  and  $\mathcal{C}^\infty(S, \Lambda)$  generated by  $\mathbf{1}_U$ . Multiplication by  $\mathbf{1}_U$  is a homomorphism  $\mathcal{C}^\infty(S, \Lambda) \rightarrow \mathcal{C}^\infty(U, \Lambda)$  of rings with unity. In this way every  $\mathcal{C}^\infty(U, \Lambda)$ -module becomes a  $\mathcal{C}^\infty(S, \Lambda)$ -module.

**Definition B.2.1.** We call a  $\mathcal{C}^\infty(S, \Lambda)$ -module  $M$

1. *smooth* if it satisfies the following equivalent conditions
  - (a) The multiplication map  $M \otimes_{\mathcal{C}_c^\infty(S, \Lambda)} \mathcal{C}^\infty(S, \Lambda) \rightarrow M$  is an isomorphism.
  - (b) The natural map  $\varinjlim(\mathbf{1}_U \cdot M) \rightarrow M$  is an isomorphism, where the colimit runs over the open compact subsets  $U \subset S$  and the transition map  $\mathbf{1}_U \cdot M \rightarrow \mathbf{1}_V \cdot M$  for  $U \subset V$  is given by the natural inclusion.
2. *complete* if the natural map  $M \rightarrow \varprojlim_U(\mathbf{1}_U \cdot M)$  is an isomorphism, where again  $U$  runs over the open compact subsets of  $S$  and the transition map  $\mathbf{1}_U \cdot M \rightarrow \mathbf{1}_V \cdot M$  for  $V \subset U$  is multiplication by  $\mathbf{1}_V$ .

**Lemma B.2.2.** *Let  $V \subset S$  be compact open and let  $M$  be any  $\mathcal{C}^\infty(S, \Lambda)$ -module. Then*

1.  $\mathbf{1}_V \cdot M$  is a submodule of  $\varinjlim_U(\mathbf{1}_U \cdot M)$  and equals  $\mathbf{1}_V \cdot \varinjlim(\mathbf{1}_U \cdot M)$ .
2.  $\mathbf{1}_V \cdot M$  is a submodule of  $\varprojlim_U(\mathbf{1}_U \cdot M)$  and equals  $\mathbf{1}_V \cdot \varprojlim(\mathbf{1}_U \cdot M)$ .

**Lemma B.2.3.** 1. *The functor  $M \mapsto M^s := \varinjlim(\mathbf{1}_U \cdot M)$  is a projector onto the category of smooth modules.*

2. *The functor  $M \mapsto M^c := \varprojlim(\mathbf{1}_U \cdot M)$  is a projector onto the category of complete modules.*

3. *The two functors give mutually inverse equivalences of categories between the categories of smooth and complete modules.*

Let  $\mathcal{B}$  be the set of open compact subsets of  $S$ . Then  $\mathcal{B}$  is a basis for the topology of  $S$  and is closed under taking finite intersections and finite unions. Restriction gives an equivalence between the category of sheaves on  $S$  and the category of sheaves on  $\mathcal{B}$ . Define  $R(U) = \mathcal{C}^\infty(U, \Lambda)$ . This is a sheaf of rings on  $S$ .

Let  $\mathcal{F}$  be an  $R$ -module sheaf on  $S$ . For  $U \in \mathcal{B}$  we extend the  $R(U)$ -module structure on  $\mathcal{F}(U)$  to an  $\mathcal{C}^\infty(S, \Lambda)$ -module structure as remarked above. Then the restriction map  $\mathcal{F}(S) \rightarrow \mathcal{F}(U)$  becomes a morphism of  $\mathcal{C}^\infty(S, \Lambda)$ -modules.

**Lemma B.2.4.** *1. For any  $U \in \mathcal{B}$  the restriction map  $\mathcal{F}(S) \rightarrow \mathcal{F}(U)$  is surjective and its restriction to  $\mathbf{1}_U \cdot \mathcal{F}(S)$  is an isomorphism  $\mathbf{1}_U \cdot \mathcal{F}(S) \rightarrow \mathcal{F}(U)$ .*

2. *We have  $\mathcal{F}(S) = \varinjlim_U \mathcal{F}(U)$ , where the transition maps are the restriction maps.*

Let  $M$  be an  $\mathcal{C}^\infty(S, \Lambda)$ -module. Let  $\mathcal{F}_M(U) = R(U)M = \mathbf{1}_U M$ . This is a  $\mathcal{C}^\infty(S, \Lambda)$ -submodule of  $M$ . Given  $V, U \in \mathcal{B}$  with  $V \subset U$  we have the map  $\mathcal{F}_M(U) \rightarrow \mathcal{F}_M(V)$  defined by multiplication by  $\mathbf{1}_V$ . In this way  $\mathcal{F}_M$  becomes an  $R$ -module sheaf.

Let  $f : M \rightarrow N$  be a morphism of  $\mathcal{C}^\infty(S, \Lambda)$ -modules. We define for each  $U$  the morphism  $f_U : \mathcal{F}_M(U) \rightarrow \mathcal{F}_N(U)$  simply by restricting  $f$  to  $\mathcal{F}_M(U)$ . One checks immediately that  $(f_U)_U$  is a morphism of sheaves of  $R$ -modules. Therefore we obtain a functor from the category of  $\mathcal{C}^\infty(S, \Lambda)$ -modules to the category of sheaves of  $R$ -modules.

Given a sheaf  $\mathcal{F}$  on  $S$  we can define the smooth module  $M_{\mathcal{F}}^s$  and the complete module  $M_{\mathcal{F}}^c$  by

$$M_{\mathcal{F}}^s = \varinjlim_U \mathcal{F}(U) \quad M_{\mathcal{F}}^c = \varprojlim_U \mathcal{F}(U)$$

where the limit is taken over the restriction maps, and the colimit is taken over their sections given by Lemma B.2.4, and in both cases  $U$  runs over  $\mathcal{B}$ . Conversely, given any  $\mathcal{C}^\infty(S, \Lambda)$ -module  $M$  we have the sheaf  $\mathcal{F}_M$ .

**Lemma B.2.5.** *These functors give mutually inverse equivalences of categories from the category of smooth (resp. complete)  $\mathcal{C}^\infty(S, \Lambda)$ -modules to the category of  $R$ -module sheaves. These equivalences commute with the equivalence between the categories of smooth and complete modules. Furthermore  $\mathcal{F}_M(S) = M^c$ .*

## C A primer on reflexive sheaves, by David Hansen

In this appendix we discuss some basic examples and non-examples of reflexive sheaves, mostly in the context of classical rigid geometry. Although not strictly necessary in the main text of the paper, we hope these results might partially illuminate the hypotheses of reflexivity and strong reflexivity which recur throughout the paper. We also note that some closely related ideas were worked out almost simultaneously by Gaisin and Welliaveetil [GW17].

### C.1 Results

Throughout what follows we fix an algebraically closed nonarchimedean base field  $C$  which we assume (for simplicity) is of mixed characteristic  $(0, p)$ . By an *adic space* we shall mean an adic space  $X$  over  $S = \mathrm{Spa}(C, \mathcal{O}_C)$  which is locally of +-weakly finite type, separated, taut and finite-dimensional. By a *rigid analytic space* we shall mean an adic space of the aforementioned type which is locally of topologically finite type and reduced; note that this last condition is harmless, since replacing a rigid space by its nilreduction leaves the étale site unchanged.

Fix a prime power  $\ell^n$  with  $\ell \neq p$ , and set  $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$ . For any separated taut finite-dimensional morphism  $f : X \rightarrow Y$  of adic spaces which is locally of +-weakly finite type, Huber [Hub96, §5.5 & §7.1] defined a functor  $Rf_! : D(X_{\text{ét}}, \Lambda) \rightarrow D(Y_{\text{ét}}, \Lambda)$  admitting a right adjoint  $Rf^!$ . In particular, if  $X$  is an adic space with structure morphism  $f : X \rightarrow S$ , we may consider the dualizing complex  $\kappa_X \stackrel{\text{def}}{=} Rf^!\Lambda$  and the duality functor

$$\begin{aligned} D(X_{\text{ét}}, \Lambda) &\rightarrow D(X_{\text{ét}}, \Lambda) \\ \mathcal{F} &\mapsto \mathbf{D}\mathcal{F} \stackrel{\text{def}}{=} \underline{\mathrm{RHom}}(\mathcal{F}, \kappa_X). \end{aligned}$$

**Definition C.1.1.** An object  $\mathcal{F} \in D(X_{\text{ét}}, \Lambda)$  is *reflexive* if the natural biduality map

$$\mathcal{F} \rightarrow \mathbf{D}\mathbf{D}\mathcal{F}$$

is an isomorphism.

As in the main text of the paper, this property is clearly étale-local on  $X$ , and is preserved under pullback along smooth maps and derived pushforward along proper maps. Moreover, reflexivity satisfies a 2-out-of-3 property: if  $\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow$  is an exact triangle in  $D(X_{\text{ét}}, \Lambda)$  such that two terms in the triangle are reflexive, then all three terms are reflexive. We also note that if  $i : Z \rightarrow X$  is a closed embedding and  $\mathcal{F} \in D(Z_{\text{ét}}, \Lambda)$  is such that  $i_*\mathcal{F}$  is

reflexive, then  $\mathcal{F}$  is reflexive. Finally, we observe that if  $\mathcal{F}$  is bounded with reflexive cohomology sheaves, then  $\mathcal{F}$  is reflexive itself.

One can make an analogous definition in the world of classical algebraic geometry, and it's a standard fact that *constructible* sheaves are reflexive in that setting. We remind the reader that if  $\mathcal{X}$  is a separated finite type  $C$ -scheme with associated rigid analytic space  $X$ , then pullback along the natural map of sites  $\mu : X_{\acute{e}t} \rightarrow \mathcal{X}_{\acute{e}t}$  does not preserve constructibility, essentially because Zariski-open subsets of  $X$  are not quasicompact. Instead, the  $\mu$ -pullback of a constructible sheaf on  $\mathcal{X}_{\acute{e}t}$  is an example of a *Zariski-constructible* sheaf on  $X_{\acute{e}t}$ . There is also an intrinsic notion of constructible sheaf in the rigid analytic world, which is of a rather different flavor. Our first order of business is to check that these examples are all reflexive:

**Proposition C.1.2.** *If  $X$  is a rigid analytic space, then any object  $\mathcal{F} \in D^b(X_{\acute{e}t}, \Lambda)$  with constructible or Zariski-constructible cohomology sheaves is reflexive.*

In §C.3 below, we sketch a direct proof that constructible sheaves are reflexive. The idea is to first show that the constant sheaf  $\Lambda$  is reflexive on any rigid space  $X$ , which we then upgrade to the reflexivity of  $j_! \Lambda$  where  $j : U \rightarrow X$  is any separated étale map with affinoid source. For the reflexivity of  $\Lambda$ , we reduce to the smooth case using resolution of singularities.

However, it is more conceptual to deduce Proposition C.1.2 from a general criterion for reflexivity which was explained to us by Peter Scholze. To state this result, recall that for any (reduced) affinoid rigid space  $U = \mathrm{Spa}(A, A^\circ)$  with its natural formal model  $\mathfrak{U} = \mathrm{Spf}(A^\circ)$  over  $\mathrm{Spf}(\mathcal{O}_C)$ , there is a natural map of sites  $\lambda_U : U_{\acute{e}t} \rightarrow \mathfrak{U}_{\acute{e}t}$  which induces a “nearby cycles” map  $R\lambda_{U*} : D^b(U_{\acute{e}t}, \Lambda) \rightarrow D^b(\mathfrak{U}_{\acute{e}t}, \Lambda)$ .

**Proposition C.1.3** (Scholze). *Let  $X$  be a rigid analytic space. Suppose  $\mathcal{F} \in D^b(X_{\acute{e}t}, \Lambda)$  has the property that for every affinoid rigid space  $U = \mathrm{Spa}(A, A^\circ)$  with an étale map  $a : U \rightarrow X$ , the nearby cycles  $R\lambda_{U*} a^* \mathcal{F}$  are constructible. Then  $\mathcal{F}$  is reflexive.*

Combining this with a result of Huber, we deduce

**Corollary C.1.4.** *If  $X$  is a rigid analytic space and  $\mathcal{F} \in D^b(X_{\acute{e}t}, \Lambda)$  has quasi-constructible or oc-quasi-constructible cohomology sheaves in the sense of [Hub98b, Hub98c], then  $\mathcal{F}$  is reflexive.*

In the setting of sheaves on a finite type  $C$ -scheme, it may be true that reflexivity and constructibility coincide. One can thus ask whether reflexivity on rigid spaces is characterized by some variant of constructibility; however, this seems unlikely:

**Proposition C.1.5.** *There is an example of a reflexive sheaf on  $\mathrm{Spa} C \langle T_1, \dots, T_6 \rangle$  with (some) infinite-dimensional stalks.*

Finally, we illustrate the failure of the Lefschetz fixed-point formula with an example of a reflexive sheaf which is not strongly reflexive.

**Proposition C.1.6.** *Let  $X = \mathrm{Spa}(C \langle T \rangle, \mathcal{O}_C \langle T \rangle)$  be the one-dimensional rigid disk, and let  $\overline{X} = \mathrm{Spa}(C \langle T \rangle, \mathcal{O}_C + T \cdot C \langle T \rangle^{\circ\circ})$  be its canonical adic compactification, with  $j : X \rightarrow \overline{X}$  the natural inclusion. Then the sheaves  $\Lambda_{\overline{X}}$  and  $j_! \Lambda_X$  are reflexive but not strongly reflexive.*

This stands in contrast to the situation in classical algebraic geometry, where any constructible sheaf on a finite type  $C$ -scheme is strongly reflexive. In the course of building this example, we determine the dualizing complex of  $\overline{X}$ ; rather strangely, it turns out that  $\kappa_{\overline{X}} \simeq j_! \Lambda_X[2](1)$ . In particular, the dualizing complex of  $\overline{X}$  is not overconvergent, and some of its stalks vanish identically, in stark contrast with the case of rigid analytic spaces, cf. Proposition C.3.4. Morally, the failure of  $\kappa_{\overline{X}}$  to overconverge on the locus lying over the topological fixed points of  $T \mapsto T + 1$  is “responsible” for the failure of the Lefschetz formula for this automorphism.

## C.2 Nearby cycles and reflexivity

In this section we deduce Proposition C.1.3 from the following result, elaborating on a sketch explained to us by Peter Scholze.

**Proposition C.2.1.** *Let  $X = \mathrm{Spa}(A, A^\circ)$  be an affinoid rigid analytic space over  $\mathrm{Spa}(C, \mathcal{O}_C)$  as before; set  $\mathfrak{X} = \mathrm{Spf}(A^\circ)$ , so we get a nearby cycles map  $R\lambda_{X^*} : D^b(X_{\acute{e}t}, \Lambda) \rightarrow D^b(\mathfrak{X}_{\acute{e}t}, \Lambda)$  as in the introduction. Then there is a natural equivalence  $R\lambda_{X^*} \mathbf{D}_X \cong \mathbf{D}_{\mathfrak{X}} R\lambda_{X^*}$  compatible with étale localization and with the biduality maps, where  $\mathbf{D}_X$  and  $\mathbf{D}_{\mathfrak{X}}$  denote the natural Verdier duality functors on  $X_{\acute{e}t}$  and  $\mathfrak{X}_{\acute{e}t}$ .*

In most other settings where a nearby cycles functor is defined, commutation with Verdier duality is well-known (cf. [Ill94] and [Mas16], for example). However, the present situation is somewhat unique in that  $R\lambda_{X^*}$  admits a useful left adjoint, which we’ll exploit heavily in the proof of Proposition C.2.1.

*Proof of Proposition C.1.3.* Fix  $\mathcal{F} \in D^b(X_{\acute{e}t}, \Lambda)$  satisfying the conditions of the proposition. We need to show that the cone of the biduality map  $\beta : \mathcal{F} \rightarrow \mathbf{D}_X \mathbf{D}_X \mathcal{F}$  is acyclic. Given any étale map  $a : U = \mathrm{Spa}(B, B^\circ) \rightarrow X$ , the

constructibility hypothesis in the proposition guarantees that the biduality map

$$R\lambda_{U*}a^*\mathcal{F} \rightarrow \mathbf{D}_{\mathfrak{U}}\mathbf{D}_{\mathfrak{U}}R\lambda_{U*}a^*\mathcal{F}$$

is an isomorphism by Théorème 4.3 in [Del77, Th. de finitude]. We then see that the biduality map  $a^*\mathcal{F} \rightarrow \mathbf{D}_U\mathbf{D}_Ua^*\mathcal{F} \cong a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}$  induces a map

$$R\lambda_{U*}a^*\mathcal{F} \rightarrow R\lambda_{U*}\mathbf{D}_U\mathbf{D}_Ua^*\mathcal{F} \cong \mathbf{D}_{\mathfrak{U}}R\lambda_{U*}\mathbf{D}_Ua^*\mathcal{F} \cong \mathbf{D}_{\mathfrak{U}}\mathbf{D}_{\mathfrak{U}}R\lambda_{U*}a^*\mathcal{F} \cong R\lambda_{U*}a^*\mathcal{F}$$

whose composition is the identity; here the first two isomorphisms are obtained by applying Proposition C.2.1 twice, and the third isomorphism is given by the inverse of the biduality map for  $\mathfrak{U}$ . In particular, the map

$$R\lambda_{U*}a^*\beta : R\lambda_{U*}a^*\mathcal{F} \rightarrow R\lambda_{U*}a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}$$

is an isomorphism. Passing to derived global sections on  $\mathfrak{U}$  gives an isomorphism

$$R\Gamma(U, \mathcal{F}|_U) \cong R\Gamma(\mathfrak{U}, R\lambda_{U*}a^*\mathcal{F}) \cong R\Gamma(\mathfrak{U}, R\lambda_{U*}a^*\mathbf{D}_X\mathbf{D}_X\mathcal{F}) \cong R\Gamma(U, \mathbf{D}_X\mathbf{D}_X\mathcal{F}|_U),$$

so in particular  $R\Gamma(U, \text{Cone}(\beta)) \simeq 0$  for all affinoid étale maps  $U \rightarrow X$ . Since the stalks of the cohomology sheaves of  $\text{Cone}(\beta)$  can be computed as colimits of  $H^i(R\Gamma(U_j, \text{Cone}(\beta)))$  over suitable cofiltered inverse systems of affinoid étale maps  $U_j \rightarrow X$ , we deduce that  $\text{Cone}(\beta)$  is acyclic, as desired.  $\square$

This criterion is very useful in practice:

*Proof of Proposition C.1.2 and Corollary C.1.4 .* In [Hub98b, Hub98c], Huber defines classes of étale sheaves which he calls quasi-constructible and oc-quasi-constructible, which are preserved under arbitrary pullback and with the property that any constructible (resp. Zariski-constructible) sheaf is quasi-constructible (resp. oc-quasi-constructible). Moreover, he proves that the nearby cycles of such sheaves are always constructible, cf [Hub98b, Prop. 3.11] and [Hub98c, Prop. 2.12]. Combining these results with Proposition C.1.3, we get the result.  $\square$

The proof of Proposition C.2.1 requires some work. Throughout the rest of this subsection, we fix  $X$  and  $\mathfrak{X}$  as in the statement of the result, and we let  $f : X_{\text{ét}} \rightarrow \text{Spa}(C, \mathcal{O}_C)_{\text{ét}}$  and  $\mathfrak{f} : \mathfrak{X}_{\text{ét}} \rightarrow \text{Spec}(\mathcal{O}_C/p)_{\text{ét}}$  denote the natural morphisms of étale sites. Note that we can identify the derived categories  $D(\text{Spa}(C, \mathcal{O}_C)_{\text{ét}}, \Lambda)$  and  $D(\text{Spec}(\mathcal{O}_C/p)_{\text{ét}}, \Lambda)$  with the derived category  $D(\Lambda)$  of  $\Lambda$ -modules. The key non-formal ingredient in the argument is the existence of a natural equivalence  $Rf_! \cong R\mathfrak{f}_!R\lambda_{X*}$ , which can be proved as in [Hub98c, Lemma 2.13].

*Proof.* We first construct a natural transformation  $\gamma_X : R\lambda_{X*}\mathbf{D}_X \rightarrow \mathbf{D}_{\mathfrak{X}}R\lambda_{X*}$ . To do this, observe that for any  $A \in D(\mathfrak{X}_{\acute{e}t}, \Lambda)$  and  $B \in D(X_{\acute{e}t}, \Lambda)$ , we have a natural series of morphisms

$$\begin{aligned}
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*}\mathbf{D}_X B) &\stackrel{(1)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \mathbf{D}_X B) \\
&\stackrel{(2)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} B, Rf_! \Lambda) \\
&\stackrel{(3)}{\cong} \mathrm{Hom}_{D(\Lambda)}(Rf_!(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \\
&\stackrel{(4)}{\cong} \mathrm{Hom}_{D(\Lambda)}(Rf_! R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) \\
&\stackrel{(5)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B), Rf_! \Lambda) \\
&\stackrel{(6)}{\rightarrow} \mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} R\lambda_{X*} B, Rf_! \Lambda) \\
&\stackrel{(7)}{\cong} \mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, \mathbf{D}_{\mathfrak{X}}R\lambda_{X*} B)
\end{aligned}$$

obtained as follows: (1) follows from adjointness of  $R\lambda_{X*}$  and  $\lambda_X^*$ ; (2) and (7) are tensor-hom adjunction; (3) follows from adjointness of  $Rf_!$  and  $Rf_!$ ; (4) follows from the natural equivalence  $Rf_! \cong Rf_! R\lambda_{X*}$  explained above; (5) follows from adjointness of  $Rf_!$  and  $Rf_!$ ; and (6) is dual to the natural ‘‘projection map’’  $A \otimes^{\mathbf{L}} R\lambda_{X*} B \rightarrow R\lambda_{X*}(\lambda_X^* A \otimes^{\mathbf{L}} B)$  obtained as the adjoint to the composition

$$\lambda_X^*(A \otimes^{\mathbf{L}} R\lambda_{X*} B) \cong \lambda_X^* A \otimes^{\mathbf{L}} \lambda_X^* R\lambda_{X*} B \rightarrow \lambda_X^* A \otimes^{\mathbf{L}} B,$$

cf. [Sta17b, Tag 0943]. The composition of these morphisms induces a map

$$\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*}\mathbf{D}_X B) \rightarrow \mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, \mathbf{D}_{\mathfrak{X}}R\lambda_{X*} B)$$

which is functorial in  $A$  and  $B$ , so we obtain the desired natural transformation from the Yoneda lemma. Next we observe that when  $A$  is perfect, the map (6) is an isomorphism by [Sta17b, Tag 0943]. In particular, taking  $A = \Lambda[n]$  for varying  $n$ , we see that  $\gamma_X$  induces an isomorphism  $R\Gamma(\mathfrak{X}, R\lambda_{X*}\mathbf{D}_X B) \xrightarrow{\sim} R\Gamma(\mathfrak{X}, \mathbf{D}_{\mathfrak{X}}R\lambda_{X*} B)$  for any  $B$ .

Now let  $\mathfrak{Y} = \mathrm{Spf}(B^\circ)$  be any affine formal scheme with an étale map  $j : \mathfrak{Y} \rightarrow \mathfrak{X}$ , and let  $j : Y = \mathrm{Spa}(B, B^\circ) \rightarrow X$  denote the induced étale map on rigid generic fibers. We then claim that formation of  $\gamma_X$  is compatible with étale localization, in the sense that the natural diagram

$$\begin{array}{ccc}
j^*R\lambda_{X*}\mathbf{D}_X B & \xrightarrow{\sim} & R\lambda_{Y*}\mathbf{D}_Y j^* B \\
\downarrow j^*\gamma_X & & \downarrow \gamma_Y \\
j^*\mathbf{D}_{\mathfrak{X}}R\lambda_{X*}B & \xrightarrow{\sim} & \mathbf{D}_{\mathfrak{Y}}R\lambda_{Y*}j^* B
\end{array}$$

commutes; here the horizontal isomorphisms are induced by the natural isomorphisms  $j^*\mathbf{D}_{\mathfrak{X}} \cong \mathbf{D}_{\mathfrak{Y}}j^*$  and  $j^*\mathbf{D}_X \cong \mathbf{D}_Y j^*$  together with the (easy) base change isomorphism  $j^*R\lambda_{X*} \cong R\lambda_{Y*}j^*$ . Granted the commutativity of this diagram, passing to derived global sections on  $\mathfrak{Y}$  induces a commutative diagram

$$\begin{array}{ccc}
R\Gamma(\mathfrak{Y}, (R\lambda_{X*}\mathbf{D}_X B)|_{\mathfrak{Y}}) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, R\lambda_{Y*}\mathbf{D}_Y j^* B) \\
\downarrow & & \downarrow \wr \\
R\Gamma(\mathfrak{Y}, (\mathbf{D}_{\mathfrak{X}}R\lambda_{X*}B)|_{\mathfrak{Y}}) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, \mathbf{D}_{\mathfrak{Y}}R\lambda_{Y*}j^* B)
\end{array}$$

where, crucially, the righthand vertical arrow is an isomorphism by arguing as in the previous paragraph with  $\gamma_Y$  in place of  $\gamma_X$ . Going around the diagram, we see that  $\gamma_X$  induces an isomorphism  $R\Gamma(\mathfrak{Y}, R\lambda_{X*}\mathbf{D}_X B) \xrightarrow{\sim} R\Gamma(\mathfrak{Y}, \mathbf{D}_{\mathfrak{X}}R\lambda_{X*}B)$  for any affine étale map  $\mathfrak{Y} \rightarrow \mathfrak{X}$ , and therefore  $\gamma_X$  is an equivalence, as desired.

It remains to check the commutativity of the aforementioned square. This follows from a rather horrible diagram chase. More precisely, choose any  $C \in D(\mathfrak{Y}_{\text{ét}}, \Lambda)$ , and set  $A = j_!C$ ; we then need to check that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*} \mathbf{D}_X B) & \xrightarrow[\mathrm{i.}]{\sim} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(C, R\lambda_{Y*} \mathbf{D}_Y j^* B) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \mathbf{D}_X B) & \xrightarrow[\mathrm{ii.}]{} & \mathrm{Hom}_{D(Y_{\acute{e}t}, \Lambda)}(\lambda_Y^* C, \mathbf{D}_Y j^* B) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} B, Rf^! \Lambda) & \xrightarrow[\mathrm{iii.}]{} & \mathrm{Hom}_{D(Y_{\acute{e}t}, \Lambda)}(\lambda_Y^* C \otimes^{\mathbf{L}} j^* B, j^! Rf^! \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\Lambda)}(Rf_! (\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) & \xrightarrow[\mathrm{iv.}]{} & \mathrm{Hom}_{D(\Lambda)}(Rf_! j_! (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\Lambda)}(Rf_! R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B), \Lambda) & \xrightarrow[\mathrm{v.}]{} & \mathrm{Hom}_{D(\Lambda)}(Rf_! j_! R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B), Rf^! \Lambda) & \xrightarrow[\mathrm{vi.}]{} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B), j^! Rf^! \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} R\lambda_{X*} B, Rf^! \Lambda) & \xrightarrow[\mathrm{vii.}]{} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(C \otimes^{\mathbf{L}} R\lambda_{Y*} j^* B, j^! Rf^! \Lambda) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(\mathfrak{X}_{\acute{e}t}, \Lambda)}(A, \mathbf{D}_X R\lambda_{X*} B) & \xrightarrow[\mathrm{viii.}]{\sim} & \mathrm{Hom}_{D(\mathfrak{Y}_{\acute{e}t}, \Lambda)}(C, \mathbf{D}_Y R\lambda_{Y*} j^* B)
\end{array}$$

commutes, functorially in  $C$  and  $B$ . Here the top and bottom horizontal arrows correspond to the horizontal isomorphisms in our original square, and the composition of all left and right vertical maps define  $j^* \gamma_X$  and  $\gamma_Y$ , respectively, by Yoneda. Let us explain the intermediate horizontal arrows together with some of the commutativity checks, leaving a few details to the reader. The idea is to repeatedly use adjointness of the pairs  $(j_!, j^* = j^!)$  and  $(j_!, j^* = j^!)$ , together with the base change isomorphism  $j^* R\lambda_{X*} \cong R\lambda_{Y*} j^*$  and its adjoint incarnation  $\lambda_X^* j_! \cong j_! \lambda_Y^*$ . In particular, applying the latter to  $A = j_! C$  gives a natural isomorphism  $\lambda_X^* A \cong j_! \lambda_Y^* C$ . Combining this isomorphism with the adjunction of  $j_!$  and  $j^*$  induces the arrow labeled ii.; on the other hand, tensoring this isomorphism with  $B$  and applying the projection formula for  $j_!$  gives

$$\lambda_X^* A \otimes^{\mathbf{L}} B \cong j_! (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B),$$

which induces arrows iii. and iv. (here we've again used the adjunction of  $j_!$  and  $j^*$ ).

Next, by Lemma C.2.2 below, there is a natural equivalence  $\tau : j_! R\lambda_{Y*} \rightarrow R\lambda_{X*} j_!$ , which moreover is compatible with  $Rf_!$  in the sense that the composite map

$$Rf_! j_! R\lambda_{Y*} \xrightarrow{Rf_! \tau} Rf_! R\lambda_{X*} j_! \cong Rf_! j_!$$

induces the natural equivalence

$$R(f \circ j)_! R\lambda_{Y*} \cong R(f \circ j)_!$$

Applying this transformation to  $\lambda_Y^* C \otimes^{\mathbf{L}} j^* B$  induces a map

$$j_! R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B) \rightarrow R\lambda_{X*} j_! (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B) \cong R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B),$$

which gives rise to arrows v. and vi. via suitable adjunctions; commutativity of the square spanned by arrows iv. and v. follows from the aforementioned compatibility of this transformation with  $Rf_!$ , and commutativity of the square spanned by arrows v. and vi. is straightforward. Next, we observe that the previous map together with the projection maps

$$A \otimes^{\mathbf{L}} R\lambda_{X*} B \xrightarrow{\pi_X} R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B)$$

and

$$C \otimes^{\mathbf{L}} R\lambda_{Y*} j^* B \xrightarrow{\pi_Y} R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B)$$

fit together sits in a commutative square

$$\begin{array}{ccc} R\lambda_{X*} (\lambda_X^* A \otimes^{\mathbf{L}} B) & \longleftarrow & j_! R\lambda_{Y*} (\lambda_Y^* C \otimes^{\mathbf{L}} j^* B) \\ \pi_X \uparrow & & \uparrow j_! \pi_Y \\ A \otimes R\lambda_{X*} B & \longleftarrow & j_! (C \otimes R\lambda_{Y*} j^* B) \end{array}$$

where the lower horizontal arrow is given by the inverse of the composition

$$A \otimes R\lambda_{X*} B = j_! C \otimes R\lambda_{X*} B \cong j_! (C \otimes j^* R\lambda_{X*} B) \cong j_! (C \otimes R\lambda_{Y*} j^* B).$$

Applying  $\mathrm{Hom}_{D(\mathfrak{X}_{\text{ét}}, \Lambda)}(-, Rf^! \Lambda)$  to this square and using the adjunction of  $j_!$  and  $j^!$  on the righthand column, we get arrows vi. and vii. together with the commutativity of the relevant square.  $\square$

In the course of these arguments, we used the following lemma.

**Lemma C.2.2.** *Let  $X$  and  $\mathfrak{X}$  be as above, and let  $j : \mathfrak{Y} = \mathrm{Spf}(B^\circ) \rightarrow \mathfrak{X}$  be an étale map of affine formal schemes, with  $j : Y = \mathrm{Spa}(B, B^\circ) \rightarrow X$  the induced map on rigid generic fibers. Then the natural transformation  $\tau : j_! R\lambda_{Y*} \rightarrow R\lambda_{X*} j_!$  defined as the adjoint to the composition*

$$R\lambda_{Y*} \rightarrow R\lambda_{Y*} j^* j_! \cong j^* R\lambda_{X*} j_!$$

*is an equivalence, and is compatible with  $Rf_!$  in the sense that the composite map*

$$Rf_! j_! R\lambda_{Y*} \xrightarrow{Rf_! \tau} Rf_! R\lambda_{X*} j_! \cong Rf_! j_!$$

*coincides with the natural equivalence*

$$R(f \circ j)_! R\lambda_{Y*} \cong R(f \circ j)_!.$$

*Proof.* By a standard argument (cf. [Sta17b, Tag 0AN8]), we can find an étale ring map  $A^\circ \rightarrow B_0$  such that  $B^\circ = \lim_{\leftarrow} B_0/p^n B_0$  as  $A^\circ$ -algebras. By Zariski's main theorem, we can find a module-finite ring map  $A^\circ \rightarrow D$  fitting into a diagram

$$\begin{array}{ccc} \mathrm{Spec}(B_0) & \xrightarrow{i} & \mathrm{Spec}(D) \\ & \searrow j & \downarrow h \\ & & \mathrm{Spec}(A^\circ) \end{array}$$

where  $i$  is an open immersion. Passing to  $p$ -adic completions induces a corresponding diagram

$$\begin{array}{ccc} \mathfrak{Y} = \mathrm{Spf}(B^\circ) & \xrightarrow{i} & \overline{\mathfrak{Y}} = \mathrm{Spf}(D) \\ & \searrow j & \downarrow h \\ & & \mathfrak{X} = \mathrm{Spf}(A^\circ) \end{array}$$

of  $p$ -adic formal schemes; here we used the fact that  $D \cong \lim_{\leftarrow} D/p^n D$ , which holds since  $A^\circ \rightarrow D$  is module-finite and  $A^\circ$  is  $p$ -adically separated and complete and Noetherian outside its ideal of definition, cf. [FGK11, Proposition 6.1.1(1)]. Passing to a similar diagram on the generic fibers and

going to étale sites, we get a commutative diagram

$$\begin{array}{ccccc}
& & Y_{\text{ét}} & \xrightarrow{\lambda_Y} & \mathfrak{Y}_{\text{ét}} \\
& \swarrow i & \downarrow j & & \swarrow i \\
\bar{Y}_{\text{ét}} & \xrightarrow{j} & \bar{\mathfrak{Y}}_{\text{ét}} & \xrightarrow{\lambda_{\bar{Y}}} & \bar{\mathfrak{Y}}_{\text{ét}} \\
& \searrow h & \downarrow \lambda_X & & \searrow \mathfrak{h} \\
& & X_{\text{ét}} & \xrightarrow{\lambda_X} & \mathfrak{X}_{\text{ét}} \\
& & & & \downarrow j
\end{array}$$

where the  $\lambda_\bullet$ 's are the evident nearby cycles maps. Note that  $h_* \cong Rh_*$  and  $\mathfrak{h}_* \cong R\mathfrak{h}_*$  since both morphisms are finite. We then compute that

$$\begin{aligned}
j_! R\lambda_{Y*} &= \mathfrak{h}_* i_! R\lambda_{Y*} \\
&\cong \mathfrak{h}_* R\lambda_{\bar{Y}} i_! \\
&\cong R\lambda_{X*} h_* i_! \\
&= R\lambda_{X*} j_!
\end{aligned}$$

where the first isomorphism follows from [Hub96, Corollary 3.5.11.ii] and the second isomorphism is induced by the natural equivalence  $\mathfrak{h}_* \lambda_{\bar{Y}*} = \lambda_{X*} h_*$ .  $\square$

Although not strictly necessary, let us develop a little more theory around the nearby cycles map  $R\lambda_{X*}$ .

**Proposition C.2.3.** *Let  $X = \text{Spa}(A, A^\circ)$  and  $\mathfrak{X} = \text{Spf}(A^\circ)$  be as above. Then  $R\lambda_{X*}$  sends reflexive objects to reflexive objects, and the functor  $R\lambda_X^! \stackrel{\text{def}}{=} \mathbf{D}_X \lambda_X^* \mathbf{D}_{\mathfrak{X}}$  defines a “weak” right adjoint of  $R\lambda_{X*}$  on reflexive objects, in the sense that it induces functorial isomorphisms*

$$R\lambda_{X*} \underline{\mathbf{R}}\mathbf{H}\mathbf{om}_X(B, R\lambda_X^! C) \cong \underline{\mathbf{R}}\mathbf{H}\mathbf{om}_{\mathfrak{X}}(R\lambda_{X*} B, C)$$

and

$$\mathbf{H}\mathbf{om}_{D(\mathfrak{X}_{\text{ét}}, \Lambda)}(R\lambda_{X*} B, C) \cong \mathbf{H}\mathbf{om}_{D(X_{\text{ét}}, \Lambda)}(B, R\lambda_X^! C)$$

for any objects  $B \in D(X_{\text{ét}}, \Lambda)$  and  $C \in D(\mathfrak{X}_{\text{ét}}, \Lambda)$  with  $C$  reflexive.

Setting  $C = \kappa_{\mathfrak{X}}$  recovers Proposition C.2.1 as a special case of this result, although we use Proposition C.2.1 in the proof. We call  $R\lambda_X^!$  a “weak” right adjoint because we’re not sure how it behaves on non-reflexive objects, or whether it preserves reflexivity; amusingly, the argument doesn’t require the latter fact.

*Proof.* The local result implies the global result upon applying  $H^0(R\Gamma(\mathfrak{X}, -))$ . For the local result, we calculate that

$$\begin{aligned}
\mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, R\lambda_{X*} \underline{\mathrm{RHom}}_X(B, R\lambda_X^! C)) &\stackrel{(1)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \underline{\mathrm{RHom}}_X(B, R\lambda_X^! C)) \\
&\stackrel{(2)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A, \underline{\mathrm{RHom}}_X(\lambda_X^* \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_X B)) \\
&\stackrel{(3)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* A \otimes^{\mathbf{L}} \lambda_X^* \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_X B) \\
&\stackrel{(4)}{\cong} \mathrm{Hom}_{D(X_{\acute{e}t}, \Lambda)}(\lambda_X^* (A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C), \mathbf{D}_X B) \\
&\stackrel{(5)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C, R\lambda_{X*} \mathbf{D}_X B) \\
&\stackrel{(6)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A \otimes^{\mathbf{L}} \mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B) \\
&\stackrel{(7)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(\mathbf{D}_{\mathfrak{X}} C, \mathbf{D}_{\mathfrak{X}} R\lambda_{X*} B)) \\
&\stackrel{(8)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, \mathbf{D}_{\mathfrak{X}} \mathbf{D}_{\mathfrak{X}} C)) \\
&\stackrel{(9)}{\cong} \mathrm{Hom}_{D(\mathfrak{x}_{\acute{e}t}, \Lambda)}(A, \underline{\mathrm{RHom}}_{\mathfrak{X}}(R\lambda_{X*} B, C))
\end{aligned}$$

functorially in  $A \in D(\mathfrak{x}_{\acute{e}t}, \Lambda)$ . Here (1) and (5) follow from adjointness of  $\lambda_X^*$  and  $R\lambda_{X*}$ ; (3) and (7) follow from tensor-hom adjunction; (4) is trivial; (6) follows from Proposition C.2.1; (9) follows from the reflexivity of  $C$ ; and, finally, in (2) and (8) we've used the fact that for  $\bullet = X, \mathfrak{X}$  and any objects  $C, D \in D(\bullet_{\acute{e}t}, \Lambda)$  we have

$$\underline{\mathrm{RHom}}_{\bullet}(C, \mathbf{D}_{\bullet} D) \cong \underline{\mathrm{RHom}}_{\bullet}(D, \mathbf{D}_{\bullet} C)$$

functorially in  $C$  and  $D$ , which is an easy exercise in tensor-hom adjunction.  $\square$

### C.3 Constructible sheaves on rigid spaces

In this section we sketch an alternative proof that constructible sheaves on rigid spaces are reflexive, which is in some ways more naive. The first non-formal input is the following claim.

**Proposition C.3.1.** *If  $X$  is a rigid analytic space, then the constant sheaf  $\Lambda$  is reflexive.*

Note that by definition,  $\Lambda$  is reflexive if and only if the natural map  $\Lambda \rightarrow \underline{\mathrm{RHom}}(\kappa_X, \kappa_X)$  is an isomorphism.

*Proof.* The result clearly holds when  $X$  is smooth. For the general case, we argue by induction on the dimension of  $X$ . Thus, fix an integer  $d \geq 1$ . Assume the result holds for all rigid spaces of dimension  $< d$ , and let  $X$  be a  $d$ -dimensional (separated taut) rigid analytic space. We can assume that  $X = \mathrm{Spa}(A, A^\circ)$  is affinoid and reduced. The ring  $A$  is an excellent Noetherian ring, so by Temkin [Tem12] we can find a projective birational morphism  $f : \mathcal{X}' \rightarrow \mathrm{Spec}(A)$  where  $\mathcal{X}'$  is a regular  $C$ -scheme, such that  $f$  is an isomorphism over the regular locus of its target. This analytifies to a proper surjective map of rigid spaces

$$\pi : X' \rightarrow X$$

such that  $X' \rightarrow S$  is smooth. In particular,  $\Lambda_{X'} = \pi^* \Lambda_X$  is a reflexive sheaf on  $X'_{\acute{e}t}$ , so  $R\pi_* \Lambda_{X'}$  is reflexive by stability under proper pushforward. Now, writing  $K$  for the cone of the adjunction map

$$\alpha : \Lambda_X \rightarrow R\pi_* \pi^* \Lambda_X \cong R\pi_* \Lambda_{X'},$$

the 2-out-of-3 property shows that  $\Lambda_X$  is reflexive if  $K$  is reflexive.

For reflexivity of  $K$ , consider the diagram

$$\begin{array}{ccccc} U & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow \wr & & \downarrow \pi & & \downarrow \tau \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z \end{array}$$

where  $U$  is the smooth locus in  $X$  with closed complement  $Z$ , and both squares are cartesian. Since  $\pi|_{\pi^{-1}(U)}$  is an isomorphism,  $j^* \alpha$  is an isomorphism in  $D(U_{\acute{e}t}, \Lambda)$ , so  $j^* K \simeq 0$ ; applying the usual exact triangle  $j_! j^* \rightarrow \mathrm{id} \rightarrow i_* i^*$ , we get an isomorphism  $K \simeq i_* i^* K$ . Since  $i_*$  is a closed immersion and thus proper, we're now reduced to showing that  $i^* K$  is a reflexive object of  $D(Z_{\acute{e}t}, \Lambda)$ . This pullback can be computed as

$$\begin{aligned} i^* K &= i^* \mathrm{Cone}(\Lambda_X \rightarrow R\pi_* \pi^* \Lambda_X) \\ &\cong \mathrm{Cone}(i^* \Lambda_X \rightarrow i^* R\pi_* \pi^* \Lambda_X) \\ &\cong \mathrm{Cone}(\Lambda_Z \rightarrow R\tau_* i'^* \pi^* \Lambda_X) \\ &\cong \mathrm{Cone}(\Lambda_Z \rightarrow R\tau_* \tau^* \Lambda_Z) \\ &\cong \mathrm{Cone}(\Lambda_Z \rightarrow R\tau_* \Lambda_{Z'}), \end{aligned}$$

where the third line follows by proper base change. Since  $Z$  and  $Z'$  are both of dimension  $< d$ , the sheaves  $\Lambda_Z$  and  $\Lambda_{Z'}$  are reflexive by the induction hypothesis, and then  $R\tau_* \Lambda_{Z'}$  is reflexive as well since  $\tau$  is proper. Applying the 2-out-of-3 property again, we deduce that  $i^* K$  is reflexive, as desired.  $\square$

**Corollary C.3.2.** *If  $X$  is a rigid space and  $j : U \rightarrow X$  is the inclusion of a Zariski-open subset with Zariski-closed complement  $i : Z \rightarrow X$ , then  $j_!\Lambda$  and  $i_*\Lambda$  are reflexive.*

*Proof.* Writing  $i : Z \rightarrow X$  for the inclusion of the closed complement, the claim for  $i_*\Lambda$  is immediate from preservation of reflexivity under proper pushforward. The exact triangle  $j_!\Lambda \rightarrow \Lambda \rightarrow i_*\Lambda \rightarrow$  and the 2-out-of-3 property then imply reflexivity of  $j_!\Lambda$ .  $\square$

**Proposition C.3.3.** *Let  $X$  be an adic space, and let  $U \subset X$  be an open constructible subset with closure  $\bar{U} \subset X$ ; let  $j : U \rightarrow X$  and  $\bar{j} : \bar{U} \rightarrow X$  denote the evident inclusions. Then for any overconvergent sheaf  $\mathcal{F} \in Sh(X_{\acute{e}t}, \Lambda)$ , we have natural identifications*

$$Rj_*j^*\mathcal{F} \cong j_*j^*\mathcal{F} \cong \bar{j}_*\bar{j}^*\mathcal{F}$$

in  $Sh(X_{\acute{e}t}, \Lambda) \subset D(X_{\acute{e}t}, \Lambda)$ .

Here we say a sheaf  $\mathcal{F} \in Sh(X_{\acute{e}t}, \Lambda)$  is *overconvergent* if for any specialization of geometric points  $\bar{x} \rightsquigarrow \bar{y}$ , the associated map on stalks  $\mathcal{F}_{\bar{y}} \rightarrow \mathcal{F}_{\bar{x}}$  is an isomorphism, cf. [Hub96, Definition 8.2.1]. We also say that  $\mathcal{F} \in D(X_{\acute{e}t}, \Lambda)$  is overconvergent if it has overconvergent cohomology sheaves.

*Proof.* By [Hub96, Lemma 2.2.6], the functor  $j_*$  is exact, so the first isomorphism is clear. For the second, let  $h : U \rightarrow \bar{U}$  be the evident open embedding, so

$$Rj_*j^*\mathcal{F} \cong \bar{j}_*Rh_*h^*\bar{j}^*\mathcal{F} \cong \bar{j}_*h_*h^*\bar{j}^*\mathcal{F}.$$

Here we used the fact that  $h_* = Rh_*$  by another application of [Hub96, Lemma 2.2.6]. Now we need to see that  $\bar{j}^*\mathcal{F} \cong h_*h^*\bar{j}^*\mathcal{F}$ . Since  $\bar{j}^*\mathcal{F}$  is overconvergent, this reduces us to checking that the natural map  $\alpha : \mathcal{G} \rightarrow h_*h^*\mathcal{G}$  is an isomorphism for  $\mathcal{G}$  any overconvergent sheaf on  $\bar{U}$ . By [Hub96, Proposition 8.2.3], the sheaf  $h_*h^*\mathcal{G}$  is overconvergent, so it suffices to check that  $\alpha$  induces an isomorphism on stalks over any rank one (geometric) point. But every rank one point of  $\bar{U}$  is contained in  $U$ , so this is trivial.  $\square$

**Proposition C.3.4.** *If  $X$  is a rigid analytic space, then the dualizing complex  $\kappa_X$  is overconvergent.*

*Proof.* The proof is “dual” to the proof of Proposition C.3.1. More precisely, the result clearly holds for smooth  $X$ ; for a general  $X$ , we take a (global) resolution  $\pi : X' \rightarrow X$  and consider the adjunction map

$$R\pi_!\kappa_{X'} \cong R\pi_!\pi^!\kappa_X \rightarrow \kappa_X.$$

One now argues by induction as in the proof of Proposition C.3.1, making crucial use of the following facts:

- i. Overconvergence satisfies a 2-out-of-3 property.
- ii. Overconvergence is preserved by derived pushforward along proper maps [Hub96, Corollary 8.2.4].  $\square$

Granted these results, we now deduce the following key intermediate case.

**Proposition C.3.5.** *Let  $X$  be a rigid analytic space, and let  $j : U \rightarrow X$  be the inclusion of an open constructible subset  $U$ . Then  $j_!\Lambda$  is reflexive.*

The argument which follows is easily adapted to prove the more general statement that  $j_!M$  is reflexive, where  $M$  is any finitely generated constant sheaf of  $\Lambda$ -modules on  $X_{\text{ét}}$ .

*Proof.* Set  $\bar{V} = X \setminus U$ , and let  $V$  be the interior of  $\bar{V}$ ; write  $h : V \rightarrow X$  and  $\bar{h} : \bar{V} \rightarrow X$  for the evident inclusions. Note that  $\bar{U} = X \setminus V$ . In particular, writing  $\bar{j} : \bar{U} \rightarrow X$  for the evident inclusion, we get a canonical exact triangle

$$h_!h^*\kappa_X \rightarrow \kappa_X \rightarrow \bar{j}_*\bar{j}^*\kappa_X \rightarrow .$$

By Propositions C.3.3 and C.3.4, the canonical map

$$\bar{j}_*\bar{j}^*\kappa_X \rightarrow Rj_*j^*\kappa_X = Rj_*\kappa_U$$

is an isomorphism. Moreover,  $Rj_*\kappa_U \cong \mathbf{D}j_!\Lambda_U$ , and  $h^*\kappa_X = \kappa_V$ . Thus we can rewrite the above triangle as

$$h_!\kappa_V \rightarrow \kappa_X \rightarrow \mathbf{D}j_!\Lambda_U \rightarrow ,$$

so dualizing this gives an exact triangle

$$\mathbf{D}^2j_!\Lambda_U \rightarrow \mathbf{D}\kappa_X \rightarrow \mathbf{D}h_!\kappa_V \rightarrow .$$

Since  $\mathbf{D}\kappa_X \cong \Lambda_X$  and  $\mathbf{D}h_!\kappa_V \cong Rh_*\mathbf{D}\kappa_V \cong Rh_*\Lambda_V$ , we can rewrite the latter triangle as

$$\mathbf{D}^2j_!\Lambda_U \rightarrow \Lambda_X \rightarrow Rh_*\Lambda_V \rightarrow .$$

This sits in a commutative diagram of exact triangles

$$\begin{array}{ccccc} j_!\Lambda_U & \longrightarrow & \Lambda_X & \longrightarrow & \bar{h}_*\Lambda_{\bar{V}} \longrightarrow \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{D}^2j_!\Lambda_U & \longrightarrow & \Lambda_X & \longrightarrow & Rh_*\Lambda_V \longrightarrow \end{array}$$

where the lefthand vertical map is the biduality map, the central vertical map is the identity, and the righthand vertical map is the canonical map  $a : \bar{h}_* \Lambda_{\bar{V}} \rightarrow Rh_* \Lambda_V$ . By [Hub98b, Theorem 3.7], the map  $a$  is an isomorphism. Therefore the biduality map  $j_! \Lambda_U \rightarrow \mathbf{D}^2 j_! \Lambda_U$  is an isomorphism, as desired.  $\square$

**Corollary C.3.6.** *Let  $f : U \rightarrow X$  be any étale map of affinoid rigid spaces, and let  $M$  be a constant sheaf of finitely generated  $\Lambda$ -modules on  $U_{\text{ét}}$ . Then  $f_! M$  is reflexive.*

*Proof.* The claim is local on  $X$ . However, locally on  $X$ , we can factor  $f$  as the composite of an open embedding  $j : U \rightarrow W$  and a finite étale map  $g : W \rightarrow X$ , cf. [Hub96, Lemma 2.2.8]. By the previous proposition,  $j_! M$  is a reflexive sheaf on  $W_{\text{ét}}$ , and  $g$  is finite, hence proper, so  $g_* = g_!$  preserves reflexivity. Therefore  $f_! M = g_! j_! M$  is reflexive.  $\square$

Now, fix an affinoid rigid space  $X$ . Let us say a constructible sheaf  $\mathcal{G}$  on  $X$  is *elementary* if there exists an affinoid rigid space  $U$  and an étale map  $j : U \rightarrow X$  such that  $\mathcal{G} \simeq j_! \Lambda$ . Note that any finite direct sum of elementary sheaves is elementary. By the previous corollary, any elementary sheaf is reflexive. Moreover, any bounded complex of elementary sheaves is reflexive; this follows by an easy induction on the length of the complex, using the exact triangle associated with the “stupid” truncation functors together with the 2-out-of-3 property. Arguing as in the schemes case, one easily checks that any constructible sheaf  $\mathcal{F}$  admits a surjection  $s_0 : \mathcal{G}^0 \rightarrow \mathcal{F}$  from an elementary sheaf  $\mathcal{G}^0$ . The kernel of  $s_0$  is again constructible, so we may choose a surjection  $s_{-1} : \mathcal{G}^{-1} \rightarrow \ker s_0$  with  $\mathcal{G}^{-1}$  elementary; iterating this procedure, we can find an isomorphism  $\mathcal{F} \simeq \mathcal{G}^\bullet = [\dots \xrightarrow{s_{-2}} \mathcal{G}^{-1} \xrightarrow{s_{-1}} \mathcal{G}^0]$  in the derived category, where all the  $\mathcal{G}^i$ 's are elementary. Set  $\mathcal{H}_n = \ker s_{-n}$ ; playing with truncations, we get an exact triangle

$$\tau_{\leq -1} \sigma_{\geq -n}(\mathcal{G}^\bullet) \simeq \mathcal{H}_n[n] \rightarrow \sigma_{\geq -n}(\mathcal{G}^\bullet) \rightarrow \tau_{\geq 0} \sigma_{\geq -n}(\mathcal{G}^\bullet) \simeq \mathcal{F} \rightarrow$$

for any  $n \geq 1$ . Note that  $\sigma_{\geq -n}(\mathcal{G}^\bullet)$  is a bounded complex of elementary sheaves, and hence is reflexive. Since any exact triangle induces a corresponding exact triangle whose terms are the cones of the evident biduality maps, we get an isomorphism

$$\text{Cone}(\mathcal{F} \rightarrow \mathbf{D}^2 \mathcal{F}) \simeq \text{Cone}(\mathcal{H}_n \rightarrow \mathbf{D}^2 \mathcal{H}_n)[n+1]$$

for any  $n$ . Using the cohomological dimension bounds proved in [Hub96], one easily checks that there is an integer  $N$  depending only on  $X$  such that

the  $i$ th cohomology sheaf of  $\text{Cone}(\mathcal{H}_n \rightarrow \mathbf{D}^2\mathcal{H}_n)$  is zero for any  $i \notin [-N, N]$  (in fact,  $N = 1 + 2 \dim X$  is sufficient). Taking  $n$  arbitrarily large, we then see that the cohomology sheaves of  $\text{Cone}(\mathcal{F} \rightarrow \mathbf{D}^2\mathcal{F})$  are zero in any given degree. Therefore  $\text{Cone}(\mathcal{F} \rightarrow \mathbf{D}^2\mathcal{F})$  is acyclic, as desired.

## C.4 Some counterexamples

In this section we prove Propositions C.1.5 and C.1.6.

*Proof of Proposition C.1.5.* We construct an explicit example as follows. Let  $X = \text{Gr}(2, 5)$  be the usual rigid analytic Grassmannian over  $C$ , which we regard as parametrizing modifications of the bundle  $\mathcal{O}(2/5)$  on the Fargues-Fontaine curve at the distinguished point. Let  $X^{\text{adm}}$  be the admissible locus inside  $X$ . According to an unpublished computation of the author and Jared Weinstein, the closed subdiamond  $Z \subset X^\diamond$  corresponding to the closed subset  $|X| \setminus |X^{\text{adm}}|$  can be explicitly described by an isomorphism

$$Z \simeq \left( \text{Spa} \left( \mathcal{O}_C[[T^{1/p^\infty}]] \right) \setminus V(pT) \right)^\diamond / \underline{D_{1/3}^\times},$$

for a certain free action of  $\underline{D_{1/3}^\times}$  on the diamond  $(\text{Spa}(\mathcal{O}_C[[T^{1/p^\infty}]]) \setminus V(pT))^\diamond$ ; here  $D_{1/3}$  denotes the division algebra over  $\mathbf{Q}_p$  of invariant  $1/3$ . In particular, there is a natural smooth map  $Z \rightarrow [\text{pt}/\underline{D_{1/3}^\times}]$ . Let  $\pi$  be any infinite-dimensional admissible representation of  $D_{1/3}^\times$ , and let  $\mathcal{L}_\pi$  be the corresponding pro-étale local system on  $Z$ . The sheaf  $\mathcal{L}_\pi$  is then reflexive, since it's the pullback of a reflexive sheaf on  $[\text{pt}/\underline{D_{1/3}^\times}]$  along a smooth map, and therefore its pushforward along the closed embedding  $i : Z \rightarrow X^\diamond$  is a reflexive sheaf on  $X_{\text{ét}} \cong X_{\text{ét}}^\diamond$ . Now, choose any open affinoid subset  $j : U \rightarrow X$  meeting  $Z$  together with a finite map  $f : U \rightarrow \text{Spa } C \langle T_1, \dots, T_6 \rangle$ . The sheaf  $\mathcal{F} \stackrel{\text{def}}{=} f_* j^* i_* \mathcal{L}_\pi$  is then an example of the type we seek.  $\square$

We now turn to Proposition C.1.6. For brevity, we prove that  $\mathcal{F} = \Lambda_{\overline{X}}$  is reflexive but not strongly reflexive; the case of  $j_! \Lambda_X$  is dual. The only non-formal ingredient we need is

**Proposition C.4.1.** *The sheaves  $\Lambda_{\overline{X}}$  and  $j_! \Lambda_X$  are reflexive, and the dualizing complex of  $\overline{X}$  coincides with  $j_! \Lambda_X[2](1)$  where  $j : X \rightarrow \overline{X}$  is the natural inclusion.*

*Proof.* We first show that  $j_! \Lambda_X$  is reflexive. To see this, let  $i : \overline{X} \rightarrow \mathbf{P}_C^1$  be the natural closed embedding; then  $i_* j_! \Lambda_X \cong (i \circ j)_! \Lambda_X$  is a constructible

sheaf on  $\mathbf{P}_C^1$ , and therefore is reflexive. Thus  $j_!\Lambda$  is reflexive by our previous remarks. We now calculate

$$\mathbf{D}_{\overline{X}}j_!\Lambda_X \simeq j_*\mathbf{D}_X\Lambda_X \simeq j_*\kappa_X \simeq j_*\Lambda_X[2](1) \simeq \Lambda_{\overline{X}}[2](1),$$

where we've used the smoothness of  $X$  to identify  $\kappa_X$ . Since this calculation exhibits  $\Lambda_{\overline{X}}$  as the dual of a reflexive sheaf, it is reflexive itself. Applying  $\mathbf{D}_{\overline{X}}$  again and using reflexivity, we get

$$j_!\Lambda_X \simeq \mathbf{D}_{\overline{X}}\mathbf{D}_{\overline{X}}j_!\Lambda_X \simeq \mathbf{D}_{\overline{X}}(\Lambda_{\overline{X}}[2](1)) \simeq \kappa_{\overline{X}}[-2](-1),$$

as desired.  $\square$

Consider the cartesian diagram

$$\begin{array}{ccc} & X \times X & \\ f_1 \swarrow & & \searrow f_2 \\ X \times \overline{X} & & \overline{X} \times X \\ h_1 \searrow & & \swarrow h_2 \\ & \overline{X} \times \overline{X} & \end{array}$$

of adic spaces, where all fiber products are taken over  $\mathrm{Spa} C$ . Using the previous proposition, it is easy to see that

$$\mathbf{D}\mathcal{F} \boxtimes \mathcal{F} \simeq h_{1!}\Lambda_{X \times \overline{X}}[2](1).$$

By the symmetry of the situation, we have

$$\mathcal{F} \boxtimes \mathbf{D}\mathcal{F} \simeq h_{2!}\Lambda_{\overline{X} \times X}[2](1),$$

so then

$$\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \simeq \mathbf{D}(h_{2!}\Lambda_{\overline{X} \times X}[2](1)) \simeq h_{2*}\kappa_{\overline{X} \times X}[-2](-1).$$

To calculate  $\kappa_{\overline{X} \times X}$ , we use that the projection  $\mathrm{pr} : \overline{X} \times X \rightarrow \overline{X}$  is smooth of relative dimension one, so

$$\kappa_{\overline{X} \times X} = \mathrm{pr}^!K_{\overline{X}} = \mathrm{pr}^*\kappa_{\overline{X}}[2](1) \simeq f_{2!}\Lambda_{X \times X}[4](2)$$

by the previous proposition. Thus

$$\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}) \simeq h_{2*}f_{2!}\Lambda_{X \times X}[2](1).$$

It is now clear that  $\mathbf{D}\mathcal{F} \boxtimes \mathcal{F}$  and  $\mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F})$  cannot be isomorphic. For example, let  $U$  be the open subspace of  $\overline{X} \times \overline{X}$  defined by the conditions  $|T_1| \leq |T_2| \neq 0$ . Then

$$H^{-2}(R\Gamma(U, \mathbf{D}\mathcal{F} \boxtimes \mathcal{F})) \simeq h_{1!}\Lambda_{X \times \overline{X}}(U) = 0,$$

since  $U \not\subseteq X \times \overline{X}$ , while on the other hand

$$H^{-2}(R\Gamma(U, \mathbf{D}(\mathcal{F} \boxtimes \mathbf{D}\mathcal{F}))) \simeq (f_{2!}\Lambda_{X \times X})(U \cap (\overline{X} \times X)) \simeq \Lambda,$$

which one easily checks using the fact that  $U \cap (\overline{X} \times X)$  is a nonempty connected open subset of  $X \times X$ .

## References

- [BBD82] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.
- [BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251.
- [BT84] ———, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d’une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376.
- [Del77] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 41øer2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.
- [Fal94] Gerd Faltings, *The trace formula and Drinfeld’s upper halfplane*, Duke Math. J. **76** (1994), no. 2, 467–481.
- [Far] L. Fargues, *Geometrization of the local Langlands correspondence: an overview*.
- [Far15] ———, *G-torseurs en théorie de Hodge p-adique*, preprint, <http://webusers.imj-prg.fr/~laurent.fargues/Gtorseurs.pdf>, 2015.
- [FF] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, To appear in Astérisque.

- [FGK11] Kazuhiro Fujiwara, Ofer Gabber, and Fumiharu Kato, *On Hausdorff completions of commutative rings in rigid geometry*, J. Algebra **332** (2011), 293–321.
- [FS] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, in preparation.
- [GW17] Ildar Gaisin and John Welliaveetil, *Constructibility and reflexivity in non-Archimedean geometry*, preprint, 2017.
- [Har15] Michael Harris, *Mathematics without apologies: Portrait of a problematic vocation*, Science Essentials, Princeton University Press, 2015.
- [Hub96] Roland Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.
- [Hub98a] R. Huber, *A comparison theorem for  $l$ -adic cohomology*, Compositio Math. **112** (1998), no. 2, 217–235.
- [Hub98b] ———, *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 359–403.
- [Hub98c] ———, *A finiteness result for the compactly supported cohomology of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 313–357.
- [Ill94] Luc Illusie, *Autour du théorème de monodromie locale*, Astérisque (1994), no. 223, 9–57, Périodes  $p$ -adiques (Bures-sur-Yvette, 1988).
- [Kal] Tasho Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. (to appear).
- [Kal13] ———, *Genericity and contragredience in the local Langlands correspondence*, Algebra Number Theory **7** (2013), no. 10, 2447–2474.
- [Kal14] ———, *Supercuspidal  $L$ -packets via isocrystals*, Amer. J. Math. **136** (2014), no. 1, 203–239.
- [Kal16a] ———, *The local Langlands conjectures for non-quasi-split groups*, Families of automorphic forms and the trace formula, Simons Symp., Springer, [Cham], 2016, pp. 217–257. MR 3675168

- [Kal16b] ———, *Rigid inner forms of real and  $p$ -adic groups*, Ann. of Math. (2) **184** (2016), no. 2, 559–632. MR 3548533
- [Kal18] ———, *Rigid inner forms vs isocrystals*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 1, 61–101. MR 3743236
- [Kot] Robert E. Kottwitz,  *$B(G)$  for all local and global fields*, arXiv:1401.5728.
- [Kot83] ———, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 289–297.
- [Kot84a] ———, *Shimura varieties and twisted orbital integrals*, Math. Ann. **269** (1984), no. 3, 287–300.
- [Kot84b] ———, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650.
- [Kot85] ———, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220.
- [Kot86] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399.
- [Kot97] ———, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), no. 3, 255–339.
- [KS99] Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190.
- [LO09] Yves Laszlo and Martin Olsson, *Perverse  $t$ -structure on Artin stacks*, Math. Z. **261** (2009), no. 4, 737–748.
- [LZ] Yifeng Liu and Weizhe Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*, <https://arxiv.org/abs/1211.5948>.
- [Mas16] David Massey, *Natural commuting of vanishing cycles and the Verdier dual*, Pacific J. Math. **284** (2016), no. 2, 431–437.
- [Mie] Yoichi Mieda, *Lefschetz trace formula and  $\ell$ -adic cohomology of Rapoport-Zink tower for  $GSp(4)$* , <http://www.ms.u-tokyo.ac.jp/~mieda/pdf/RZ-LTF.pdf>.

- [Mie12] Yoichi Mieda, *Lefschetz trace formula and  $\ell$ -adic cohomology of Lubin-Tate tower*, Math. Res. Lett. **19** (2012), no. 1, 95–107.
- [Mie14a] ———, *Geometric approach to the local Jacquet-Langlands correspondence*, Amer. J. Math. **136** (2014), no. 4, 1067–1091.
- [Mie14b] ———, *Lefschetz trace formula for open adic spaces*, J. Reine Angew. Math. **694** (2014), 85–128.
- [MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.
- [NP01] B. C. Ngô and P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, J. Algebraic Geom. **10** (2001), no. 3, 515–547.
- [Rap95] Michael Rapoport, *Non-Archimedean period domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 423–434.
- [Ric14] Timo Richarz, *A new approach to the geometric Satake equivalence*, Doc. Math. **19** (2014), 209–246.
- [RV14] Michael Rapoport and Eva Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326.
- [RZ96] M. Rapoport and Th. Zink, *Period spaces for  $p$ -divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.
- [Sch17] Peter Scholze, *The étale cohomology of diamonds*, ARGOS Seminar in Bonn, 2017.
- [SGA77] *Cohomologie  $l$ -adique et fonctions  $L$* , Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie.
- [She14] Xu Shen, *Cell decomposition of some unitary group rapoport-zink spaces*, Math. Ann. **360** (2014), 825–899.
- [Sta17a] The Stacks Project Authors, *stacks project*, <http://stacks.math.columbia.edu>, 2017.

- [Sta17b] ———, *Stacks Project*, <http://stacks.math.columbia.edu>, 2017.
- [Ste65] Robert Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 25, 49–80.
- [Str05] Matthias Strauch, *On the Jacquet-Langlands correspondence in the cohomology of the Lubin-Tate deformation tower*, Astérisque (2005), no. 298, 391–410, Automorphic forms. I.
- [Str08] ———, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217** (2008), no. 3, 889–951.
- [SW13] Peter Scholze and Jared Weinstein, *Moduli of  $p$ -divisible groups*, Camb. J. Math. **1** (2013), no. 2, 145–237.
- [SW14] ———, *Berkeley lectures on  $p$ -adic geometry*, Lecture notes at <http://math.bu.edu/people/jsweinst/Berkeley.pdf>, 2014.
- [Tem12] Michael Temkin, *Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case*, Duke Math. J. **161** (2012), no. 11, 2207–2254.
- [Tit79] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.
- [Var07] Yakov Varshavsky, *Lefschetz-Verdier trace formula and a generalization of a theorem of Fujiwara*, Geom. Funct. Anal. **17** (2007), no. 1, 271–319.