Gal($\overline{\mathbf{Q}}_p/\mathbf{Q}_p$) as a geometric fundamental group

Jared Weinstein*

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Abstract

Let p be a prime number. In this article we present a theorem, suggested by Peter Scholze, which states that $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is the étale fundamental group of a certain object Z which is defined over an algebraically closed field. This object is the quotient of the "punctured perfectoid open disk" by an action of the group $\mathbf{Q}_p^\times$. The proof of this fact combines two themes: the tilting equivalence for perfectoid spaces, and the Fargues-Fontaine curve.

1 Introduction

Let p be a prime number. In this article we present a theorem, suggested by Peter Scholze, which states that $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is the étale fundamental group of certain object Z which is *defined over an algebraically closed field*. As a consequence, representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ correspond to local systems on Z.

The precise theorem involves perfectoid spaces [Sch12]. Let $C/\mathbf{Q}_p$ be complete and algebraically closed. Let D be the open unit disk centered at 1, considered as a rigid space over C, and given the structure of a $\mathbf{Z}_p$-module where the composition law is multiplication, and $a \in \mathbf{Z}_p$ acts by $x \mapsto x^a$. Let

$$\widetilde{D} = \varprojlim_{x \mapsto x^p} D.$$

Then $\widetilde{D}$ is no longer a classical rigid space, but it does exist in Huber's category of adic spaces, and is in fact a perfectoid space. Note that $\widetilde{D}$ has the structure of a $\mathbf{Q}_p$-vector space. Let $\widetilde{D}^* = \widetilde{D} \backslash \{1\}$; this admits an action of $\mathbf{Q}_p^\times$.

*Boston University Department of Mathematics and Statistics, 111 Cummington Mall, Boston, MA 02115. E-mail: jsweinst@math.bu.edu

Theorem A. *The category of* $\mathbf{Q}_p^\times$*-equivariant finite étale covers of* $\widetilde{D}^*$ *is equivalent to the category of finite étale* $\mathbf{Q}_p$*-algebras.*

The object Z of the first paragraph is then the quotient $Z = \widetilde{D}^*/\mathbf{Q}_p^\times$. This quotient doesn't belong to the category of adic spaces. Instead there is a sheaf-theoretic interpretation, which makes Z into something like an algebraic space. The category Perf_C of perfectoid spaces over C carries a topology, the *pro-étale topology*, in which (roughly speaking) a cover of an object X is a surjective morphism from an inverse limit $\varprojlim X_i$, with each $X_i \to X$ an étale morphism. See §4.2 for details. Each object in Perf_C becomes a sheaf on Perf_C via the Yoneda embedding (Proposition 4.2.5).

In this larger category of sheaves on Perf_C, there is a notion of finite étale morphism, which agrees with the usual notion when the target is representable. The category of finite étale morphisms onto a given sheaf X on Perf_C becomes (after choosing a base point) a Galois category, with associated étale fundamental group $\pi_1^{\text{ét}}(X)$.

Theorem B. *Let* Z *be the quotient* $\widetilde{D}^*/\mathbf{Q}_p^\times$*, where the quotient is taken in the category of sheaves on* Perf_C*. There is an equivalence of categories between finite étale morphisms to* Z *and étale* $\mathbf{Q}_p$*-algebras. Thus* $\pi_1^{\acute{e}t}(Z) \cong \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$*.*

Theorems A and B can be generalized to a finite extension $E/\mathbf{Q}_p$. Let $\pi \in E$ be a uniformizer, and let H be the corresponding Lubin-Tate formal $\mathcal{O}_E$-module. Then D gets replaced by the generic fiber H_C^{ad}; this is the open unit disc centered at 0, considered as an adic space over C. H_C^{ad} is endowed with the $\mathcal{O}_E$-module structure coming from H. Form the *universal cover* $\widetilde{H}_C^{\mathrm{ad}} = \varprojlim H_C^{\mathrm{ad}}$, where the inverse limit is taken with respect to multiplication by π in H. Then $\widetilde{H}_C^{\mathrm{ad}}$ is an E-vector space object in the category of perfectoid spaces over C. Let $\widetilde{H}_C^{\mathrm{ad},*} = \widetilde{H}_C^{\mathrm{ad}} \backslash \{0\}$. Form the quotient

$$Z_E = \widetilde{H}_C^{\mathrm{ad},*}/E^\times$$

in the category of sheaves on Perf_C. Then the categories of finite étale covers of Z_E and $\operatorname{Spec} E$ are equivalent, so that $\pi_1^{\text{ét}}(Z_E) \cong \mathrm{Gal}(\overline{E}/E)$.

The proof of Theorem A hinges on a combination of two themes: the *fundamental curve of* p*-adic Hodge theory*, due to Fargues-Fontaine, and the *tilting equivalence*, due to Scholze. Let us sketch the proof in the case $E = \mathbf{Q}_p$. Let $C^\flat$ be the tilt of C, a perfectoid field in characteristic p. Consider the punctured open disc $D_{C^\flat}^*$ (with parameter t) and let $\widetilde{D}_{C^\flat}^* = \varprojlim_{x \mapsto x^p} D_{C^\flat}^*$. Then $\widetilde{D}_{C^\flat}^*$ is simultaneously a perfectoid space over two perfectoid fields:

$$
\begin{tikzcd}
 & \operatorname{Spa} \mathbf{F}_p(\!(t^{1/p^\infty})\!) \\
\widetilde{D}^*_{C^\flat} \arrow[ur] \arrow[dr] & \\
 & \operatorname{Spa} C^\flat
\end{tikzcd}
$$

Scholze shows that if K is any perfectoid field, there is an equivalence of categories (the *tilting equivalence*) $X \mapsto X^\flat$ between perfectoid spaces over K and $K^\flat$, and that the categories of finite étale covers of X and $X^\flat$ are equivalent. Considered as a perfectoid space over $C^\flat$, $\widetilde{D}^*_{C^\flat}$ has an obvious "un-tilt", namely $\widetilde{D}^*_C$. But $\widetilde{D}^*_{C^\flat}$ is also a perfectoid space over $\mathbf{F}_p(\!(t^{1/p^\infty})\!)$, which is the tilt of the cyclotomic field $\widehat{\mathbf{Q}}_p(\mu_{p^\infty})$. Thus $\widetilde{D}^*_{C^\flat}$ also has an un-tilt to a perfectoid space over $\widehat{\mathbf{Q}}_p(\mu_{p^\infty})$, and this is where the Fargues-Fontaine curve comes in.

The construction of the Fargues-Fontaine curve X is reviewed in §3. X is an integral noetherian scheme of dimension 1 over $\mathbf{Q}_p$, whose closed points parametrize un-tilts of $C^\flat$ modulo Frobenius. For our purposes we need the adic version $\mathcal{X}$, which is the quotient of another adic space $\mathcal{Y}$ by a Frobenius automorphism ϕ. The extension of scalars $\mathcal{Y}\widehat{\otimes}\widehat{\mathbf{Q}}_p(\mu_{p^\infty})$ is a perfectoid space; by a direct calculation (Proposition 3.5.4) we show that its tilt is isomorphic to $\widetilde{D}^*_{C^\flat}$. In this isomorphism, the action of $\mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \cong \mathbf{Z}_p^\times$ on the field of scalars $\widehat{\mathbf{Q}}_p(\mu_{p^\infty})$ corresponds to the geometric action of $\mathbf{Z}_p^\times$ on $\widetilde{D}^*_{C^\flat}$, and the automorphism ϕ corresponds (up to absolute Frobenius) to the action of p on $\widetilde{D}^*_{C^\flat}$.

Therefore under the tilting equivalence, finite étale covers of $\widetilde{D}^*_C$ and $\mathcal{Y}\widehat{\otimes}\widehat{\mathbf{Q}}_p(\mu_{p^\infty})$ are identified. The same goes for finite étale covers of $\widetilde{D}^*_C/p^{\mathbf{Z}}$ and $\mathcal{X}\widehat{\otimes}\widehat{\mathbf{Q}}_p(\mu_{p^\infty})$. Now we apply the key fact that $\mathcal{X}$ is geometrically simply connected. The same statement is proved in [FF11] for the algebraic curve X; we have adapted the proof for $\mathcal{X}$ in Proposition 3.4.3. Thus finite étale covers of $\widetilde{D}^*_C/p^{\mathbf{Z}}$ are equivalent to finite étale $\mathbf{Q}_p(\mu_{p^\infty})$-algebras. Now we can descend to $\mathbf{Q}_p$: $\mathbf{Z}_p^\times$-equivariant finite étale covers of $\widetilde{D}^*_C/p^{\mathbf{Z}}$ are equivalent to finite étale $\mathbf{Q}_p$-algebras, which is Theorem A.

To derive Theorem B from Theorem A, we prove a descent statement for finite étale morphisms relative to pro-étale covers in Perf_C, Proposition 4.2.4. That is, whenever $Y' \to X'$ is a finite étale morphism in Perf_C equipped with a descent datum relative to $X' \to X$, then $Y' \to X'$ descends to a finite étale morphism $Y \to X$. The same descent statement holds in the category of sheaves on Perf_C, Proposition 4.2.9. As a corollary, whenever $Y \to X$ is a *pro-étale $\underline{G}$-torsor* in the category of sheaves on Perf_C, where G is a profinite group, there is an equivalence of categories between $\underline{G}$-equivariant finite étale covers of Y and finite étale covers of X. After checking that $\widetilde{D}_C^*/p^{\mathbf{Z}} \to Z$ is a pro-étale $\mathbf{Z}_p^\times$-torsor, we can conclude Theorem B.

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2 Generalities on perfectoid spaces

Perfectoid spaces and the tilting equivalence play a crucial role in this article, so we review some foundational results on them here. The main result of this section is Theorem 2.5.5, which concerns affine formal schemes $\mathrm{Spf}\, R$ over $\mathrm{Spf}\, \mathcal{O}_K$ (where K is a perfectoid field). Under certain hypotheses on R, the adic generic fiber of $\mathrm{Spf}\, R$ is a perfectoid space over K. Theorem 2.5.5 shows that the tilt of this perfectoid space is the adic generic fiber of $\mathrm{Spf}\, R^\flat$, where $R^\flat = \varprojlim_{x \mapsto x^p} R$.

This result will immediately allow us to identify the tilt of the Fargues-Fontaine curve in Proposition 3.2.1, which plays an important role in the proof of our main theorem. Proposition 3.2.1 had already been proved in [Far13] "by hand", so the reader who is only interested in the main theorem can proceed directly to §3.

2.1 Review of adic spaces

The category of adic spaces is introduced in [Hub94]. We quickly review the main definitions.

Definition 2.1.1. A topological ring R is *Huber* (*f-adic* in [Hub94]) if it contains an open subring R_0 whose topology is generated by a finitely generated ideal $I \subset R_0$. Such an R_0 (resp., an I) is called a *ring of definition* (resp., *ideal of definition*) of R. A *Huber pair* (*affinoid ring* in [Hub94]) is a pair (R, R^+), where R is a Huber ring and $R^+ \subset R$ is an open and integrally closed subring consisting of power-bounded elements. A morphism of Huber pairs $(R, R^+) \to (S, S^+)$ is a continuous homomorphism $R \to S$ which sends $R^+ \to S^+$.

Let (R, R^+) be a Huber pair, and let R_0 (resp., I) be a ring of definition (resp., ideal of definition) of R. Huber defines a topological space $X = \mathrm{Spa}(R, R^+)$ whose points x are equivalence classes of *continuous valuations* $f \mapsto |f(x)|$ on R which are ≤ 1 for all $f \in R^+$. The topology on X is generated by *rational subsets*

$$U\left(\frac{T_1}{s_1},\dots,\frac{T_n}{s_n}\right)=\left\{x\in X\;\middle|\;|f_i(x)|\leq|s_i(x)|\neq 0,\ f_i\in T_i\right\}$$

where for each $i=1,\dots,n$, $T_i\subset R$ is a finite subset which generates an open ideal in R, and $s_i\in R$. Given such data $T_1,\dots,T_n,s_1,\dots,s_n$, one can give $R[T_1/s_1,\dots,T_n/s_n]\subset R[1/s_1,\dots,1/s_n]$ a ring topology making $R_0[T_1/s_1,\dots,T_n/s_n]$ an open subring equipped with the I-adic topology. Let $R\langle T_1/s_1,\dots,T_n/s_n\rangle$ be the completion of $R[T_1/s_1,\dots,T_n/s_n]$. Then one defines the structure presheaf $(\mathcal{O}_X,\mathcal{O}_X^+)$ on $X=\operatorname{Spa}(R,R^+)$. If $U=U(T_1/s_1,\dots,T_n/s_n)$, then $\mathcal{O}_X(U)=R\langle T_1/s_1,\dots,T_n/s_n\rangle$ and $\mathcal{O}_X^+(U)$ is the completion of the integral closure of the subring $R^+[T_1/s_1,\dots,T_n/s_n]\subset\mathcal{O}_X(U)$. For each $x\in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring; the continuous valuation $f\mapsto|f(x)|$ on R extends in a natural way to a continuous valuation on $\mathcal{O}_{X,x}$.

It is important to note that $\mathcal{O}_X$ is not necessarily a sheaf. Let us call a Huber pair (R,R^+) *sheafy* if $\mathcal{O}_X$ is a sheaf. Huber shows that (R,R^+) is sheafy when R is "strongly noetherian", meaning that $R\langle X_1,\dots,X_n\rangle$ is noetherian for all $n\geq 0$. In brief, an *adic space* is a topological space X equipped with a sheaf of complete topological rings $\mathcal{O}_X$ whose stalks come with distinguished continuous valuations, such that X is locally isomorphic to $\operatorname{Spa}(R,R^+)$ for a sheafy Huber pair (R,R^+). We refer to [Hub94] for the precise definition.

In §2 of [SW13] we constructed a larger category of "general" adic spaces, whose objects are sheaves on the category of complete Huber pairs. (This category can be given the structure of a site, in which the coverings of (R,R^+) correspond to coverings of $\operatorname{Spa}(R,R^+)$ by rational subsets.) If (R,R^+) is a (not necessarily sheafy) affinoid ring, then $\operatorname{Spa}(R,R^+)$ belongs to this larger category. If X is an adic space in the general sense, let us call X an *honest adic space* if it belongs to the category of adic spaces in the sense of Huber; *i.e.* if it is locally $\operatorname{Spa}(R,R^+)$ for a sheafy (R,R^+).

If R is a Huber ring, write R° for its subring of power-bounded elements. Then (R,R°) is a Huber pair, and we write $\operatorname{Spa}R=\operatorname{Spa}(R,R^\circ)$.

2.2 The adic generic fiber of a formal scheme over $\operatorname{Spf}\mathcal{O}_K$

Let K be a nonarchimedean field, and let $\varpi\in K$ be a *pseudo-uniformizer* (a topologically nilpotent unit). Let $\mathcal{O}_K$ be the valuation ring of K. Suppose R is an $\mathcal{O}_K$-algebra which is separated and complete with respect to the topology induced by a finitely generated ideal. Then $\operatorname{Spf}R$ is a formal scheme over $\operatorname{Spf}\mathcal{O}_K$. Also $R=R^\circ$, and (R,R) is a Huber pair over $(\mathcal{O}_K,\mathcal{O}_K)$. One can form the adic space $\operatorname{Spa}R$, which is fibered over the two-point space $\operatorname{Spa}(\mathcal{O}_K,\mathcal{O}_K)$. By [SW13, Proposition 2.2.1], $\operatorname{Spf} R \mapsto \operatorname{Spa} R$ extends to a fully faithful functor $M \mapsto M^{\mathrm{ad}}$ from the category of formal schemes over $\operatorname{Spf} \mathcal{O}_K$ locally admitting a finitely generated ideal of definition, to the category of adic spaces over $(\mathcal{O}_K, \mathcal{O}_K)$.

Let $\eta = \operatorname{Spa}(K, \mathcal{O}_K)$ be the generic point of $\operatorname{Spa}(\mathcal{O}_K, \mathcal{O}_K)$. The *adic generic fiber* of $\operatorname{Spf} R$ is $(\operatorname{Spa} R)_\eta = (\operatorname{Spa} R) \backslash \{\varpi = 0\}$. Suppose that the elements $f_1, \ldots, f_n \in R$ generate an ideal of definition for R. Then for every $x \in (\operatorname{Spa} R) \backslash \{\varpi = 0\}$, and each $i = 1, \ldots, n$, we have $|f_i(x)|^m \to 0$ as $m \to \infty$, whereas $|\varpi(x)| \neq 0$. Therefore there exists $N \geq 1$ such that $|f_i(x)|^N \leq |\varpi| \neq 0$, and thus $x \in U(f_1^N/\varpi, \ldots, f_n^N/\varpi)$. Note that $(f_1^N, \ldots, f_n^N)$ is also an ideal of definition of R. We have proved:

Lemma 2.2.1. *The adic generic fiber* $(\operatorname{Spa} R) \backslash \{\varpi = 0\}$ *is covered by rational subsets* $U(f_1/\varpi, \ldots, f_n/\varpi)$*, where* $(f_1, \ldots, f_n)$ *runs through tuples of elements which generate an ideal of definition of* R*.*

Example 2.2.2. The formal open unit disc is $\operatorname{Spf} \mathcal{O}_K[\![T]\!]$. Its adic generic fiber is covered by rational subsets $U_n = U((T^n, \varpi)/\varpi) = \{|T|^n \leq |\varpi| \neq 0\}$ for $n = 1, 2, \ldots$. This is the adic open unit disc over K. Note that $U_n = \operatorname{Spa}(R_n, R_n^+)$, where $R_n = \mathcal{O}_K[\![T]\!] \langle T^n/\varpi \rangle [1/\varpi]$, and R_n^+ is the integral closure of $\mathcal{O}_K[\![T]\!] \langle T^n/\varpi \rangle$ in R_n.

2.3 Perfectoid algebras and perfectoid spaces

We review here some definitions from [Sch12] concerning perfectoid algebras and perfectoid spaces.

Definition 2.3.1. A *perfectoid field* is a complete nonarchimedean field K with non-discrete rank 1 value group and characteristic p residue field, such that the pth power Frobenius map $\Phi \colon \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.

Remark 2.3.2. It will be convenient to use the symbol Φ to denote the pth power Frobenius map $A/I \to A/J$ whenever A is a ring and $I, J \subset A$ are ideals such that J contains p and $I^p \subset J$.

Let K be a perfectoid field with absolute value $|\,|$ and valuation ring $\mathcal{O}_K = K^\circ$. One can always find an element $\varpi \in \mathcal{O}_K$ such that $|p| \leq |\varpi| < 1$, with pth root $\varpi^{1/p} \in \mathcal{O}_K$. Then $\Phi \colon \mathcal{O}_K/\varpi^{1/p} \to \mathcal{O}_K/\varpi$ is an isomorphism.

If M is an $\mathcal{O}_K$-module, we say that M is *almost zero* if for all $m \in M$, we have $\varpi^{1/p^n} m = 0$ for all n. The category of $\mathcal{O}_K^{\mathrm{a}}$-modules is obtained as the quotient of the category of $\mathcal{O}_K$-modules by the thick subcategory of modules which are almost zero. Let $M \mapsto M^{\mathrm{a}}$ be the localization functor from $\mathcal{O}_K$-modules to $\mathcal{O}_K^{\mathrm{a}}$-modules; this admits a right adjoint $M \mapsto M_*$. One can define the category of $\mathcal{O}_K^{\mathrm{a}}$-algebras, as well as A-algebras, where A is an $\mathcal{O}_K^{\mathrm{a}}$-algebra.

Definition 2.3.3. A Huber ring R is *uniform* if the subring $R^\circ \subset R$ of power-bounded elements is bounded.

Definition 2.3.4. Let K be a perfectoid field.

1. A *perfectoid K-algebra* is Banach K-algebra R such that R is uniform and such that $\Phi\colon R^\circ/\varpi \to R^\circ/\varpi$ is surjective.
2. A *perfectoid $\mathcal{O}_K^{\mathrm{a}}$-algebra* is a ϖ-adically complete flat $\mathcal{O}_K^{\mathrm{a}}$-algebra A such that $\Phi\colon A/\varpi^{1/p} \to A/\varpi$ is an isomorphism.
3. A *perfectoid $\mathcal{O}_K^{\mathrm{a}}/\varpi$-algebra* is a flat $\mathcal{O}_K^{\mathrm{a}}$-algebra $\overline{A}$ such that $\Phi\colon \overline{A}/\varpi^{1/p} \to \overline{A}$ is an isomorphism.

Lemma 2.3.5 ([Sch12, Proposition 5.9])**.** *Suppose K has characteristic p, and let R be a uniform Banach K-algebra. Then R is a perfectoid K-algebra if and only if R is perfect.*

Definition 2.3.6. A *perfectoid space* over K is an honest adic space over K which can be covered by rational subsets of the form $\mathrm{Spa}(R, R^+)$, where R is a perfectoid K-algebra.

2.4 Tilting

The following is a review of §5 of [Sch12].

Let K be a perfectoid field of characteristic 0. For a perfectoid K-algebra R, the *tilt* $R^\flat$ is a ring whose underlying topological multiplicative semigroup is the set of sequences $(f_0, f_1, \dots)$ of elements of R satisfying $f_i^p = f_{i-1}$ for all $i \geq 1$. The addition law on $R^\flat$ is defined by $(f_i) + (g_i) = (h_i)$, where

$$h_i = \lim_{n\to\infty} (f_{n+i} + g_{n+i})^{p^n}.$$

Then $R^\flat$ is a ring of characteristic p. If $f = (f_0, f_1, \dots) \in R^\flat$ we write $f^\sharp = f_0$.

In particular if $R = K$, then $K^\flat$ is a perfectoid field of characteristic p. After replacing $\varpi \in K$ with an element of the same norm, one can find $\varpi^\flat \in K^\flat$ such that $\varpi = (\varpi^\flat)^\sharp$. In general, $R^\flat$ is a perfectoid $K^\flat$-algebra, and $R^{\flat\circ}/\varpi^\flat \cong R^\circ/\varpi$. We remark that we could also have defined $R^\flat$ as $R_*^{\circ\,\mathrm{a}\,\flat}[1/\varpi]$, where

$$R^{\circ\,\mathrm{a}\,\flat} = \varprojlim_{\Phi} R^{\circ\,\mathrm{a}}/\varpi;$$

the two definitions of $R^\flat$ are equivalent by [Sch12, Proposition 5.17].

Theorem 2.4.1 ([Sch12, Theorem 5.2])**.** *The functor $R \mapsto R^\flat$ is an equivalence between the category of perfectoid K-algebras and the category of perfectoid $K^\flat$-algebras. In fact, all of the following categories are equivalent:*

1. *Perfectoid K-algebras,*
2. *Perfectoid $\mathcal{O}_K^{\mathrm{a}}$-algebras,*
3. *Perfectoid $\mathcal{O}_K^{\mathrm{a}}/\varpi$-algebras,*
4. *Perfectoid $\mathcal{O}_{K^\flat}^{\mathrm{a}}/\varpi^\flat$-algebras,*
5. *Perfectoid $\mathcal{O}_{K^\flat}^{\mathrm{a}}$-algebras,*
6. *Perfectoid $K^\flat$-algebras.*

In Theorem 2.4.1, the equivalence between (1) and (2) is $R \mapsto R^{\circ\,\mathrm{a}}$ in one direction, and $A \mapsto A_*[1/\varpi]$ in the other. The equivalence between (2) and (3) follows from the vanishing of the cotangent complex for a perfectoid $\mathcal{O}_K^{\mathrm{a}}/\varpi$-algebra, which contains obstructions for lifting $\mathcal{O}_K^{\mathrm{a}}/\varpi$-algebras to $\mathcal{O}_K^{\mathrm{a}}$. The equivalence between (3) and (4) is immediate from $\mathcal{O}_K/\varpi \cong \mathcal{O}_{K^\flat}/\varpi^\flat$. The remaining equivalences involving $K^\flat$ are parallel to the ones involving K.

Definition 2.4.2. A *perfectoid affinoid K-algebra* is a Huber pair (R, R^+), where R is a perfectoid K-algebra.

Lemma 2.4.3 ([Sch12, Lemma 6.2])**.** *The categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras are equivalent. If (R, R^+) corresponds to $(R^\flat, R^{\flat+})$ under this equivalence, then $R^{\flat+}/\varpi^\flat \cong R^+/\varpi$.*

Theorem 2.4.4 ([Sch12], Theorem 6.3)**.** *Let (R, R^+) be a perfectoid affinoid K-algebra. Then (R, R^+) is sheafy. Let $R^{\flat+} \subset R^\flat$ be the subring consisting of sequences in R^+. There is a homeomorphism*

$$\begin{array}{ccc} \left|\mathrm{Spa}(R, R^+)\right| & \to & \left|\mathrm{Spa}(R^\flat, R^{\flat+})\right| \\ x & \mapsto & x^\flat, \end{array}$$

defined by the relation $\left|f(x^\flat)\right| = \left|f^\sharp(x)\right|$ for $f \in R^\flat$. Furthermore, rational subsets of $\mathrm{Spa}(R, R^+)$ and $\mathrm{Spa}(R^\flat, R^{\flat+})$ are identified. Finally, the categories of finite étale algebras over R and $R^\flat$ are equivalent.

2.5 Formal schemes over $\mathcal{O}_K$ with perfectoid generic fiber

Let K be a perfectoid field, and let $\varpi \in K$ be an element with $|p| \leq |\varpi| < 1$. Assume that $\varpi = (\varpi^\flat)^\sharp$ for some $\varpi^\flat \in \mathcal{O}_{K^\flat}$, so that $\varpi^{1/p^n} \in \mathcal{O}_K$ for all $n \geq 1$.

Definition 2.5.1. Let S be a ring in characteristic p. S is *semiperfect* if the Frobenius map $\Phi\colon S \to S$ is surjective. If S is semiperfect, let $S^\flat = \varprojlim_\Phi S$, a perfect ring which surjects onto S.

If R is a topological $\mathcal{O}_K$-algebra with R/ϖ semiperfect, then write $R^\flat = (R/\varpi)^\flat$, a perfect topological $\mathcal{O}_{K^\flat}$-algebra. Note that $R^\flat/\varpi^\flat \to R/\varpi$ is surjective.

Lemma 2.5.2. *Suppose that an $\mathcal{O}_K$-algebra R is ϖ-adically separated and complete, and that R/ϖ is semiperfect. The natural map*

$$\varprojlim_{x\mapsto x^p} R \to \varprojlim_{\Phi} R/\varpi = R^\flat$$

is an isomorphism of multiplicative monoids.

Proof. Injectivity follows from R being ϖ-adically separated, and surjectivity follows from R being ϖ-adically complete, cf. the argument in [Sch12, Lemma 3.4]. $\square$

Given $f \in R^\flat$, we may use Lemma 2.5.2 to identify f with a sequence $(f_0, f_1, \dots) \in \varprojlim_{x\mapsto x^p} R$. Let $f^\sharp = f_0$. Then $f \mapsto f^\sharp$ is a continuous map of multiplicative monoids $R^\flat \to R$. Note that the image of f under $R^\flat \to R^\flat/\varpi^\flat \to R/\varpi$ equals $f^\sharp \pmod{\varpi}$.

Proposition 2.5.3. *Let R be a topological $\mathcal{O}_K$-algebra which is separated and complete for the topology induced by a finitely generated ideal of definition. Assume that:*

1. *$\varpi R \subset R$ is closed,*
2. *R/ϖ is semiperfect,*
3. *If $f \in R$ satisfies $f^{p^n} \in \varpi R$ for some $n \geq 1$, then $f \in \varpi^{1/p^n} R$.*

Then $R^\flat$ is also separated and complete for the topology induced by a finitely generated ideal of definition, and it satisfies the same conditions (1)-(3) (with $R^\flat$ and $\varpi^\flat$ replacing R and ϖ.) Furthermore if $f_1, \dots, f_n \in R^\flat$ generate an ideal of definition, then the elements $f_1^\sharp, \dots, f_n^\sharp, \varpi$ generate an ideal of definition of R.

Remark 2.5.4. A systematic setting in which Prop. 2.5.3 holds has been developed in [GR15, Defn. 14.1.14 and Prop. 14.1.22].

Proof. We denote by $\mathrm{pr}_r\colon R^\flat \to R/\varpi$ the projection onto the rth coordinate: $\mathrm{pr}_r(x_0, x_1, \dots) = x_r$. Thus we have $\mathrm{pr}_r(x) = \mathrm{pr}_0(x^{1/p^r})$. Note that condition (3) implies that $\ker \mathrm{pr}_0(x) = \varpi^\flat R^\flat$. Indeed, if $f = (f_0, f_1, \dots) \in R^\flat$ and $\mathrm{pr}_0(f_0, f_1, \dots) = 0$, then for all $n \geq 1$, $f_n^{p^n} \in \varpi R$, and so by condition (3) $f_n \in \varpi^{1/p^n} R$, which means that $f \in \varpi^\flat R^\flat$.

Let $J = (g_1, \dots, g_m)$ be an ideal of definition for R, and let $\overline{J}$ denote its image in R/ϖ. We claim that R/ϖ is $\overline{J}$-adically separated and complete. Completeness follows from the completeness of R, and separatedness is the statement that $\bigcap_{k\geq 1}(J^k + \varpi R) = \varpi R$, which is equivalent to condition (1).

By condition (2), the projection $\mathrm{pr}_0\colon R^\flat \to R/\varpi$ is surjective, and we can find elements $g_1^\flat, \dots, g_m^\flat \in R^\flat$ which lift $g_1, \dots, g_m \pmod{\varpi}$. For ease of notation we assume that $g_m = 0$ and $g_m^\flat = \varpi^\flat$. We claim that $J^\flat = (g_1^\flat, \dots, g_m^\flat)$ induces the topology on $R^\flat$. By definition of the inverse limit topology, a system of neighborhoods of the origin in $R^\flat$ is given by $\left\{\mathrm{pr}_r^{-1}(\overline{J}^k)\right\}_{r,k\geq 0}$. We will show that each of these neighborhoods contains some $(\overline{J}^\flat)^N$, and vice versa.

Given $r, k \geq 0$, let N be large enough so that $(J^\flat)^N$ is contained in $((g_1^\flat)^{kp^r}, \dots, (g_m^\flat)^{kp^r})$. We have $\mathrm{pr}_r(g_i^\flat)^{kp^r} = g_i^k \pmod{\varpi}$, and therefore $(J^\flat)^N \subset \mathrm{pr}_r^{-1}(\overline{J}^k)$.

In the other direction, let $N \geq 1$. Let r be large enough so that $p^r \geq N$. Suppose $x \in \mathrm{pr}_r^{-1}(J)$. Write $\mathrm{pr}_r(x) = \sum_{i=1}^m g_i h_i$ with $h_i \in R/\varpi$. Let $h_i^\flat \in R^\flat$ lift h_i, and let $y = \sum_{i=1}^m g_i^\flat h_i^\flat \in J^\flat$. Then $\mathrm{pr}_0(x^{1/p^r} - y) = \mathrm{pr}_r(x) - \mathrm{pr}_0(y) = 0$. Therefore $x^{1/p^r} - y \in \varpi^\flat R^\flat$, and so $x^{1/p^r} \in J^\flat$, which implies $x \in (J^\flat)^{p^r} \subset J^N$. We conclude that $\mathrm{pr}_r^{-1}(\overline{J}) \subset (J^\flat)^N$.

Thus $J^\flat$ is an ideal of definition for $R^\flat$. We claim that $R^\flat$ is $J^\flat$-adically separated and complete, which is to say that the map $\alpha\colon R^\flat \to \varprojlim R^\flat/(J^\flat)^N$ is an isomorphism.

For injectivity of α: Suppose $x \in R^\flat$ lies in the kernel. Let $k, r \geq 1$ be arbitrary. Let N be large enough so that $(J^\flat)^N$ is contained in $((g_1^\flat)^{kp^r}, \dots, (g_m^\flat)^{kp^r})$. Since $x \in (J^\flat)^N$ we can write $x = \sum_{i=1}^m (g_i^\flat)^{kp^r} h_i$, with $h_i \in R^\flat$. Then $\mathrm{pr}_r(x) = \sum_{i=1}^n g_i^k \mathrm{pr}_r(h_i) \in \overline{J}^k$. Since k was arbitrary, and R/ϖ is $\overline{J}$-adically separated, we have $\mathrm{pr}_r(x) = 0$. Since r was arbitrary, $x = 0$.

For surjectivity of α: The mod $\varpi^\flat$ reduction of α is surjective, since we have an isomorphism $R^\flat/\varpi^\flat \cong R/\varpi$ carrying the image of $J^\flat$ onto $\overline{J}$, and R/ϖ is $\overline{J}$-adically complete. By an inductive argument, we are reduced to showing that $R^\flat$ is $\varpi^\flat$-adically complete. This follows from the isomorphism of inverse systems

$$R^\flat = \varprojlim_{\Phi} R^\flat/\varpi^\flat \cong \varprojlim_{N} R^\flat/(\varpi^\flat)^N,$$

which sends $(x_0, x_1, x_2, \dots)$ to $(x_0, x_1^p, x_2^{p^2}, \dots)$.

Now suppose that $I = (f_1, \dots, f_n) \subset R^\flat$ is an ideal of definition. We claim that $I^\sharp = (f_1^\sharp, \dots, f_n^\sharp, \varpi^\sharp)$ is an ideal of definition for R. Since J is an ideal of definition of R, this claim means that there exists $N \geq 1$ such that $(I^\sharp)^N \subset J$ and $J^N \subset I^\sharp$. The existence of an N for which $(I^\sharp)^N \subset J$ follows from the fact that the f_i are topologically nilpotent, and $f \mapsto f^\sharp$ is continuous. For the other containment, it suffices to show that $g_i^N \in I^\sharp$ for large N. Let $r \geq 1$ be large enough so that $(g_i^\flat)^{p^r} \in I$ for $i = 1, \dots, m$ (this can be done because the $g_i^\flat$ are topologically nilpotent and I is an ideal of definition). Since $R^\flat$ is perfect,

$$g_i^\flat \in (f_1^{1/p^r}, \dots, f_n^{1/p^r}).$$

Since $g_i \equiv g_i^{\flat\sharp} \pmod{\varpi}$, we have

$$g_i \in ((f_1^\sharp)^{1/p^r}, \dots, (f_n^\sharp)^{1/p^r}, \varpi).$$

From here it is easy to see that some power of g_i lies in $I^\sharp = (f_1^\sharp, \dots, f_n^\sharp, \varpi)$.

Now we can check conditions (1)-(3) for $R^\flat$. The map $R^\flat \to R$ sending $f \mapsto f^\sharp$ is continuous and pulls back ϖR to $\varpi^\flat R^\flat$, which is therefore closed. $R^\flat/\varpi^\flat$ is semiperfect because $R^\flat$ is perfect. Finally if $f = (f_0, f_1, \dots) \in R^\flat$ satisfies $f^{p^n} \in \varpi^\flat R^\flat$, then we have $f_i^{p^n} \in \varpi^{1/p^i} R$ for all $i \geq 0$, so that $f_i \in \varpi^{1/p^{n+i}} R$ and therefore $f \in (\varpi^\flat)^{1/p^n} R^\flat$. $\square$

We now come to the main theorem of the section.

Theorem 2.5.5. *Let R be an $\mathcal{O}_K$-algebra which is separated and complete for the topology induced by a finitely generated ideal. Assume conditions (1) and (2) of Proposition 2.5.3. Also assume that R is ϖ-torsion free and that R is integrally closed in $R[1/\varpi]$ (this implies condition (3) of Proposition 2.5.3). Then* $(\operatorname{Spa} R) \backslash \{\varpi = 0\}$ *is a perfectoid space over K, with tilt* $(\operatorname{Spa} R^\flat) \backslash \{\varpi^\flat = 0\}$.

Example 2.5.6. Let $R = \mathcal{O}_K[\![T^{1/p^\infty}]\!]$ be the (ϖ, T)-adic completion of $\mathcal{O}_K[T^{1/p^\infty}]$; then $R^\flat = \mathcal{O}_{K^\flat}[\![T^{1/p^\infty}]\!]$. Then $(\operatorname{Spa} R) \backslash \{\varpi = 0\}$ is the "perfectoid open unit disc" over K. It is the union of rational subsets $U((T^n, \varpi)/\varpi) = \operatorname{Spa}(R_n, R_n^+)$ for $n = 1, 2, \dots$, where $R_n = R\langle T^n/\varpi \rangle$. We have $R_n^+ = R_n^\circ = R\left\langle (T^n/\varpi)^{1/p^\infty} \right\rangle$. In this case it is easy to verify Theorem 2.5.5: the tilt of $(\operatorname{Spa} R) \backslash \{\varpi = 0\}$ is the perfectoid open unit disc over $K^\flat$, which is $(\operatorname{Spa} R^\flat) \backslash \{\varpi^\flat = 0\}$.

Remark 2.5.7. It is possible to remove the condition that R is integrally closed in $R[1/\varpi]$, but keeping it simplifies the proof considerably.

Proof. Condition (2) implies that the pth power map $R/\varpi^{1/p} \to R/\varpi$ is surjective, and condition (3) implies that it is injective. Let R_ϖ be the ring R endowed with the ϖ-adic topology. Then R_ϖ is ϖ-adically separated and complete, and so by definition R_ϖ^{a} is a perfectoid $\mathcal{O}_K^{\mathrm{a}}$-algebra. Therefore by Theorem 2.4.1 $R_\varpi[1/\varpi]$ is a perfectoid K-algebra. Since R is integrally closed in $R[1/\varpi]$, we may define $X = \operatorname{Spa}(R_\varpi[1/\varpi], R_\varpi)$, a perfectoid affinoid over K.

Let $X_0 \subset X$ be the subset consisting of those $x \in X$ for which $f \mapsto |f(x)|$ is continuous for the original topology on R. Let $f_1, \dots, f_n$ generate an ideal of definition for R. Then for all $x \in X_0$, there exists $N \geq 1$ such that $\left|f_i^N(x)\right| \leq |\varpi(x)|$ for $i = 1, \dots, n$, so that x lies in the rational subset $U((f_1^N, \dots, f_n^N)/\varpi) \subset X$. Conversely if $x \in U((f_1^N, \dots, f_n^N)/\varpi)$, then for all $\varepsilon > 0$, the ideal $\{f \in R | \, |f(x)| < \varepsilon\}$ is open, since it contains the ideal $(f_1^M, \dots, f_n^M)$ for M sufficiently large. Thus $x \in X_0$.

This shows that $X_0 \subset X$ is open, and is thus a perfectoid space over K. We claim that there is an isomorphism of adic spaces $X_0 \cong (\operatorname{Spa} R)\backslash \{\varpi = 0\}$ which sends x to x. This map is bijective, because any $x \in \operatorname{Spa} R$ with $|\varpi(x)| \neq 0$ induces a continuous valuation on $R^{\varpi}[1/\varpi]$ which lies in X_0. It is a homeomorphism because it carries the rational subset $U((f_1^N, \dots, f_n^N)/\varpi)$ in X_0 onto the same rational subset in $\operatorname{Spa} R$. Finally, it is an isomorphism of adic spaces, because on either side the sections of the structure sheaf on $U((f_1^N, \dots, f_n^N)/\varpi)$ are $R\left\langle f_1^N/\varpi, \dots, f_n^N/\varpi\right\rangle[1/\varpi]$.

We had defined $R^\flat = \varprojlim_\Phi R/\varpi$. Let $R^\flat_\varpi$ be the ring $R^\flat$ endowed with the $\varpi^\flat$-adic topology. The tilt of the perfectoid $\mathcal{O}_K^{\mathrm{a}}$-algebra R^{a}_ϖ is $\varprojlim_\Phi R^{\mathrm{a}}_\varpi/\varpi = R^{\flat\mathrm{a}}_\varpi$, which shows that $R^\flat_\varpi[1/\varpi^\flat]$ is a perfectoid algebra. By [Sch12, Lemma 6.2], $R^\flat$ is integrally closed in $R^\flat[1/\varpi^\flat]$ and $X^\flat = \operatorname{Spa}(R^\flat_\varpi[1/\varpi^\flat], R^\flat_\varpi)$. As before, let $(X^\flat)_0 \subset X^\flat$ be the subset consisting of those $x \in X^\flat$ for which $f \mapsto |f(x)|$ is continuous for the original topology on $R^\flat$. Since $R^\flat$ satisfies the same hypotheses as R, the argument above shows that $(X^\flat)_0 \cong \operatorname{Spa} R^\flat \backslash \left\{\varpi^\flat = 0\right\}$.

We claim that the tilt of X_0 is $(X^\flat)_0$. This will follow from the claim that the homeomorphism $|X| \xrightarrow{\sim} \left|X^\flat\right|$ carries X_0 onto $X_0^\flat$. Indeed, choose generators $f_1, \dots, f_n \in R^\flat$ for the ideal of definition. By Proposition 2.5.3, $f_1^\sharp, \dots, f_n^\sharp$ generate an ideal of definition for R. Then $x \in X$ belongs to X_0 if and only if $\left|f_i^\sharp(x)\right| < 1$ for $i = 1, \dots, n$. But $\left|f_i^\sharp(x)\right| = \left|f_i(x^\flat)\right|$, so this is true if and only if $x^\flat$ belongs to $(X^\flat)_0$. This finishes the proof. $\square$

3 The adic Fargues-Fontaine curve

3.1 The adic spaces $\mathcal{Y}_{F,E}$ and $\mathcal{X}_{F,E}$

Here we review the construction of the adic Fargues-Fontaine curve. The construction requires the following two inputs:

- A finite extension $E/\mathbf{Q}_p$, with uniformizer π and residue field $\mathbf{F}_q/\mathbf{F}_p$,
- An algebraically closed nonarchimedean field F whose residue field contains $\mathbf{F}_q$.

Let $|\,|$ be a valuation inducing the topology on F, and let $\varpi_F \in F$ be a pseudo-uniformizer. For $a \in \mathbf{Q}$ we let ϖ_F^a denote any element of F with $|\varpi_F^a| = |\varpi_F|^a$.

Let $W(\mathcal{O}_F)$ denote the ring of Witt vectors of $\mathcal{O}_F$. Let $W_{\mathcal{O}_E}(\mathcal{O}_F) = W(\mathcal{O}_F) \otimes_{W(\mathbf{F}_q)} \mathcal{O}_E$. Thus a typical element of $W_{\mathcal{O}_E}(\mathcal{O}_F)$ is a series

$$x = \sum_{n \gg -\infty} [x_n]\pi^n,$$

where $x_n \in \mathcal{O}_F$. We equip $W_{\mathcal{O}_E}(\mathcal{O}_F)$ with the $(\pi, [\varpi_F])$-adic topology.

Definition 3.1.1. Let $\mathcal{Y}_{F,E} = \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_F) \backslash \{\pi[\varpi_F] = 0\}$.

We claim that $\mathcal{Y}_{F,E}$ is the union of rational subsets

$$U\left(\frac{\{\pi, [\varpi_F^a]\}}{[\varpi_F^a]}, \frac{\{\pi, [\varpi_F^b]\}}{\pi}\right) = \left\{\left|[\varpi_F^b]\right| \leq |\pi| \leq |[\varpi_F^a]|\right\} \tag{3.1.1}$$

as a and b range through positive rational numbers with $a \leq b$. Indeed, suppose $x \in \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_F)$ satisfies $|\pi[\varpi_F](x)| \neq 0$. Since $[\varpi_F] \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ is topologically nilpotent, $\left|[\varpi_F]^b(x)\right| \to 0$ as $b \to \infty$. Since $|\pi(x)| \neq 0$, there exists $b > 0$ with $\left|[\varpi_F^b](x)\right| \leq |\pi(x)|$. Similar reasoning shows that there exists $a > 0$ with $|\pi(x)| \leq |[\varpi_F^a](x)|$.

For a subinterval $I = [a, b] \subset (0, \infty)$ with $a, b \in \mathbf{Q}$, let $\mathcal{Y}_{F,E}^I$ be the rational subset defined in Eq. (3.1.1). Explicitly, $\mathcal{Y}_{F,E}^I = \operatorname{Spa}(B_I, B_I^+)$, where B_I and B_I^+ are obtained as follows. Let R be the π-adic completion of $W_{\mathcal{O}_E}(\mathcal{O}_F)\left[\frac{\pi}{[\varpi_F^a]}, \frac{[\varpi_F^b]}{\pi}\right]$. Then $B_I = R[1/\pi]$, and B_I^+ is the integral closure of R in B_I. We have

$$\mathcal{Y}_{F,E} = \varinjlim_I \mathcal{Y}_{F,E}^I = \varinjlim_I \operatorname{Spa}(B_I, B_I^+).$$

This shows that our definition of $\mathcal{Y}_{F,E}$ agrees with the definition in [Far13], Définition 2.5.

Theorem 3.1.2 ([Ked, Theorem 3.10])**.** *B_I is strongly noetherian. (In particular $\mathcal{Y}_{F,E}$ is an honest adic space.)*

Since F is perfect, the qth power map $\phi \colon F \to F$ is an automorphism. We use the same symbol ϕ to denote the induced automorphism of $\mathcal{Y}_{F,E}$. Note that $\phi(\mathcal{Y}_{F,E}^I) = \mathcal{Y}_{F,E}^{qI}$. For I narrow enough, I and qI are disjoint. Thus ϕ is totally discontinuous.

Definition 3.1.3. The adic space $\mathcal{X}_{F,E}$ is the quotient of $\mathcal{Y}_{F,E}$ by the automorphism ϕ.

Let $B = H^0(\mathcal{Y}_{F,E}, \mathcal{O}_{\mathcal{Y}_{F,E}})$. We remark that the (schematic) Fargues-Fontaine curve $X_{F,E}$ defined in [FF11] is defined as $\operatorname{Proj} P$, where $P = \bigoplus B^{\phi=\pi^n}$. (There is no schematic version of $\mathcal{Y}_{F,E}$.)

3.2 The adic Fargues-Fontaine curve in equal characteristic

Analogous spaces $\mathcal{Y}_{F,E}$ and $\mathcal{X}_{F,E}$ can be constructed when $E = \mathbf{F}_q(\!(\pi)\!)$ is a local field in positive characteristic. The construction is very similar, except that $W_{\mathcal{O}_E}(\mathcal{O}_F)$ is replaced by $\mathcal{O}_F \widehat{\otimes}_{\mathbf{F}_q} \mathcal{O}_E = \mathcal{O}_F[\![\pi]\!]$. Put

$$\mathcal{Y}_{F,E} = \operatorname{Spa} \mathcal{O}_F[\![\pi]\!] \backslash \{\pi\varpi_F = 0\}.$$

This is nothing but the punctured rigid open disc D_F^*, with parameter π. As before, $\mathcal{X}_{F,E}$ is defined as the quotient of $\mathcal{Y}_{F,E}$ by the automorphism ϕ coming from the Frobenius on F. Note that since ϕ does not act F-linearly, $\mathcal{X}_{F,E}$ does not make sense as a rigid space over F.

Suppose once again that E has characteristic 0. Let K be a perfectoid field containing E, and let $\varpi_K \in K$ be an element with $|\varpi_K| = |\pi|$, such that $\varpi_K = \varpi_K^{\flat\sharp}$ for some $\varpi_K^\flat \in K^\flat$. Let $\mathcal{Y}_{F,E}\widehat{\otimes}_E K$ be the base change to K of $\mathcal{Y}_{F,E}$. That is:

$$\mathcal{Y}_{F,E}\widehat{\otimes}_E K = \operatorname{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_F)\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_K)\backslash\{[\varpi_F]\otimes\varpi_K = 0\}.$$

Proposition 3.2.1 ([Far13, Theorem 2.2 and Theorem 2.7])**.** *The adic space $\mathcal{Y}_{F,E}\widehat{\otimes}_E K$ is perfectoid, and*

$$\left(\mathcal{Y}_{F,E}\widehat{\otimes}_E K\right)^\flat \cong \operatorname{Spa}(\mathcal{O}_F\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_{K^\flat})\backslash\left\{\varpi_F\otimes\varpi_K^\flat = 0\right\}.$$

Proof. This follows from Theorem 2.5.5 applied to the $\mathcal{O}_K$-algebra

$$R = W_{\mathcal{O}_E}(\mathcal{O}_F)\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_K.$$

The ring $R/\varpi_K = \mathcal{O}_F\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_K/\varpi_K$ is semiperfect, and

$$R^\flat = \varprojlim_{\Phi} R/\varpi_K \cong \mathcal{O}_F\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_{K^\flat}/\varpi_K^\flat.$$

It is easy to check that the hypotheses of Theorem 2.5.5 are satisfied. Therefore $X = (\operatorname{Spa} R)\backslash\{\varpi_K = 0\}$ is a perfectoid space over K with tilt $X^\flat = (\operatorname{Spa} R)\backslash\left\{\varpi_K^\flat = 0\right\}$. Under the homeomorphism $|X| \cong \left|X^\flat\right|$, the locus $\{[\varpi_F]\neq 0\}$ in X gets identified with $\{\varpi_F \neq 0\}$. The latter is $\mathcal{Y}_{F,E}\widehat{\otimes}_E K$ and the former is $\operatorname{Spa}(\mathcal{O}_F\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_{K^\flat})\backslash\left\{\varpi_F\otimes\varpi_K^\flat = 0\right\}$. $\square$

As a special case, let E_n be the field obtained by adjoining the π^n-torsion in a Lubin-Tate formal group over E, and let $E_\infty = \bigcup_{n\geq 1} E_n$. Then $\widehat{E}_\infty$ is a perfectoid field. Let $L(E)$ be the imperfect field of norms for the arithmetically profinite extension E_∞/E. As a multiplicative monoid we have

$$L(E) = \varprojlim E_n,$$

where the inverse limit is taken with respect to the norm maps $E_{n+1}\to E_n$. $L(E)\cong \mathbf{F}_q(\!(t)\!)$ is a local field, and $\widehat{E}_\infty \cong \mathbf{F}_q(\!(t^{1/q^\infty})\!)$ is the completed perfection of $L(E)$.

It follows from Theorem 3.2.1 that we have an isomorphism

$$\left(\mathcal{Y}_{F,E}\widehat{\otimes}_E\widehat{E}_\infty\right)^\flat \cong \mathcal{Y}_{F,L(E)}\widehat{\otimes}_{L(E)}\widehat{E}_\infty^\flat. \tag{3.2.1}$$

3.3 Classification of vector bundles

From now on we fix E of characteristic 0 and we abbreviate $\mathcal{Y} = \mathcal{Y}_{F,E}$, $\mathcal{X} = \mathcal{X}_{F,E}$. Since $\mathcal{X}$ is the quotient of $\mathcal{Y}$ by the totally discontinuous action of $\phi^{\mathbf{Z}}$, a vector bundle on $\mathcal{X}$ is the same thing as pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a vector bundle on $\mathcal{Y}$ and $\phi_{\mathcal{E}} \colon \phi^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an isomorphism.

Let k be the residue field of F, and let $L = W_{\mathcal{O}_E}(k)[1/\pi]$. Note that $\mathcal{Y}$ is fibered over $\operatorname{Spa} L$, and the automorphism ϕ of $\mathcal{Y}$ lies over the automorphism ϕ of L. Recall that an *isocrystal* over L is a finite-dimensional L-vector space M equipped with an isomorphism $\phi_M \colon \phi^*M \xrightarrow{\sim} M$. The category of isocrystals over L is semisimple, and there is a bijection $\lambda \mapsto M(\lambda)$ between rational numbers and isomorphism classes of simple objects. Explicitly, if $\lambda = d/h$ for relatively prime d and h with $h > 0$, then $M(\lambda) = Le_1 \oplus \cdots \oplus Le_h$, with

$$\phi_{M(\lambda)}(e_i) = \begin{cases} e_{i+1}, & i = 1, 2, \ldots, h-1 \\ \pi^d e_1, & i = h. \end{cases}.$$

For an isocrystal $\mathcal{M}$ over L we let $\mathcal{E}_M$ be the vector bundle on $\mathcal{X}$ corresponding to the pair $(\mathcal{O}_Y \otimes_L M, \phi \otimes \phi_M)$. For $\lambda \in \mathbf{Q}$ we let $\mathcal{O}(\lambda) = \mathcal{E}_{M(-\lambda)}$.

Theorem 3.3.1 ([Far13, Theorem 3.1])**.** *Every vector bundle $\mathcal{E}$ on $\mathcal{X}$ is isomorphic to a vector bundle of the form $\bigoplus_{i=1}^n \mathcal{O}(\lambda_i)$, for a unique sequence of rational numbers ("slopes") $\lambda_1 \le \cdots \le \lambda_n$.*

Remark 3.3.2. Before [Ked] appeared, it was not known that the rings B_I were strongly noetherian, which makes it difficult to define a good notion of coherent sheaf or vector bundle on $\mathcal{X}$. Fargues gives *ad hoc* definitions of these, and shows that the categories of coherent sheaves on $\mathcal{X}$ and its schematic version X are equivalent ("GAGA for the curve", [Far13, Theorem 3.5]). The classification of vector bundles on X is the main theorem of [FF11]. *A posteriori*, Fargues' definition of vector bundles agrees with the expected one. We also remark that a generalization of the GAGA principle has been proved in [KL, Theorem 8.7.7] in the context of the relative Fargues-Fontaine curve. Finally, the analogue of Theorem 3.3.1 for the equal characteristic curve appears in [HP04].

Proposition 3.3.3. *Let $\lambda \in \mathbf{Q}$. Then $H^0(\mathcal{X}, \mathcal{O}(\lambda)) \neq 0$ if and only if $\lambda \ge 0$.*

Proof. GAGA for the curve shows that $H^0(\mathcal{X}, \mathcal{O}(\lambda)) \cong H^0(X, \mathcal{O}(\lambda))$, and then [FF11, Theorem 12.2(2)] shows that this is nonzero if and only if $\lambda \ge 0$. □

3.4 $\mathcal{X}$ is geometrically simply connected

Let us recall the following definition from [Sch12, Definition 7.1].

Definition 3.4.1. Let K be a nonarchimedean field. A morphism $(R, R^+) \to (S, S^+)$ of affinoid K-algebras is called *finite étale* if S is a finite étale R-algebra with the induced topology, and S^+ is the integral closure of R^+ in S. A morphism $f\colon X \to Y$ of adic spaces over K is called finite étale if there is a cover of Y by open affinoids $V \subset Y$ such that the preimage $U = f^{-1}(V)$ is affinoid, and the associated morphism of affinoid K-algebras

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is finite étale.

If $R \to S$ is finite étale, then S is flat and of finite presentation, hence locally free ("locally" referring to the Zariski topology on $\operatorname{Spec} R$). Thus there exist $f_1, \dots, f_n \in R$ which generate the unit ideal, such that $S[1/f_i]$ is a free $R[1/f_i]$-module for all i. Since f_i is invertible in $R\langle f_1/f_i, \dots, f_n/f_i\rangle$, we find that $S\langle f_1/f_i, \dots, f_n/f_i\rangle$ is a free $R\langle f_1/f_i, \dots, f_n/f_i\rangle$-module for all i. The rational subsets $U(f_1/f_i, \dots, f_n/f_i)$ cover $\operatorname{Spa} R$. We conclude that $R \to S$ is locally free for the topology on $\operatorname{Spa} R$ as well.

Globalizing, we see that if $f\colon Y \to X$ is a finite étale morphism of adic spaces over K, then $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module. It therefore makes sense to talk about the degree of f as a locally constant function on X.

Lemma 3.4.2. *Let $f\colon Y \to X$ be a finite étale morphism of adic spaces of degree d, and let $\mathcal{F} = f_*\mathcal{O}_Y$. Then*

$$\left(\bigwedge^d \mathcal{F}\right)^{\otimes 2} \cong \mathcal{O}_X.$$

Proof. If R is a ring, and if S is a finite étale R-algebra, then the trace map

$$\operatorname{tr}_{S/R}\colon S \otimes_R S \to R$$

is perfect, so that S is self-dual as an R-module. Globalizing, we get that $\mathcal{F} \cong \mathcal{F}^* = \operatorname{Hom}(\mathcal{F}, \mathcal{O}_X)$. Taking top exterior powers shows that $\bigwedge^d \mathcal{F} \cong \bigwedge^d \mathcal{F}^* = \left(\bigwedge^d \mathcal{F}\right)^*$ is a self-dual line bundle, so that the tensor square of $\bigwedge^d \mathcal{F}$ must be trivial. $\square$

Theorem 3.4.3. *The functor $E' \mapsto \mathcal{X} \otimes E'$ is an equivalence between the category of finite étale E-algebras and the category of finite étale covers of $\mathcal{X}$.*

Proof. It suffices to show that if $f\colon Y \to \mathcal{X}$ is a finite étale cover of degree n with Y geometrically irreducible, then $n = 1$. Given such a cover, let $\mathcal{F} = f_*\mathcal{O}_Y$. This is a sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras, so we have a multiplication morphism $\mu\colon \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$.

By Theorem 3.3.1, $\mathcal{F} \cong \bigoplus_{i=1}^n \mathcal{O}(\lambda_i)$, for a collection of slopes $\lambda_i \in \mathbf{Q}$ (possibly with multiplicity). Assume that $\lambda_1 \geq \cdots \geq \lambda_n$.

The proof now follows that of [FF11], Theorem 18.1. After replacing E with a finite unramified extension we may assume that $\lambda_i \in \mathbf{Z}$ for all i. We claim that $\lambda_1 \leq 0$. Assume otherwise, so that $\lambda_1 > 0$. The restriction of μ to $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1)$ is the direct sum of morphisms

$$\mu_{1,1,k}\colon \mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to \mathcal{O}(\lambda_k)$$

for $k = 1, \dots, n$. The morphism $\mu_{1,1,k}$ is tantamount to a global section of $\mathcal{O}(\lambda_k) \otimes \mathcal{O}(-\lambda_1)^{\otimes 2}$, whose slopes are all negative. Proposition 3.3.3 shows that $\mu_{1,1,k} = 0$. Thus the multiplication map $\mathcal{O}(\lambda_1) \otimes \mathcal{O}(\lambda_1) \to 0$ is 0. This means that $H^0(Y, \mathcal{O}_Y)$ contains zero divisors, which is a contradiction because Y is irreducible.

Thus $\lambda_1 \leq 0$, and thus $\lambda_i \leq 0$ for all i. By Lemma 3.4.2, $(\bigwedge^n \mathcal{F})^{\otimes 2} \cong \mathcal{O}_{\mathcal{X}}$, from which we deduce $\sum_{i=1}^n \lambda_i = 0$. This shows that $\lambda_i = 0$ for all i, and therefore $\mathcal{F} \cong \mathcal{O}_{\mathcal{X}}^n$. We find that $E' = H^0(Y, \mathcal{O}_Y)$ is an étale E-algebra of degree n. Since Y is geometrically irreducible, $E' \otimes_E E''$ must be a field for every separable field extension E''/E, which implies that $E' = E$ and $n = 1$. $\square$

Corollary 3.4.4. *Let K/E be an algebraic extension. The functor $K' \mapsto \mathcal{X} \widehat{\otimes} \widehat{K}'$ is an equivalence between the category of finite étale K-algebras and the category of finite étale covers of $\mathcal{X} \widehat{\otimes} \widehat{K}$. In particular, every finite étale cover of $\mathcal{X} \widehat{\otimes} \widehat{\overline{E}}$ is split.*

Proof. The result will follow from Thm. 3.4.3 as soon as we can show that every finite étale cover of $\mathcal{X} \widehat{\otimes} \widehat{K}$ descends to some $\mathcal{X} \otimes E'$, where E'/E is finite subextension of K. Since $\mathcal{X}$ is quasi-compact, we can reduce this to the following statement about affinoids: if (R_i, R_i^+) is a filtered directed system of affinoid algebras, and (R, R^+) is the completion of $\varinjlim (R_i, R_i^+)$, then every finite étale cover of $\mathrm{Spa}(R, R^+)$ descends to some $\mathrm{Spa}(R_i, R_i^+)$. This is [Sch12, Lemma 7.5(i)]. $\square$

3.5 Proof of Theorem A

As in the introduction, let H_E be the Lubin-Tate formal $\mathcal{O}_E$-module attached to the uniformizer π. Let us recall the construction of H_E. Choose a power series $f(T) \in T\mathcal{O}_E[[T]]$ with $f(T) \equiv T^q \pmod{\pi}$. Then H_E is the unique formal $\mathcal{O}_E$-module satisfying $[\pi]_{H_E}(T) = f(T)$. We think of H_E as the formal scheme $\mathrm{Spf}\, \mathcal{O}_E[[T]]$ endowed with an $\mathcal{O}_E$-module structure. Let $t = t_1, t_2, \dots$ be a compatible family of roots of $f(T), f(f(T)), \dots$, and let $E_n = E(t_n)$. For each $n \geq 1$, $H_E[\pi^n]$ is a free $(\mathcal{O}_E/\pi^n)$-module of rank 1, and the action of Galois induces an isomorphism $\mathrm{Gal}(E_n/E) \cong (\mathcal{O}_E/\pi^n)^\times$.

Let $H_{E,0} = H_E \otimes_{\mathcal{O}_E} \mathbf{F}_q$.

Lemma 3.5.1. *For all $n \geq 2$ we have an isomorphism $H_{E,0}[\pi^{n-1}] \cong \operatorname{Spec} \mathcal{O}_{E_n}/t$. The inclusion $H_{E,0}[\pi^{n-1}] \to H_{E,0}[\pi^n]$ corresponds to the qth power Frobenius map $\mathcal{O}_{E_{n+1}}/t \to \mathcal{O}_{E_n}/t$.*

Proof. We have $H_{E,0}[\pi^{n-1}] = \operatorname{Spec} \mathbf{F}_q[T]/f^{(n-1)}(T) = \operatorname{Spec} \mathbf{F}_q[T]/T^{q^{n-1}}$. Meanwhile $\mathcal{O}_{E_n} = \mathcal{O}_{E_1}[T]/(f^{(n-1)}(T) - t)$, so that $\mathcal{O}_{E_n}/t = \mathbf{F}_q[T]/T^{q^{n-1}}$. Thus $H_{E,0}[\pi^{n-1}] \cong \operatorname{Spec} \mathcal{O}_{E_n}/t$. The second claim in the lemma follows from $[\pi]_{H_{E,0}}(T) = T^q$. $\square$

Let $\widetilde{H}_E$ be the *universal cover*:

$$\widetilde{H}_E = \varprojlim_{\pi} H_E.$$

Then $\widetilde{H}_E$ is an E-vector space object in the category of formal schemes over $\mathcal{O}_E$. We will call such an object a *formal E-vector space*.

Let $\widetilde{H}_{E,0} = \widetilde{H}_E \otimes_{\mathcal{O}_E} \mathbf{F}_q$, a formal E-vector space over $\mathbf{F}_q$. Since $H_{E,0} = \operatorname{Spf} \mathbf{F}_q[\![T]\!]$ and $[\pi]_{H_{E,0}}(T) = T^q$, we have

$$\begin{aligned} \widetilde{H}_{E,0} &= \varprojlim_{\pi} \operatorname{Spf} \mathbf{F}_q[\![T]\!] \\ &= \operatorname{Spf} \left(\varinjlim_{T \mapsto T^q} \mathbf{F}_q[\![T]\!] \right)^\wedge \\ &= \operatorname{Spf} \mathbf{F}_q[\![T^{1/q^\infty}]\!]. \end{aligned}$$

In fact we also have $\widetilde{H}_E = \operatorname{Spf} \mathcal{O}_E[\![T^{1/q^\infty}]\!]$, see [Wei14], Proposition 2.4.2(2).

Lemma 3.5.2. *We have an isomorphism of formal E-vector spaces over $\mathbf{F}_q$:*

$$\widetilde{H}_{E,0} = \varinjlim_{\pi} \varprojlim_{n} H_{E,0}[\pi^n].$$

Proof. For each $n \geq 1$ we have the closed immersion $H_{E,0}[\pi^n] \to H_{E,0}$. Taking inverse limits gives a map $\varprojlim_n H_{E,0}[\pi^n] \to \widetilde{H}_{E,0}$, and taking injective limits gives a map $\varinjlim_\pi \varprojlim_n H_{E,0}[\pi^n] \to \widetilde{H}_{E,0}$. The corresponding homomorphism of topological rings is

$$\mathbf{F}_q[\![T^{1/q^\infty}]\!] \to \varprojlim_{T \mapsto T^q} \varinjlim \mathbf{F}_q[T^{1/q^n}]/T = \varprojlim_{T \mapsto T^q} \mathbf{F}_q[T^{1/q^\infty}]/T,$$

which is an isomorphism. $\square$

Proposition 3.5.3. *There exists an isomorphism of formal schemes over $\mathbf{F}_q$:*

$$\widetilde{H}_{E,0} \cong \operatorname{Spf} \mathcal{O}_{\widehat{E}^\flat_\infty}.$$

This isomorphism is $E^\times$-equivariant, where the action of $E^\times$ on $\mathcal{O}_{\widehat{E}^\flat_\infty}$ is defined as follows: $\mathcal{O}_E^\times$ acts through the isomorphism $\mathcal{O}_E^\times \cong \operatorname{Gal}(E_\infty/E)$ of class field theory, and $\pi \in E^\times$ acts as the qth power Frobenius map.

Proof. Combining Lemmas 3.5.1 and 3.5.2, we get

$$\begin{aligned}
\widetilde{H}_{E,0} &\cong \varinjlim_{\pi} \varprojlim_{n} H_{E,0}[\pi^n] \\
&\cong \varinjlim_{x \mapsto x^q} \operatorname{Spec} \mathcal{O}_{E_\infty}/t \\
&\cong \operatorname{Spf} \mathcal{O}_{E^\flat_\infty}.
\end{aligned}$$

The compatibility of this isomorphism with the $\mathcal{O}_E^\times$ follows from the definition of the isomorphism $\mathcal{O}_E^\times \cong \operatorname{Gal}(E_\infty/E)$ of local class field theory. The compatibility of the action of π follows from the second statement in Lemma 3.5.1. $\square$

Let C be an algebraically closed nonarchimedean field containing E. Let $\varpi_C \in C$ be an element with $|\varpi_C| = |\pi|$; write $\varpi_C = \varpi_C^{\flat\sharp}$ for some $\varpi_C^\flat \in C^\flat$.

Write $\widetilde{H}^{\mathrm{ad}}_{E,C}$ for the adic generic fiber of $\widetilde{H}\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_C$ over $\operatorname{Spf}\mathcal{O}_C$. That is:

$$\widetilde{H}^{\mathrm{ad}}_{E,C} = \operatorname{Spa} \mathcal{O}_C[\![T^{1/p^\infty}]\!] \backslash \{\varpi_C = 0\}.$$

Proposition 3.5.4. *$\widetilde{H}^{\mathrm{ad}}_{E,C}$ is a perfectoid space. Furthermore we have an isomorphism*

$$\left(\widetilde{H}^{\mathrm{ad}}_{E,C} \backslash \{0\}\right)^\flat \cong \left(\mathcal{Y}^{\mathrm{ad}}_{C^\flat,E} \widehat{\otimes} \widehat{E}_\infty\right)^\flat$$

which is equivariant for the action of $E^\times$ (which acts on E_∞ by local class field theory). The action of $\pi \in E^\times$ on the left corresponds to the action of $\phi^{-1} \otimes 1$ on the right, up to composition with the absolute Frobenius morphism on $\left(\mathcal{Y}_{C^\flat,E}\widehat{\otimes}\widehat{E}_\infty\right)^\flat$.

Proof. Write $\widetilde{H}_E \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_C = \operatorname{Spf} R$. Then $R \cong \mathcal{O}_C[\![T^{1/p^\infty}]\!]$. Then $R^\flat/\varpi_C$ is semiperfect, and let $R^\flat = \varprojlim_\Phi R/\varpi_C$; then $R^\flat \cong \mathcal{O}_{C^\flat}[\![T^{1/p^\infty}]\!]$. Theorem 2.5.5 applies once again; we find that $\widetilde{H}^{\mathrm{ad}}_{E,C} = (\operatorname{Spa} R) \backslash \{\varpi_C = 0\}$ is a perfectoid space, with tilt $(\operatorname{Spa} R^\flat) \backslash \left\{\varpi_C^\flat = 0\right\}$.

By Proposition 3.5.3 we have an isomorphism $\widetilde{H}_{E,0} \cong \operatorname{Spf} \mathcal{O}_{\widehat{E}^\flat_\infty}$, so that $R/\varpi_C \cong \mathcal{O}_C/\varpi_C \widehat{\otimes}_{\mathbf{F}_q} \mathcal{O}_{\widehat{E}^\flat_\infty}$. Therefore $R^\flat \cong \mathcal{O}_{C^\flat} \widehat{\otimes}_{\mathbf{F}_q} \mathcal{O}_{\widehat{E}^\flat_\infty}$. After removing the "origin" from both sides of the isomorphism between $(\operatorname{Spa} R)\backslash\{\varpi_C = 0\}^\flat$ and $(\operatorname{Spa} R^\flat)\backslash\left\{\varpi_C^\flat = 0\right\}$, we get a series of $\mathcal{O}_E^\times$-equivariant isomorphisms:

$$\begin{aligned}\left(\widetilde{H}_{E,C}^{\mathrm{ad}}\backslash\{0\}\right)^\flat &\cong \operatorname{Spa}\left(\mathcal{O}_{C^\flat}\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_{\widehat{E}_\infty^\flat}\right)\backslash\{\varpi_C\otimes t=0\} \\ &\cong \left(\mathcal{Y}_{C^\flat,E}\widehat{\otimes}\widehat{E}_\infty\right)^\flat,\end{aligned}$$

where in the last step we used Proposition 3.2.1.

The adic space $\operatorname{Spa}\left(\mathcal{O}_{C^\flat}\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_{\widehat{E}_\infty^\flat}\right)\backslash\left\{\varpi_C^\flat\otimes t=0\right\}$ has two qth power "Frobenii": one coming from $\mathcal{O}_{C^\flat}$ and the other coming from $\mathcal{O}_{\widehat{E}_\infty^\flat}$. Their composition is the absolute qth power Frobenius. The action of π on $\widetilde{H}_C^{\mathrm{ad},\flat}$ corresponds to the Frobenius on $\mathcal{O}_{\widehat{E}_\infty^\flat}$. This proves the last claim of the proposition. □

Lemma 3.5.5. *Let X be a perfectoid space which is fibered over $\operatorname{Spa}\mathbf{F}_q$, and suppose $f\colon X\to X$ is an $\mathbf{F}_q$-linear automorphism. Let $\operatorname{Frob}_q\colon X\to X$ be the absolute Frobenius automorphism of X. Then the category of f-equivariant finite étale covers of X is equivalent to the category of $f\circ\operatorname{Frob}_q$-equivariant finite étale covers of X.*

Proof. First observe that a perfectoid algebra in characteristic p is necessarily perfect ([Sch12], Proposition 5.9), which implies that absolute Frobenius is an automorphism of any perfectoid space X in characteristic p. Then note that if $Y\to X$ is a finite étale cover, then Y is also perfectoid ([Sch12], Theorem 7.9(iii)), so that Frob_q is an automorphism of Y.

The proof of the lemma is now formal: if $Y\to X$ is a finite étale cover and $f_Y\colon Y\to Y$ lies over $f\colon X\to X$, then $f_Y\circ\operatorname{Frob}_q\colon Y\to Y$ lies over $f\circ\operatorname{Frob}_q\colon X\to X$. Thus we have a functor from f-equivariant covers to $f\circ\operatorname{Frob}_q$ equivariant covers. Since Frob_q is invertible on Y, the functor is invertible. □

We can now prove the following theorem, which specializes to Theorem A when $E=\mathbf{Q}_p$.

Theorem 3.5.6. *There is an equivalence between the category of $E^\times$-equivariant étale covers of $\widetilde{H}_{E,C}^{\mathrm{ad}}\backslash\{0\}$ and the category of finite étale E-algebras.*

Proof. In the following chain of equivalences, "G-cover of X" is an abbreviation for "G-equivariant finite étale cover of X".

$$
\begin{array}{lll}
 & \left\{E^\times\text{-covers of } \widetilde{H}^{\mathrm{ad}}_{E,C}\backslash\{0\}\right\} & \\
\cong & \left\{E^\times\text{-covers of } \widetilde{H}^{\mathrm{ad},\flat}_{E,C}\backslash\{0\}\right\} & \text{[Sch12], Theorem 7.12} \\
\cong & \left\{\mathcal{O}_E^\times \times (\phi^{-1}\circ \mathrm{Frob}_q)^{\mathbf{Z}}\text{-covers of } \mathcal{Y}_{C^\flat,E}\widehat{\otimes}\widehat{E}_\infty\right\} & \text{Proposition 3.5.4} \\
\cong & \left\{\mathcal{O}_E^\times \times \phi^{\mathbf{Z}}\text{-covers of } \mathcal{Y}_{C^\flat,E}\widehat{\otimes}\widehat{E}_\infty\right\} & \text{Lemma 3.5.5} \\
\cong & \left\{\mathcal{O}_E^\times\text{-covers of } \mathcal{X}_{C^\flat,E}\widehat{\otimes}\widehat{E}_\infty\right\} & \text{Defn. of } \mathcal{X}_{C^\flat,E} \\
\cong & \left\{\mathcal{O}_E^\times\text{-equivariant finite étale } \widehat{E}_\infty\text{-algebras}\right\} & \text{Proposition 3.4.3} \\
\cong & \{\text{Finite étale } E\text{-algebras}\} & A \mapsto A^{\mathcal{O}_E^\times}.
\end{array}
$$

□

Remark 3.5.7. It would also have been possible to prove Theorem 3.5.6 using only the Fargues-Fontaine curve in characteristic p (even though we are interested in a Galois group of a field of characteristic 0!). Indeed, in the chain of equivalences above, the tilting equivalence could be applied to $\mathcal{Y}_{C^\flat,E}\widehat{\otimes}\widehat{E}_\infty$, whose tilt is $\mathcal{Y}_{C^\flat,L(E)}\widehat{\otimes}_{L(E)}\widehat{E}^\flat_\infty$ (3.2.1). Then we could have used only the classification of vector bundles on the characteristic p curve, which already appears in [HP04], and avoided appealing to [FF11].

4 Sheaves on the pro-étale site

4.1 Descent of finite étale morphisms through étale covers

Let X be a perfectoid space, and let $X' \to X$ be a surjective étale morphism. Let $Y' \to X'$ be a finite étale morphism. A *descent datum* for $Y' \to X'$ relative to $X' \to X$ is an isomorphism $\phi\colon Y' \times_X X' \cong X' \times_X Y'$ lying over $X' \times_X X'$ satisfying the cocycle condition. The descent datum ϕ is *effective* if there exists a finite étale morphism $Y \to X$ and an isomorphism $Y' \cong Y \times_X X'$ such that the composite

$$Y' \times_X X' \cong (Y \times_X X') \times_X X' \cong X' \times_X (Y \times_X X') \cong X' \times_X Y'$$

equals ϕ.

Proposition 4.1.1. *Descent data for finite étale morphisms relative to étale surjections of perfectoid spaces are effective.*

In other words, if X is a perfectoid space, then $Y \mapsto \{Y'/Y \text{ finite étale}\}$ is a stack on $X_{\text{ét}}$.

Proof. We claim that it is enough to show that descent data are effective relative to the following kinds of étale surjections:

1. $X' = \coprod_i X_i \to X$, where $\{X_i\}$ is an open cover of X in its analytic topology, and
2. Finite étale morphisms of affinoids.

This reduction step is explained in the proof of [dJvdP96, Proposition 3.2.2]. (It applies to rigid spaces there, but the proof proceeds verbatim in our case.)

The effectiveness of a descent datum relative to an open cover $\{X_i\}$ in the analytic topology is immediate: a family of finite étale covers $Y_i \to X_i$ equipped with gluing data over the $X_i \cap X_j$ glues together to produce a finite étale cover $Y \to X$.

Now suppose $X = \mathrm{Spa}(A, A^+)$ is an affinoid perfectoid space, and $X' \to X$ is a finite étale cover. By [Sch12, Theorem 7.9(ii)], $X' = \mathrm{Spa}(A', (A')^+)$, where $A \to A'$ is finite étale and $(A')^+$ is the integral closure of A^+ in A'. (Theorem 7.9(ii) applies to "strongly finite étale morphisms", but these are equivalent to finite étale morphisms by Theorem 7.9.) Suppose that $Y' \to X'$ is a finite étale morphism equipped with a descent datum relative to $X' \to X$. By the same reasoning, $Y' = \mathrm{Spa}(B', (B')^+)$, where $A' \to B'$ is finite étale and $(B')^+$ is the integral closure of $(A')^+$ in B'. The descent datum for the finite étale morphism $Y' \to X'$ relative to $X' \to X$ induces a descent datum for the finite étale morphism of rings $A' \to B'$ relative to $A \to A'$, which is effective by classical descent theorems. Thus $B' \cong A' \otimes_A B$ for a finite étale $A \to B$. Let B^+ be the integral closure of A^+ in B, and let $Y = \mathrm{Spa}(B, B^+)$. By [Sch12, Theorem 7.9(iii)], B is a perfectoid algebra, and so Y is a perfectoid affinoid. We claim that $Y' \cong Y \times_X X'$. This is immediate from $B' \cong A' \times_A B$ and the fact that $(B')^+$ is the integal closure of $(A')^+ \otimes_{A'} B^+$ in B' (since both are the integral closure of A^+ in B). $\square$

We remark that Proposition 4.1.1 applies not just to perfectoid spaces but to general analytic adic spaces, using the descent theorems found in [KL, §2.6].

4.2 Pro-étale morphisms and the pro-étale topology on Perf_C

Recall that Perf_C is the category of perfectoid spaces over the complete algebraically closed field $C/\mathbf{Q}_p$.

Definition 4.2.1. A morphism $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces is *affinoid pro-étale* if

$$(B, B^+) = \left[\varinjlim (A_i, A_i^+)\right]^\wedge$$

for a filtered directed system of pairs (A_i, A_i^+), where A_i is perfectoid, and $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale. A morphism $f\colon Y \to X$ of perfectoid spaces is *pro-étale* if for every $y \in Y$ there exists an affinoid $V \subset Y$ containing y and an affinoid $U \subset X$ such that $f(V) \subset U$ and such that $f\colon V \to U$ is affinoid pro-étale. Finally, a morphism $f\colon Y \to X$ is a *pro-étale cover* if it is pro-étale and if for all quasi-compact $U \subset X$ there exists a quasi-compact $V \subset Y$ such that $U = f(V)$.

Remark 4.2.2. In the context of Definition 4.2.1, if $X_i = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(B, B^+)$, then $Y \cong \varprojlim X_i$ in the category of perfectoid affinoids over C.

Remark 4.2.3. We can also define what it means for *family* of morphisms $Y_i \to X$ to be a pro-étale cover: it just means that $\coprod_i Y_i \to X$ is a pro-étale cover. Since it usually simplifies notation, we will always consider covers consisting of only one morphism.

The pro-étale covers endow Perf_C with the structure of a site, the *pro-étale site*. For a perfectoid space X over C, let $X_{\text{pro-ét}}$ be the category of morphisms $Y \to X$ in Perf_C with the pro-étale topology.

Proposition 4.2.4. *Descent data are effective for finite étale morphisms in* Perf_C *relative to pro-étale covers.*

Proof. Let $f\colon X' \to X$ be a pro-étale cover, and let $Y' \to X'$ be a finite étale morphism equipped with a descent datum relative to $X' \to X$. We want to show that $Y' \to X'$ descends to a finite étale morphism X'. We can cover X by affinoids U_i. For each i, there exists a quasi-compact open $U_i' \subset X'$ such that $f(U_i') = U_i$. Then $Y' \times_{X'} U_i' \to U_i'$ comes equipped with a descent datum relative to $U_i' \to U_i$. If we can show that these descend to a finite étale morphism $Y_i \to U_i$, then the Y_i glue together to form the desired morphism $Y \to X$.

Therefore it suffices to assume that X is affinoid and X' is quasi-compact. By definition of pro-étale cover, there exists a cover of X' by affinoids $V_i' = \mathrm{Spa}(A_i, A_i^+)$ and for each i an affinoid $V_i \subset X$ such that $f(V_i') \subset V_i$ and such that $f\colon V_i' \to V_i$ is affinoid pro-étale. Since X' is quasi-compact, we may assume there are only finitely many V_i', say with indices $i = 1, \ldots, n$. For each i write $V_i' \sim \varprojlim V_{ij}$, where j runs through a directed system I_i, V_{ij} is affinoid and $V_{ij} \to V_i$ is étale. Then for each tuple $\underline{j} = (j_1, \ldots, j_n)$, $V_{\underline{j}} = \prod_i V_{ij_i}$ is affinoid and $V_{\underline{j}} \to X$ is surjective. We have $\prod_i V_i \cong \varprojlim V_{\underline{j}}$, where $\underline{j}$ runs over all such tuples. Let $X'' = \coprod_i V_i'$. Note that $X'' \to X$ factors through $X' \to X$. It suffices to solve the descent problem for $X'' \to X$.

After replacing X' with X'', it suffices to assume that X' is affinoid and that $X' \sim \varprojlim X_i$, where X_i is affinoid and $X_i \to X$ is an étale surjection. Write $X = \mathrm{Spa}(A, A^+)$, $X' = \mathrm{Spa}(A', (A')^+)$, and $X_i = \mathrm{Spa}(A_i, A_i^+)$. Then $Y' = \mathrm{Spa}(B', (B')^+)$ is affinoid, and $A' \to B'$ is finite étale. By [Sch12, Lemma 7.5(i)], the category of finite étale A'-algebras is the 2-limit of the categories of finite étale A_i-algebras. Thus

$Y' \to X'$ descends to a finite étale morphism $Y_i \to X_i = \mathrm{Spa}(A_i, A_i^+)$, as does the descent datum. By Proposition 4.1.1, Y_i descends to a finite étale morphism $Y \to X$. □

Proposition 4.2.5. *The presheaf $\mathcal{O}_X$ on $X_{\text{pro-ét}}$ defined by $Y \mapsto \mathcal{O}_Y(Y)$ is a sheaf.*

Proof. Given $X \in \mathrm{Perf}_C$ and a pro-étale cover $X' \to X$, we need to show that the descent complex

$$0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(X') \to \mathcal{O}_X(X') \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(X') \to \cdots$$

is exact. Without loss of generality $X = \mathrm{Spa}(A, A^+)$ is a perfectoid affinoid, and $X' = \mathrm{Spa}(A', (A')^+)$ with $(A', (A')^+) = [\varinjlim (A_i, A_i^+)]^\wedge$, with $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ an étale surjection. Let $\varpi \in C$ be a pseudo-uniformizer. For all i, the complex

$$0 \to A^+ \to A_i^+ \to A_i^+ \otimes_{A^+} A_i^+ \to \cdots$$

is almost exact, because $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ is almost zero for $i > 0$ [Sch12, Proposition 7.13]. Taking direct limits shows that

$$0 \to A^+ \to (A')^+ \to (A')^+ \otimes_{A^+} (A')^+ \to \cdots$$

is almost exact. Invert ϖ to get the result. □

Proposition 4.2.6. *For each $X \in \mathrm{Perf}_C$, let h_X be the presheaf on Perf_C defined by $h_X(Y) = \mathrm{Hom}(Y, X)$, $Y \in \mathrm{Perf}_C$. Then h_X is a sheaf on the pro-étale site.*

Proof. Given a pro-étale cover $Y' \to Y$, and a morphism $Y' \to X$ such that the two pull-backs to $Y' \times_Y Y'$ agree, we need to show that $Y' \to X$ factors through a morphism $Y \to X$. We may assume that $X = \mathrm{Spa}(A, A^+)$ is affinoid. The morphism $Y' \to X$ induces a homomorphism $A \to \mathcal{O}_Y(Y')$, which by hypothesis factors through $H^0(Y_{\text{ét}}, \mathcal{O}_Y)$. By Proposition 4.2.5, $H^0(Y_{\text{ét}}, \mathcal{O}_Y) = \mathcal{O}_Y(Y)$; thus we get a homomorphism $A \to \mathcal{O}_Y(Y)$, which induces the desired morphism $Y \to X$. □

Definition 4.2.7. A morphism $Y \to X$ of sheaves on Perf_C is *finite étale* (resp., *pro-étale, a pro-étale cover*) if for all objects $X' \in \mathrm{Perf}_C$ and all morphisms $h_{X'} \to X$, the fiber product $h_{X'} \times_X Y$ is representable by an object $Y' \in \mathrm{Perf}_C$, where the morphism $Y' \to X'$ (corresponding to the projection $h_{Y'} \xrightarrow{\sim} h_{X'} \times_X Y \to h_{X'}$) is finite étale (resp., pro-étale, a pro-étale cover).

From now on we will confuse an object $X \in \mathrm{Perf}_C$ with its image h_X under the Yoneda embedding. The following lemma shows that no ambiguity arises when it comes to finite étale covers.

Lemma 4.2.8. *Let X be an object of* Perf_C*. The categories of finite étale covers of X and h_X are equivalent, via $Y \mapsto h_Y$.*

Proof. Indeed, if $Y \to h_X$ is a finite étale cover, then by definition its fiber product by the identity morphism $h_X \to h_X$ is representable. But this fiber product is just $Y \to h_X$, which therefore is representable by a finite étale cover of X in Perf_C. □

Lemma 4.2.9. *Descent data are effective for finite étale morphisms relative to pro-étale covers of sheaves on* Perf_C*.*

Proof. Let $X' \to X$ be a pro-étale cover of sheaves on Perf_C. Let $Y' \to X'$ be a finite étale morphism equipped with a descent datum relative to $X' \to X$. Sheaves always descend, so there exists a morphism of sheaves $Y \to X$ such that $Y' \cong Y \times_X X'$.

We just need to show that $Y \to X$ is finite étale. Suppose we are given a morphism $Z \to X$, with Z representable. We claim that $Y \times_X Z$ is representable by a finite étale cover of Z. Since $X' \to X$ is a pro-étale cover, $X' \times_X Z = Z'$ for a pro-étale cover $Z' \to Z$ in Perf_C. The pull-back of the finite étale morphism $Y' \to X'$ through $Z' \to X'$ is representable: $Y' \times_{X'} Z' = W'$, where $W' \to Z'$ is a finite étale cover. The descent datum for $Y' \to X'$ relative to $X' \to X$ induces a descent datum for $W' \to Z'$ relative to $Z' \to Z$. Now we apply Proposition 4.2.4: $W' \to Z'$ descends to a finite étale cover $W \to Z$, with $W' = W \times_Z Z'$. Furthermore, the morphism of sheaves $W' \to Y'$ descends to a morphism $W \to Y$. In the diagram

$$\begin{tikzcd}
& W' \arrow[rr] \arrow[dl] \arrow[dd] & & Z' \arrow[dl] \arrow[dd] \\
Y' \arrow[rr, crossing over] \arrow[dd] & & X' & \\
& W \arrow[rr] \arrow[dl] & & Z \arrow[dl] \\
Y \arrow[rr] & & X \arrow[from=uu, crossing over] &
\end{tikzcd}$$

it is known that every square but the bottom one is cartesian. The morphism $W \to Y \times_X Z$ becomes an isomorphism after base change along the the pro-étale cover $X' \to X$; hence it is itself an isomorphism. This shows that $Y \to X$ is finite étale as required. □

Lemma 4.2.10. *Let $X' \to X$ be a morphism of sheaves on* Perf_C*. Assume there exists a pro-étale cover $X'' \to X$ such that $X' \times_X X'' \to X''$ is a pro-étale cover. Then descent data are effective for finite étale covers of X' relative to $X' \to X$.*

Proof. Let $Y' \to X'$ be a finite étale morphism equipped with a descent datum relative to $X' \to X$. Then $Y' \times_X X'' \to X' \times_X X''$ is a finite étale morphism equipped with a descent datum relative to $X' \times_X X'' \to X''$. Since the latter is a pro-étale cover, Lemma 4.2.9 applies to give a finite étale morphism $W \to X''$. But $Y' \times_X X'' \to X' \times_X X''$ also comes equipped with a descent datum relative to $X'' \to X$, which (by functoriality of descents) induces a descent datum for $W \to X''$ relative to $X'' \to X$. Lemma 4.2.9 applies to give the required descent to a finite étale cover $Y \to X$. □

4.3 Pro-étale torsors

Let $\mathcal{G}$ be a group object in the category of sheaves on Perf_C.

Definition 4.3.1. A *pro-étale $\mathcal{G}$-torsor* is a morphism $X' \to X$ of sheaves on Perf_C equipped with an action of $\mathcal{G}$ on X' lying over the trivial action on X, such that pro-étale locally on X there is an isomorphism $X' \cong \mathcal{G} \times X$.

The condition means that there exists a pro-étale cover $Y \to X$ such that $X' \times_X Y \cong \mathcal{G} \times Y$.

For a profinite group G, let $\underline{G}$ be the corresponding constant group object in Perf_C. Explicitly, $\underline{G} = \mathrm{Spa}(A, A^+)$, where A (resp., A^+) is the ring of continuous functions on G with values in C (resp., $\mathcal{O}_C$). Then the underlying topological group of $\underline{G}$ is G itself. For each open subgroup $H \subset G$, let A_H and A_H^+ be the corresponding rings of functions which are right-invariant under H; *i.e.*, the functions factoring through the finite quotient G/H. It is easy to see that $(A, A^+) = [\varinjlim_H (A_H, A_H^+)]^\wedge$. Since $\mathrm{Spa}(A_H, A_H^+)$ is just G/H copies of $\mathrm{Spa}\, C$, this shows that $\underline{G} \to \mathrm{Spa}\, C$ is a pro-étale cover.

The following proposition shows that if an action of a profinite group on a perfectoid space is "nontrivial enough", then the quotient by that group is a pro-étale torsor.

Proposition 4.3.2. *Let G be a profinite group, let $X' \in \mathrm{Perf}_C$, and let $\underline{G} \times X' \to X'$ be an action. Let $X = X'/\underline{G}$, considered as a sheaf on Perf_C. Assume that for every complete algebraically closed field C'/C, the action of G on $X'(C') = X'(\mathrm{Spa}(C', \mathcal{O}_{C'}))$ is free. Then $X' \to X$ is a pro-étale $\underline{G}$-torsor.*

We remind the reader that $X = X'/\underline{G}$ is the sheafification of the pre-sheaf $Y \mapsto h_{X'}(Y)/\underline{G}(Y)$. Thus to give a section of X over Y is to give a pro-étale cover $Y' \to Y$, together with a section of X' over Y' modulo the action of $\underline{G}$. Two such sections $Y' \to X'$ and $Y'' \to X'$ determine the same section of X if there is a common refinement $Y''' \to X'$ on which the two pull-backs differ by the action of $\underline{G}$.

Proof. First we claim that the morphism $\underline{G} \times X' \to X' \times_X X'$ given by $(g, x) \mapsto (gx, x)$ is an isomorphism. It suffices to construct a "division map" $\gamma\colon X' \times_X X' \to \underline{G}$, which satisfies $g(x, y)x = y$ for $(x, y) \in X' \times_X X'$. (We abuse notation slightly here and throughout the proof. The equation "$g(x, y)x = y$" is meant to indicate that the appropriate diagram commutes.)

Suppose we are given a section of $X' \times_X X'$ over an object Y of Perf_C. This means we have two morphisms $f_1, f_2\colon Y \to X'$, such that there exists a pro-étale cover $g\colon Y' \to Y$ and a morphism $\gamma\colon Y' \to \underline{G}$ such that for all $y \in Y'$, $f_2(g(y)) = \gamma(y)f_1(g(y))$. We claim that γ is constant on fibers of $Y' \to Y$. This can be checked on geometric fibers, so without loss of generality we may assume that $Y = \mathrm{Spa}(C', \mathcal{O}_{C'})$ for a complete and algebraically closed field C'/C. Then f_1 and f_2 determine two elements of $X'(C')$. By hypothesis there can be at most one $\gamma \in G$ which translates one to the other. This proves the existence of the morphism γ, and consequently establishes that $\underline{G} \times X' \to X' \times_X X'$ is an isomorphism.

Now we claim that $X' \to X$ is a pro-étale cover. This will finish the proof of the proposition, because then $X' \to X$ becomes trivial after passing to a pro-étale cover, namely $X' \to X$ itself. Let $Y \to X$ be a morphism from a representable sheaf. We claim that $X' \times_X Y \to Y$ is a pro-étale morphism in Perf_C. It suffices to assume that Y is affinoid.

The morphism $Y \to X$ is given by a pro-étale cover $Y' \to X'$ together with a morphism $f\colon Y' \to X'$. We have an isomorphism $X' \times_X Y' \to \underline{G} \times Y'$, where the map to $\underline{G}$ is the composition of $X' \times_X Y' \to X' \times_X X'$ with γ, and the map to Y' is the projection. Consider the diagram

$$\begin{tikzcd} X' \times_X Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X' \times_X Y \arrow[r] & Y, \end{tikzcd}$$

in which all objects but $X' \times_X Y$ are known to be representable. The top arrow factors as $X' \times_X Y' \xrightarrow{\sim} \underline{G} \times Y' \to Y'$. If G is finite, then the top arrow is a finite étale morphism between affinoids, and then $X' \times_X Y$ is representable by an affinoid finite étale over Y by Proposition 4.2.4. If G is profinite, then

$$X' \times_X Y = \varprojlim_H X'/\underline{H} \times_X Y,$$

where H runs over open normal subgroups of G. The same argument as above shows that each $X' \times_X X'/\underline{H}$ is representable by an affinoid which is finite étale over Y, and so $X' \times_X Y \to Y$ pro-étale. $\square$

Let G be profinite, and let $X' \to X$ be a pro-étale $\underline{G}$-torsor. Let $X'' \to X$ be a pro-étale cover such that $X' \times_X X'' \cong \underline{G} \times X''$. Then

$X' \times_X X'' \to X''$ is a pro-étale cover. By Lemma 4.2.10, descent data are effective for finite étale covers of X' relative to $X' \to X$.

In particular, suppose $f\colon Y' \to X'$ is a $\underline{G}$-equivariant finite étale cover. Let $(y, x) \mapsto g(y, x)$ be the morphism $Y' \times_X X' \to X' \times_X X' \tilde{\to} G \times X' \to G$. Then the isomorphism $Y' \times_X X' \to X' \times_X Y$ defined by $(y, x) \mapsto (f(y), g(y, x)y)$ defines a descent datum for $Y' \to X'$ relative to $X' \to X$, and so $Y' \to X'$ descends to a finite étale morphism $Y \to X$. Therefore:

Proposition 4.3.3. *Let G be a profinite group. Let $X' \to X$ be a pro-étale $\underline{G}$-torsor. The following categories are equivalent:*

1. *Finite étale covers of X, and*
2. *$\underline{G}$-equivariant finite étale covers of X'.*

4.4 The object Z_E, and the proof of Theorem B

Let $\widetilde{H}^{\mathrm{ad},*}_{E,C} = \widetilde{H}^{\mathrm{ad}}_{E,C} \backslash \{0\}$, and form the quotient

$$Z_E = \widetilde{H}^{\mathrm{ad},*}_{E,C} / \underline{E}^{\times}$$

as a sheaf on Perf_C. For any algebraically closed C'/C, $\widetilde{H}^{\mathrm{ad}}_{E,C}(C') = \widetilde{H}_E(\mathcal{O}_{C'})$ is an E-vector space. In particular $\mathcal{O}_E^{\times}$ acts freely on $\widetilde{H}^{\mathrm{ad},*}_{E,C}(C')/\pi^{\mathbf{Z}}$. By Proposition 4.3.2, $\widetilde{H}^{\mathrm{ad},*}_{E,C}/\pi^{\mathbf{Z}} \to Z_E$ is a pro-étale $\underline{\mathcal{O}}_E^{\times}$-torsor. Proposition 4.3.3 shows that finite étale covers of Z_E are equivalent to $\underline{\mathcal{O}}_E^{\times}$-equivariant finite étale covers of $\widetilde{H}^{\mathrm{ad},*}_{E,C}/\pi^{\mathbf{Z}}$, which (since the action of π is totally discontinuous) are in turn equivalent to $\underline{E}^{\times}$-equivariant covers of $\widetilde{H}^{\mathrm{ad},*}_{E,C}$. Applying Theorem 3.5.6, we deduce the following.

Theorem 4.4.1. *The following categories are equivalent:*

1. *Finite étale covers of Z_E, and*
2. *Étale E-algebras.*

4.5 Functoriality in E.

In this final section we establish the functoriality of Z_E in E. First let us check that Z_E really only depends on E. The construction depends on the choice of Lubin-Tate formal $\mathcal{O}_E$-module $H = H_E$, which depends in turn on the choice of uniformizer π. If $\pi' \in E$ is a different uniformizer, with corresponding $\mathcal{O}_E$-module H', then H and H' become isomorphic after base extension to $\mathcal{O}_{\widehat{E}^{\mathrm{nr}}}$, the ring of integers in the completion of the maximal unramified extension of E. Such an isomorphism is unique up to multiplication by $E^{\times}$. Thus the adic spaces $\widetilde{H}^{\mathrm{ad}}_C$ and $(\widetilde{H}')^{\mathrm{ad}}_C$ are isomorphic, and we get a *canonical* isomorphism $(\widetilde{H}^{\mathrm{ad}}_C \backslash \{0\})/\underline{E}^{\times} \to (\widetilde{H}^{\mathrm{ad}}_C \backslash \{0\})/\underline{E}^{\times}$. Thus Z_E only depends on E.

Now suppose E'/E is an extension of degree d. There is a "norm" morphism $N_{E'/E}\colon Z_{E'} \to Z_E$, which makes the following diagram commute:

$$\begin{tikzcd} \pi_1^{\text{ét}}(Z_{E'}) \arrow[r, "N_{E'/E}"] \arrow[d, "\sim"'] & \pi_1^{\text{ét}}(Z_E) \arrow[d, "\sim"] \\ \mathrm{Gal}(\overline{E}/E') \arrow[r] & \mathrm{Gal}(\overline{E}/E). \end{tikzcd} \tag{4.5.1}$$

This morphism is induced from the *determinant* morphism on the level of π-divisible $\mathcal{O}_E$-modules. The existence of exterior powers of such modules is the subject of [Hed10]. Here is the main result we need[1] (Theorem 4.34 of [Hed10]): let G be a π-divisible $\mathcal{O}_E$-module of height h (relative to E) and dimension 1 over a noetherian ring R. Then for all $r \geq 1$ there exists a π-divisible $\mathcal{O}_E$-module $\bigwedge_{\mathcal{O}_E}^r G$ of height $\binom{h}{r}$ and dimension $\binom{h-1}{r-1}$, together with a morphism $\lambda\colon G^r \to \bigwedge_{\mathcal{O}_E}^r G$ which satisfies the appropriate universal property. In particular the determinant $\bigwedge_{\mathcal{O}_E}^h G$ has height 1 and dimension 1.

Let π' be a uniformizer of E', and let H' be a Lubin-Tate formal $\mathcal{O}_{E'}$-module. Then $H'[(\pi')^\infty]$ is a π'-divisible $\mathcal{O}_{E'}$-module over $\mathcal{O}_{E'}$ of height 1 and dimension 1. By restriction of scalars, it becomes a π-divisible $\mathcal{O}_E$-module $H'[\pi^\infty]$ of height d and dimension 1. Then $\bigwedge_{\mathcal{O}_E}^d H'[\pi^\infty]$ is a π-divisible $\mathcal{O}_E$-module of height 1 and dimension 1, so that it is the π-power torsion in a Lubin-Tate formal $\mathcal{O}_E$-module $\bigwedge^d H'$ defined over $\mathcal{O}_{E'}$. For all $n \geq 1$ we have an $\mathcal{O}_E/\pi^n$-alternating morphism

$$\lambda\colon H'[\pi^n]^d \to \bigwedge^d H'[\pi^n]$$

of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_{E'}$. Let $H_0' = H' \otimes \mathcal{O}_{E'}/\pi'$. Reducing mod π', taking inverse limits with respect to n and applying Lemma 3.5.2 gives a morphism

$$\lambda_0\colon (\widetilde{H}_0')^d \to \widetilde{\bigwedge^d H'}_0$$

of formal vector spaces over $\mathcal{O}_{E'}/\pi'$. By the crystalline property of formal vector spaces ([SW13], Proposition 3.1.3(ii)), this morphism lifts uniquely to a morphism

$$\widetilde{\lambda}\colon (\widetilde{H}')^d \to \widetilde{\bigwedge^d H'}$$

[1] This result requires the residue characteristic to be odd, but we strongly suspect this is unnecessary. See [SW13], §6.4 for a construction of the determinant map (on the level of universal covers of formal modules) without any such hypothesis.

of formal vector space over $\mathcal{O}_{E'}$.

Since $\bigwedge^d H'$ and H are both height 1 and dimension 1, they become isomorphic after passing to $\mathcal{O}_C$. Let $\alpha_1, \dots, \alpha_n$ be a basis for E'/E, and define a morphism of formal schemes

$$\begin{array}{rcl} \widetilde{H}'_{\mathcal{O}_C} & \to & \widetilde{\bigwedge^d H'}_{\mathcal{O}_{\mathbf{C}}} \cong \widetilde{H}_{\mathcal{O}_C} \\ x & \mapsto & \widetilde{\lambda}(\alpha_1 x, \dots, \alpha_d x) \end{array}$$

After passing to the generic fiber we get a well-defined map $N_{E'/E} \colon Z_{E'} \to Z_E$ which does not depend on the choice of basis for E'/E.

The commutativity of the diagram in Eq. (4.5.1) is equivalent to the following proposition.

Proposition 4.5.1. *The following diagram commutes:*

$$\begin{tikzcd} \left\{\text{\'Etale } E\text{-algebras}\right\} \arrow[r, "A \mapsto A\otimes_E E'"] \arrow[d] & \left\{\text{\'Etale } E'\text{-algebras}\right\} \arrow[d] \\ \{\text{Finite \'etale covers of } Z_E\} \arrow[r] & \{\text{Finite \'etale covers of } Z'_E\}. \end{tikzcd}$$

Here the bottom arrow is pullback via $N_{E'/E}$.

Proof. Ultimately, the proposition will follow from the functoriality of the isomorphism in Lemma 3.5.1. For $n \geq 1$, let E'_n be the field obtained by adjoining the π^n-torsion in H' to E'. (Note that the π^n-torsion is the same as the $(\pi')^{en}$ torsion, where e is the ramification degree of E'/E.) The existence of λ shows that E_n contains the field obtained by adjoining the π^n-torsion in $\bigwedge^d H'$ to E_n. Namely, let $x \in H[\pi^n](\mathcal{O}_{E'_n})$ be a primitive element, in the sense that x generates $H'[\pi^n](\mathcal{O}_{E'_n})$ as an $\mathcal{O}_{E'}/\pi^n$-module. If $\alpha_1, \dots, \alpha_d$ is a basis for $\mathcal{O}_{E'}/\mathcal{O}_E$ then $\lambda(\alpha_1 x, \dots, \alpha_d x)$ generates $\bigwedge H'[\pi^n](\mathcal{O}_{E'_n})$ as an $\mathcal{O}_E/\pi^n$-module. Since $\bigwedge^d H'$ and H become isomorphic over $\widehat{E}'^{,\mathrm{nr}}$, we get a compatible family of embeddings $\widehat{E}_n^{\mathrm{nr}} \hookrightarrow \widehat{E}_n'^{,\mathrm{nr}}$.

By construction, these embeddings are compatible with the isomorphisms in Lemma 3.5.1, so that the following diagram commutes:

$$\begin{tikzcd} \operatorname{Spf} \mathcal{O}_{E_n'^{,\mathrm{nr}}/t'} \arrow[r, "\sim"] \arrow[d] & \widetilde{H}'_0[\pi^{n-1}] \otimes \overline{\mathbf{F}}_q \arrow[d, "N_{E'/E}"] \\ \operatorname{Spf} \mathcal{O}_{E_n^{\mathrm{nr}}/t} \arrow[r, "\sim"'] & \widetilde{H}_0[\pi^{n-1}] \otimes \overline{\mathbf{F}}_q \end{tikzcd}$$

Here $t' \in \mathcal{O}_{E_1}$ is a uniformizer. From here we get the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Spf}\left(\mathcal{O}_{\widehat{E}'^{,\flat}_\infty}\widehat{\otimes}_{\mathcal{O}_{E'}/\pi'}\mathcal{O}_{C^\flat}\right) \arrow[r] \arrow[d] & \widetilde{H}'^{,\flat}_{\mathcal{O}_C} \arrow[d, "N_{E'/E}"] \\ \mathrm{Spf}\left(\mathcal{O}_{\widehat{E}^\flat_\infty}\widehat{\otimes}_{\mathbf{F}_q}\mathcal{O}_{C^\flat}\right) \arrow[r] & \widetilde{H}^\flat_{\mathcal{O}_C}. \end{tikzcd}$$

One can now trace this compatibility with the chain of equivalences in the proof of Theorem 3.5.6 to get the proposition. The details are left to the reader. □

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