Semistable models for modular curves of arbitrary level

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Abstract

We produce an integral model for the modular curve $X(Np^m)$ over the ring of integers of a sufficiently ramified extension of $\mathbb{Z}_p$ whose special fiber is a *semistable curve* in the sense that its only singularities are normal crossings. This is done by constructing a semistable covering (in the sense of Coleman) of the supersingular part of $X(Np^m)$, which is a union of copies of a Lubin-Tate curve. In doing so we tie together nonabelian Lubin-Tate theory to the representation-theoretic point of view afforded by Bushnell-Kutzko types.

For our analysis it was essential to work with the Lubin-Tate curve not at level p^m but rather at infinite level. We show that the infinite-level Lubin-Tate space (in arbitrary dimension, over an arbitrary nonarchimedean local field) has the structure of a perfectoid space, which is in many ways simpler than the Lubin-Tate spaces of finite level.

Contents

1 Introduction: The Lubin-Tate tower

Let K be a non-archimedean local field with uniformizer π, and residue field $\mathbb{F}_q$, and let $n \geq 1$. The *Lubin-Tate tower* is a projective system of affine formal schemes $\mathcal{M}_m$ which parameterize deformations with level π^m structure of a one-dimensional formal $\mathcal{O}_K$-module of height n over $\overline{\mathbb{F}}_q$. (For precise definitions, see §2.2; for a comprehensive historical overview of Lubin-Tate spaces, see the introduction to [Str08].) After extending scalars to a separable closure of K, the Lubin-Tate tower admits an action of the triple product group $\mathrm{GL}_n(K) \times D^\times \times W_K$, where D/K is the central division algebra of invariant $1/n$, and W_K is the Weil group of K. Significantly, the ℓ-adic étale cohomology of the Lubin-Tate tower realizes both the Jacquet-Langlands correspondence (between $\mathrm{GL}_n(K)$ and $D^\times$) and the local Langlands correspondence (between $\mathrm{GL}_n(K)$ and W_K). When $n = 1$, this statement reduces to classical Lubin-Tate theory [LT65]. For $n = 2$ the result was proved by Deligne and Carayol (see [Car83], [Car86]); Carayol conjectured the general phenomenon under the name "non-abelian Lubin-Tate theory". Non-abelian Lubin-Tate theory was established for all n by Boyer [Boy99] for K of positive characteristic and by Harris and Taylor [HT01] for p-adic K. In both cases, the result is established by embedding K into a global field and appealing to results from the theory of Shimura varieties or Drinfeld modular varieties.

In this paper we focus on the case that $n = 2$ and q is odd. We construct a compatible family of *semistable models* $\hat{\mathcal{M}}_m$ for each $\mathcal{M}_m$ over the ring of integers of a sufficiently ramified extension of K. For our purposes this means that the rigid generic fiber of $\hat{\mathcal{M}}_m$ is the same as that of $\mathcal{M}_m$, but that the special fiber of $\hat{\mathcal{M}}_m$ is a locally finitely presented scheme of dimension 1 with only ordinary double points as singularities. The weight spectral sequence would then allow for the computation of the cohomology of the Lubin-Tate tower of curves (along with the actions of the three relevant groups), and one could recover the result of Deligne-Carayol in a purely local manner, although we do not do this here.

The study of semistable models for modular curves begins with the Deligne-Rapoport model for $X_0(Np)$ in [DR73]. A semistable model for $X_0(Np^2)$ was constructed by Edixhoven in [Edi90]. A stable model for $X(p)$ was constructed by Bouw and Wewers in [BW04]. A stable model for $X_0(Np^3)$ was constructed by Coleman and McMurdy in [CM10], using the notion of *semistable coverings* of a rigid-analytic curve by "basic wide opens". The special fiber of their model is a union of Igusa curves together which are linked at each supersingular point of $X_0(N) \otimes \overline{\mathbb{F}}_p$ by a peculiar configuration of projective curves, including in every case a number of copies of the curve with affine model $y^p - y = x^2$. The same method was employed by Tsushima [Tsu11] and Imai-Tsushima [IT11] for the curves $X_0(p^4)$ and $X_1(p^3)$, respectively; the curve $y^p + y = x^{p+1}$ appears in the former. In each of these cases the interesting part of the special fiber of the modular curve is the supersingular locus. Inasmuch a Lubin-Tate curve (for $K = \mathbb{Q}_p$) appears as the rigid space attached to the p-adic completion of a modular curve at one of its mod p supersingular points, the problem of finding a semistable model for a modular curve is essentially the same as finding one for the corresponding Lubin-Tate curve. In this sense our result subsumes the foregoing results; however our method cannot produce the "intersection multiplicities" for the singular points of the special fiber.

We now summarize our main result. Let $\mathbb{F}_q$ be the residue field of K (so that q is odd).

Theorem 1.0.1. *For each $m \geq 1$, there is a finite extension $L_m/\hat{K}^{\mathrm{nr}}$ for which $\mathcal{M}_m$ admits a semistable model $\hat{\mathcal{M}}_m$; every connected component of the special fiber of $\hat{\mathcal{M}}_m$ admits a purely inseparable morphism to one of the following smooth projective curves over $\overline{\mathbb{F}}_q$:*

1. *The projective line $\mathbf{P}^1$,*
2. *The curve with affine model $xy^q - x^q y = 1$,*
3. *The curve with affine model $y^q + y = x^{q+1}$,*
4. *The curve with affine model $y^q - y = x^2$.*

Remark 1.0.2. The mere existence of a semistable model of $\mathcal{M}_m$ (after passing to a finite extension of the field of scalars) follows from the corresponding theorem about proper (algebraic) curves. The formal scheme $\mathcal{M}_m$ appears as the completion along a point in the special fiber of a proper curve over $\mathcal{O}_K$ (e.g., the appropriate modular curve), and a semistable model of the proper curve restricts to a semistable model of $\mathcal{M}_m$. Furthermore, the theorem Drinfeld-Carayol allows one to predict in advance the field L_m over which a semistable model appears. The real content of the theorem is the assertion about the equations for the list of curves appearing therein. A semistable model is unique up to blowing up, so the above theorem holds for all semistable models of $\mathcal{M}_m$ if it holds for one of them.

Remark 1.0.3. A purely inseparable morphism between nonsingular projective curves induces an equivalence on the level of étale sites and therefore an isomorphism on the level of ℓ-adic cohomology.

Remark 1.0.4. The equations for the curves appearing in Thm. 1.0.1 were known some time ago by S. Wewers to appear in the stable reduction of $\mathcal{M}_m$ (unpublished work). Furthermore, it so happens that $xy^q - x^q y = 1$ and $y^q + y = x^{q+1}$ determine isomorphic projective curves, but we have listed them separately because the nature of the group actions on these curves is different.

Let us explain more features of our semistable model $\hat{\mathcal{M}}_m$. It is not the case that one can arrange for the semistable models $\hat{\mathcal{M}}_m$ to be compatible: there is no tower $\cdots \to \hat{\mathcal{M}}_2 \to \hat{\mathcal{M}}_1$ with finite transition maps. Loosely speaking, the problem is that as $m \to \infty$, the singularities of the $\hat{\mathcal{M}}_m$ accumulate around the *CM points*, that is, the points corresponding to deformations of G_0 with extra endomorphisms. But once all the CM points are removed from the rigid generic fiber of $\mathcal{M}_m$, then there does exist a compatible family of semistable models $\hat{\mathcal{M}}_m^{\text{non-CM}}$.

Passing to the special fiber, we have a tower of schemes $\hat{\mathcal{M}}_{m,s}^{\text{non-CM}}$ which are locally of finite type, with finite transition maps. Each of the irreducible components C_m of $\hat{\mathcal{M}}_{m,s}^{\text{non-CM}}$ is a nonsingular projective curve. Suppose $\cdots \to C_{m+1} \to C_m \to \dots$ is a tower of irreducible components. Rather miraculously, the morphisms $C_{m+1} \to C_m$ are purely inseparable for m large enough. As a result the genus of C_m is bounded in any such chain, and the induced maps between étale cohomology groups $H^1(C_m, \mathbb{Q}_\ell)$ become isomorphisms for m large enough. For such a chain we can put $C = \varprojlim C_m$; we find that C is the *perfection* of one of the projective curves listed in Thm. 1.0.1.

One arrives at a convenient combinatorial picture for the scheme $\mathscr{X} = \varprojlim_m \hat{\mathcal{M}}_{m,s}^{\text{non-CM}}$. The *dual graph* of this scheme has vertices corresponding to chains $\{C_m\}$ as above; each vertex is labeled with the scheme $C = \varprojlim C_m$. Vertices $\{C_m\}$ and $\{C'_m\}$ are adjacent if each C_m crosses each C'_m. The resulting graph has as uncountably many connected components; one connected component is displayed in §6. The scheme $\mathscr{X}$ admits an action of (a large subgroup of) the triple product group $\mathrm{GL}_2(K) \times D^\times \times W_K$, where D/K is the quaternionic division algebra and W_K is the Weil group. As one might expect from nonabelian Lubin-Tate theory, $H^1_c(\mathscr{X}, \mathbb{Q}_\ell)$ realizes the local Langlands correspondence and the Jacquet-Langlands correspondence simultaneously for every discrete series representation of $\mathrm{GL}_2(K)$.

In theory one could draw a picture of the special fiber of any particular $\hat{\mathcal{M}}_m$ by forming the quotient of the pictures described in §6 by the congruence subgroup $1 + \pi^m M_2(\mathcal{O}_K)$. This would allow one to determine the structure of the reduction of a semistable model of the appropriate modular curve at level m, see §6.

1.1 The Lubin-Tate perfectoid space

The individual formal schemes $\mathcal{M}_m$ are rather mysterious. Even at level zero, $\mathcal{M}_0$ is merely a formal open unit ball, but the action of the group $\mathcal{O}_D^\times = \operatorname{Aut} G_0$ on $\mathcal{M}_0$ is very difficult to write down. It turns out that the infinite-level deformation space $\mathcal{M}$, by which we mean the completion of $\varprojlim \mathcal{M}_m$, is somehow simpler than all of the spaces at finite level. To prove Thm. 1.0.1 it was indispensable to work at infinite level, where a surprisingly nice description of $\mathcal{M}$ emerges. Results gathered about $\mathcal{M}$ can then be translated into results about the individual $\mathcal{M}_m$.

It will be helpful to first describe the case of $n = 1$, in which $\mathcal{M}_m$ parameterizes lifts of a Lubin-Tate formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$. These lift uniquely, so that $\mathcal{M}_0 = \operatorname{Spf} \mathcal{O}_{\hat{K}^{\mathrm{nr}}}$ is a single point, and for each $m \geq 0$ we have $\mathcal{M}_m = \operatorname{Spf} \mathcal{O}_{K_m}$, where $K_m/\hat{K}^{\mathrm{nr}}$ is the field obtained by adjoining a π^m-torsion point λ_m in a Lubin-Tate formal $\mathcal{O}_K$-module. Let $K_\infty = \cup_{m\geq 1} K_m$ and let $\hat{K}_\infty$ be its completion; then $\hat{K}_\infty$ is the completion of the maximal abelian extension of K. One finds in $\hat{K}_\infty$ an element $t = \lim_{m\to\infty} \lambda_m^{q^m}$ which admits arbitrary qth power roots. If K has positive characteristic, then in fact $\mathcal{O}_{\hat{K}_\infty} \cong \overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!]$ is a ring of *fractional power series* in t. If K has characteristic 0, then we can form the inverse limit $\mathcal{O}_{K_\infty^\flat} = \varprojlim \mathcal{O}_{K_\infty}/\pi$ along the Frobenius map, and then once again $\mathcal{O}_{K_\infty^\flat} \cong \overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!]$. The fraction field $K_\infty^\flat$ of $\mathcal{O}_{K_\infty^\flat}$ is the field of norms of K_∞, as in [FW79]. In either case the field $\hat{K}_\infty$ is an example of a *perfectoid field*; see [Sch11].

Now return to the case of general n. Let A_m be the coordinate ring of $\mathcal{M}_m$, so that $\mathcal{M}_m = \operatorname{Spf} A_m$. Each A_m is complete with respect to the topology induced by the maximal ideal I of A_0. Let A be the completion of $\varinjlim A_m$ with respect to the I-adic topology. We show in Cor. 2.9.11 that if K has positive characteristic, then

$$A \cong \overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$$

is a ring of *fractional power series* in n variables. If K has characteristic 0, then A contains topologically nilpotent elements $X_1, \dots, X_n$ admitting arbitrary qth power roots in A, and the inverse limit $A^\flat = \varprojlim A/\pi$ (limit with respect to the Frobenius map) is also a ring of formal fractional power series. In either case, the parameters $X_1, \dots, X_n$ arise from Drinfeld's parameters on A_m through a limiting process; the action of the group $\mathrm{GL}_n(\mathcal{O}_K) \times \mathcal{O}_D^\times$

on these parameters can be determined directly from the formal $\mathcal{O}_K$-module G_0 itself.

There is a morphism $\mathcal{O}_{\hat{K}_\infty} \to A$, which sends t to a rather complicated fractional power series in $X_1, \ldots, X_n$. This power series can be interpreted as the determinant of a *formal vector space*, see §2.7. Let $\mathcal{M}_\eta^{\mathrm{ad}}$ be the adic generic fiber of $\mathcal{M} = \operatorname{Spf} A$ in the sense of Huber [Hub94]. That is, $\mathcal{M}_\eta^{\mathrm{ad}}$ is the set of continuous valuations $|\,|$ on A for which $|t| \neq 0$. The above descriptions of A show that $\mathcal{M}_\eta^{\mathrm{ad}}$ is a *perfectoid space*, see [Sch11]. We call $\mathcal{M}_\eta^{\mathrm{ad}}$ the Lubin-Tate perfectoid space. In light of the above description of A it would appear that $\mathcal{M}_\eta^{\mathrm{ad}}$ is a very simple sort of space, let alone that it encodes the Langlands correspondence! In fact it is the complexity of the element $t \in A$ which accounts for the interesting cohomological behavior of $\mathcal{M}_\eta^{\mathrm{ad}}$.

Much of the above was probably known to the experts, although perhaps not in this precise form. In [FGL08], an isomorphism between the Lubin-Tate and Drinfeld towers is constructed. For this it is necessary to work with the infinite-level versions of both towers. Roughly speaking, the authors work with an integral model not of the whole Lubin-Tate space (as we have done), but rather with an integral model of a "fundamental domain", whose coordinate ring carries the structure of a perfectoid affinoid algebra. Certainly the important role of the determinant is recognized in [FGL08].

It turns out that all Rapoport-Zink spaces at infinite level are perfectoid spaces which can be described in terms of p-adic Hodge theory: this is the subject of forthcoming work with Peter Scholze. There we prove a general duality theorem relating basic Rapoport-Zink spaces to one another, and in particular we arrive at an isomorphism between the Lubin-Tate and Drinfeld perfectoid spaces which does not require any integral models at all.

1.2 The role of Bushnell-Kutzko types

The theory of types for $\mathrm{GL}_2(K)$ plays an important role in our work. This theory, developed in broad generality by Bushnell-Kutzko in [BK93], furnishes an explicit parametrization of the supercuspidal representations of $\mathrm{GL}_n(K)$ by characters of finite-dimensional subgroups. There is a similar parametrization of the smooth irreducible representations of the multiplicative group of a central division algebra of dimension n^2 over F, see [Bro95]. In certain cases it is known how to align the two parametrizations according to the Jacquet-Langlands correspondence, see [Hen93], [BH05a], [BH05b].

Let $\mathscr{X} = \varprojlim_m \hat{\mathcal{M}}_{m,s}^{\text{non-CM}}$ as before; then $\mathscr{X}$ inherits an action of a large subgroup of $\mathrm{GL}_2(K) \times D^\times$, where D/K is the quaternion algebra. As we have noted, $\mathscr{X}$ is a union of irreducible components, each of which is the perfection of a nonsingular projective curve C over $\overline{\mathbb{F}}_q$. The stabilizer of any particular irreducible component in $\mathrm{GL}_2(K) \times D^\times$ is an open compact-mod-center subgroup of the form $(L^\times \mathcal{L}^\times)^{\det=1}$, where $L \subset M_2(K) \times D$ is a quadratic extension field of K and $\mathcal{L} \subset M_2(K) \times D$ is a certain $\mathcal{O}_L$-order, and the superscript $\det = 1$ indicates the subgroup of pairs (g, b) for which the determinant of g equals the reduced norm of b. The family of orders $\mathcal{L}$ was investigated in [Wei10], where they were called "linking orders"; we find that the representation of $(L^\times \mathcal{L}^\times)^{\det=1}$ on $H^1(C, \mathbb{Q}_\ell)$ encodes the theory of types for $\mathrm{GL}_2(K)$ and $D^\times$ simultaneously, in the sense that the representation of $\mathrm{GL}_2(K) \times D^\times$ induced from $H^1(C, \mathbb{Q}_\ell)$ realizes the Jacquet-Langlands correspondence for those supercuspidal representations of $\mathrm{GL}_2(K)$ containing a certain class of simple strata corresponding to the choice of C. We therefore settle a question of Harris on whether there exist analytic subspaces of the Lubin-Tate tower which realize the Bushnell-Kutzko types in their cohomology, at least in the case of GL_2.

1.3 Outline of the paper

In §2, we review the construction of the Lubin-Tate tower, along with its connection to the Jacquet-Langlands and local Langlands correspondences. We consider the functor which assigns to a formal $\mathcal{O}_K$-module G the inverse limit $\tilde{G} = \varprojlim G$ (limit along multiplication by π), which carries the structure of a K-vector space object in the category of formal schemes. Following [FF11], we call $\tilde{G}$ a *formal vector space*. We observe that if G is a formal $\mathcal{O}_K$-module over a topological base ring R which is p-adically complete, then $\tilde{G}$ is a crystal on the nilpotent site of R. Thus if G_0 is the (unique up to isomorphism) formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$ of height n, one can choose a lift G of G_0 to $\mathcal{O}_{\hat{K}^{\mathrm{nr}}}$, and then $\tilde{G}$ will not depend on the choice of lift G. A choice of coordinate on G_0 allows us to identify $\tilde{G}$ with $\operatorname{Spf} \mathcal{O}_{\hat{K}^{\mathrm{nr}}}[\![X^{1/q^\infty}]\!]$.

We also review relevant results on determinants of formal $\mathcal{O}_K$-modules, as these play an important role. Let $\wedge G_0$ be the top exterior power of G_0. Then $\wedge G_0$ is a formal $\mathcal{O}_K$-module of height one and dimension one; *i.e.* it is a Lubin-Tate module. If $\mathcal{M}_{G_0}$ is the Lubin-Tate deformation space of G_0 at infinite level, and similarly for $\mathcal{M}_{\wedge G_0}$, then there is a determinant morphism $\mathcal{M}_{G_0} \to \mathcal{M}_{\wedge G_0}$. The main result of the section is that there is a Cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_{G_0} & \longrightarrow & \mathcal{M}_{\wedge G_0} \\ \downarrow & & \downarrow \\ \tilde{G}^n & \longrightarrow & \widetilde{\wedge G} \end{array}$$

where the horizontal arrows are determinant morphisms (Thm. 2.8.1). We then define the Lubin-Tate perfectoid space $\mathcal{M}^{\mathrm{ad}}_{G_0,\eta}$.

In §3 we review the constructions from the theory of Bushnell-Kutzko types for $\mathrm{GL}_2(K)$. We also review some of our previous work in [Wei10] concerning the technology of linking orders.

In §4 we identify a family of special open affinoid subspaces $\{\mathscr{Z}_{\mathbf{x},m}\}$ of the Lubin-Tate perfectoid space. These affinoids are parametrized by pairs $(\mathbf{x}, m)$, where $\mathbf{x}$ is a point of $\mathcal{M}_{G_0,\eta}$ representing a deformation of G_0 with endomorphisms by a degree n extension L/K, and $m \geq 0$ is an integer. For a fixed $\mathbf{x}$, the $\mathscr{Z}_{\mathbf{x},m}$ constitute a descending family of opens whose intersection is $\mathbf{x}$. In the case of $n = 2$ and q odd, the reduction of $\mathscr{Z}_{\mathbf{x},m}$ is the perfection of a nonsingular affine variety, whose compactly supported cohomology realizes a Bushnell-Kutzko type associated to the pair $(\mathbf{x}, m)$. (This is probably true in general, but it is a subtle business to compute the reduction of $\mathscr{Z}_{\mathbf{x},m}$ in a systematic way.)

Relying on the Bushnell-Kutzko classification, we show that the compactly supported cohomology of the union of the reductions of the $\mathscr{Z}_{\mathbf{x},m}$ accounts for the entire supercuspidal part of the cohomology of the Lubin-Tate tower, which is to say it is the direct sum of $(\Pi \otimes \mathrm{JL}(\hat{\Pi}))^{\oplus 2}$, where Π runs over supercuspidal representations of $\mathrm{GL}_2(K)$, and JL is the Jacquet-Langlands correspondence. At this point it would certainly be possible to analyze the action of the Weil group, in order to show that the local Langlands correspondence appears. We have not done this here, but we do note that in order to define $\mathscr{Z}_{\mathbf{x},m}$, it becomes necessary to pass to the field of definition of $\mathbf{x}$, which is the completion of the maximal abelian extension of the CM field L.

In §5, we translate the results of §4 back to finite level. We consider the images of the special affinoids in one of the finite level spaces $\mathcal{M}_{m,\eta}$ (considered as a rigid curve). The cohomology of these images exhausts the entire supercuspidal part of $H^1_c(\mathcal{M}_{m,\eta}, \overline{\mathbb{Q}}_\ell)$. At this point our argument starts to resemble the method employed in [CM10]: we find an admissible covering of $\mathcal{M}_{m,\eta}$ by "wide opens", which for cohomological reasons must be a semistable covering. This means that pairs of wide opens intersect in annuli, and that the complement in each wide open of all the others is an affinoid with nonsingular reduction. By the general theory of [Col03], a semistable covering of $\mathcal{M}_{m,\eta}$ corresponds to a semistable model $\hat{\mathcal{M}}_m$.

At infinite level, the combinatorics of our integral model are described by an infinite tree $\mathcal{T}$, whose vertices correspond to equivalence classes of pairs $(\mathbf{x}, m)$. The tree $\mathcal{T}$ is depicted in §6. We note here that $\mathcal{T}$ contains a copy of the Bruhat-Tits tree for $\mathrm{PGL}_2(K)$: this reflects the structure of fundamental domains in $\mathcal{M}_\eta$ already observed in [FGL08]. On the other hand $\mathcal{T}$ has additional ends (infinite branches) due to the existence of supercuspidal representations of positive depth; these ends are in correspondence with the CM points of $\mathcal{M}_\eta$. We sketch a procedure for drawing the special fiber of $\hat{\mathcal{M}}_m$.

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2 The Lubin-Tate perfectoid space

2.1 Moduli of one-dimensional formal modules

Let K be a nonarchimedean field with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field $\mathbb{F}_q$. Let G_0 be a one-dimensional formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$ of height n. Then G_0 is unique up to isomorphism. Let $K_0 = \hat{K}^{\mathrm{nr}}$ be the completion of the maximal unramified extension of K. Let $\mathcal{C}$ be the category of complete local Noetherian $\mathcal{O}_{K_0}$-algebras with residue field $\overline{\mathbb{F}}_q$. We consider the moduli problem $M_{G_0,0}$ which associates to each $R \in \mathcal{C}$ the set of isomorphism classes of *deformations* of G_0. A deformation of G_0 to R is a pair (G, ι), where G is a formal $\mathcal{O}_K$-module over R and $\iota \colon G_0 \tilde{\to} G \otimes_R \overline{\mathbb{F}}_q$ is an isomorphism. An isomorphism between pairs (G, ι) and (G', ι') is an isomorphism of formal $\mathcal{O}_K$-modules $f \colon G \to G'$ which intertwines ι with ι'.

By [Dri74], Prop. 4.2, the functor $\mathcal{M}_{G_0,0}$ is represented by the formal scheme $\mathrm{Spf}\, A_0$, where A_0 is (noncanonically) isomorphic to a formal power series ring $\mathcal{O}_{K_0}[\![u_1, \dots, u_{n-1}]\!]$ in one variable. Thus there is a universal formal $\mathcal{O}_K$-module G^{univ} defined over A_0. We follow the construction of

G^{univ} in [GH94], §5 and §12. Over the polynomial ring $\mathcal{O}_{K_0}[v_1, v_2, \dots]$ we can consider the universal p-typical formal $\mathcal{O}_K$-module F, whose logarithm $f(T) = \log_F(T)$ satisfies Hazewinkel's "functional equation"

$$f(T) = T + \sum_{i \geq 1} \frac{v_i}{\pi} f^{q^i}(X^{q^i}).$$

Here f^{q^i} is the series obtained from $f(X)$ by replacing each variable v_j by $v_j^{q^i}$. Then multiplication by π in F satisfies the congruences

$$[\pi]_F(T) \equiv v_k T^{q^k} \pmod{\pi, v_1, \dots, v_{k-1}, T^{q^k+1}}, \tag{2.1.1}$$

as in [GH94], Prop. 5.7. Then G^{univ} is the push-forward of F through the homomorphism $\mathcal{O}_{K_0}[v_1, v_2, \dots] \to A_0$ which sends

$$v_i \mapsto \begin{cases} u_i, & i = 1, \dots, n-1 \\ 1, & i = n \\ 0, & i > n. \end{cases}$$

Let $[\pi]_{G^{\mathrm{univ}}}(T) = c_1 T + c_2 T^2 + \dots$. (Thus $c_1 = \pi$.) It follows from Eq. (2.1.1) that in $A_1 = \mathcal{O}_{K_0}[\![u_1, \dots, u_{n-1}]\!]$ we have the congruences

$$\begin{array}{ccccc} c_q & \equiv & u_1 & & \pmod{\pi} \\ c_{q^2} & \equiv & u_2 & & \pmod{\pi, u_1} \\ & \vdots & & & \\ c_{q^{n-1}} & \equiv & u_{n-1} & & \pmod{\pi, u_1, \dots, u_{n-2}} \\ c_{q^n} & \equiv & 1 & & \pmod{\pi, u_1, \dots, u_{n-1}}. \end{array}$$

These congruences have the following immediate consequence.

Lemma 2.1.1. *The coefficients $c_1, c_q, \dots, c_{q^{n-1}}$ of $[\pi]_{G^{\mathrm{univ}}}(T)$ generate the maximal ideal of A_0, and $c_{q^n} \in A_0$ is a unit.*

2.2 Level structures

For an algebra $R \in \mathcal{C}$ and a deformation $(G, \iota) \in \mathcal{M}_{G_0,0}(R)$, a *Drinfeld level π^m structure* on G is an $\mathcal{O}_K$-module homomorphism

$$\phi\colon (\pi^{-m}\mathcal{O}_K/\mathcal{O}_K)^{\oplus n} \to G(R)$$

for which the relation

$$\prod_{x \in (\mathfrak{p}_K^{-1}/\mathcal{O}_K)^{\oplus n}} (X - \phi(x)) \;\Big|\; [\pi]_G(X)$$

holds in $R[\![X]\!]$. The images under ϕ of the standard basis elements $(\pi^{-n}, 0)$ and $(0, \pi^{-n})$ of $(\pi^{-n}/\mathcal{O}_K)^{\oplus n}$ form a *Drinfeld basis* of $G[\pi^m]$ over R.

Remark 2.2.1. Note that $x_1, \dots, x_n$ is a Drinfeld basis of $G[\pi^m](R)$ if and only if $\pi^{m-1}x_1, \dots, \pi^{m-1}x_n$ is a Drinfeld basis of $G[\pi](R)$.

Let $\mathcal{M}_{G_0,m}$ denote the functor which assigns to each $R \in \mathcal{C}$ the set of deformations (G, ι) of G_0 to R together with a Drinfeld level π^m structure on G over R.

By [Dri74], Prop. 4.3, the functor $\mathcal{M}_{G_0,m}$ is represented by a formal scheme $\operatorname{Spf} A_m$, where A_m is finite, flat, and generically étale over $A_0 \cong \mathcal{O}_{K_0}[\![u_1, \dots, u_{n-1}]\!]$.

2.3 The case of height one

In this paragraph we assume $n = 1$. Then G_0 is a Lubin-Tate formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$, which admits a unique deformation G to $\mathcal{O}_{K_0}$. In fact after choosing a suitable coordinate on G, we may assume $[\pi]_G(T) = \pi T + T^q$. For each $m \geq 1$, write $\Phi_m(T) = [\pi^m]_G(T)/[\pi^{m-1}]_G(T)$. Then $\Phi_m(T)$ is an Eisenstein polynomial of degree $q^{m-1}(q-1)$ and a unit in $\mathcal{O}_{K_0}[\![T]\!]$.

Lemma 2.3.1. *Let $R \in \mathcal{C}$. An element $x \in G[\pi^m](R)$ constitutes a Drinfeld basis if and only if it is a root of $\Phi_m(T)$.*

Proof. By Remark 2.2.1, and because $\Phi_1([\pi^{m-1}]_G(T)) = \Phi_m(T)$, we may assume $m = 1$. The condition for x to be is a Drinfeld basis of $G[\pi](R)$ is the condition that $T\prod_{a\in k^\times}(T - [a]_G(x))$ is divisible by $[\pi]_G(T) = T\Phi_1(T)$. This is equivalent to the condition that x is a root of $\Phi_1(T)$. □

Let K_m be the field obtained by adjoining the π^m-torsion in G to K_0. Lemma 2.3 implies that $\mathcal{M}_{G_0,m} = \operatorname{Spf} \mathcal{O}_{K_m}$. Note that by local class field theory, the union $K_\infty = \bigcup_m K_m$ is the compositum of K_0 with the maximal abelian extension of K. The following fact will be useful later.

Lemma 2.3.2. *The qth power Frobenius map is surjective on $\mathcal{O}_{K_\infty}/\pi$.*

Proof. Let $\lambda_1, \lambda_2, \dots \in \mathcal{O}_{K_\infty}$ be a compatible sequence of roots of $[\pi^m]_G(T)$, $m \geq 1$. Then λ_m generates $\mathcal{O}_{K_m}$ over $\mathcal{O}_{K_0}$. Since $\lambda_m = [\pi]_G(\lambda_{m+1})$, and $[\pi]_G(T) \pmod{\pi}$ is a power series in T^q, we have that every element of $\mathcal{O}_{K_m}/\pi$ is a qth power in $\mathcal{O}_{K_{m+1}}/\pi$. The result follows. □

Let $\hat{K}_\infty$ be the π-adic completion of K_∞.

Proposition 2.3.3. *If K has positive characteristic, then $\mathcal{O}_{\hat{K}_\infty} \cong \overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!]$, where $\overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!]$ is the t-adic completion of $\overline{\mathbb{F}}_q[t^{1/q^\infty}]$. If K has characteristic 0, let $\mathcal{O}_{K^\flat_\infty} = \varprojlim \mathcal{O}_{K_\infty}/\pi$, where the inverse limit is taken with respect to the qth power Frobenius map. Then $\mathcal{O}_{K^\flat_\infty} \cong \overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!]$.*

Proof. The element $\varpi = \lim_{m\to\infty} \lambda_m^{q^{m-1}}$ belongs to $\hat{K}_\infty$ and has an obvious system of qth power roots, which we write as ϖ^{1/q^m}, $m \geq 1$. We have the congruences $\lambda_m \equiv \lambda_{m-1}^q \equiv \lambda_{m_2}^{q^2} \equiv \dots$ modulo $\pi\mathcal{O}_{K_\infty}$, which shows that $\lambda_m \equiv \varpi^{1/q^{m-1}} \pmod{\pi\mathcal{O}_{\hat{K}_\infty}}$, and therefore (since $\mathcal{O}_{K_\infty}$ is generated by the λ_m) there is a surjection $\overline{\mathbb{F}}_q[t^{1/q^\infty}] \to \mathcal{O}_{K_\infty}/\pi$ which sends t to ϖ.

Assume K has positive characteristic. Then there is a continuous $\overline{\mathbb{F}}_q$-algebra homomorphism $\phi\colon \overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ which sends t to ϖ and which is a surjection modulo π. In particular it is a surjection modulo ϖ, because in $\hat{K}_\infty$ we have $|\varpi| = |\pi|^{1/(q-1)}$. Thus any $b \in \mathcal{O}_{\hat{K}_\infty}$ can be written $b = \phi(a_1) + \varpi b_1 = \phi(a_1) + \phi(t)b_1$, with $a_1 \in \overline{\mathbb{F}}_q[t^{1/q^\infty}]$ and $b_1 \in \mathcal{O}_{\hat{K}_\infty}$. But then we can write $b_1 = \phi(a_2) + \phi(t)b_2$, and so forth, the result being that $b = \phi(a_1 + ta_2 + \dots)$. Thus ϕ is surjective. The injectivity of ϕ follows from the fact that any nonzero element of $\overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!]$ equals t^α times a unit for some $\alpha \in \mathbb{Z}[1/q]$, so that if ϕ has a nonzero kernel, we would have $\phi(t^\alpha) = \varpi^\alpha = 0$ for some α, which is absurd. Thus ϕ is an isomorphism.

Now assume K has characteristic 0. We have put $\mathcal{O}_{K^\flat_\infty} = \varprojlim \mathcal{O}_{K_\infty}/\pi$; this makes $\mathcal{O}_{K^\flat}$ the ring of integers in a complete nonarchimedean valuation field $K^\flat$ containing $\overline{\mathbb{F}}_q$. We have a continuous $\overline{\mathbb{F}}_q$-algebra homomorphism $\phi\colon \overline{\mathbb{F}}_q[\![t^{1/q^\infty}]\!] \to \mathcal{O}_{K^\flat}$ which sends t to the sequence $\varpi^\flat = (\varpi, \varpi^{1/q}, \dots)$. We have an isomorphism $\mathcal{O}_{K^\flat_\infty}/\varpi^\flat \to \mathcal{O}_{\hat{K}_\infty}/\varpi$ given by projection onto the first coordinate. We see that ϕ is once again surjective modulo ϖ. The argument now continues as in the previous paragraph. □

2.4 Nonabelian Lubin-Tate theory

We return to the case of general n. Let D be the algebra of endomorphisms of G_0 up to isogeny. Then D is a division algebra over K of invariant $1/n$. Let $\mathcal{O}_D$ be the ring of integers of D. The formal scheme $\mathcal{M}_{G_0,m}$ has a right action of $\mathrm{GL}_n(\mathcal{O}_K/\pi^m\mathcal{O}_K) \times \mathcal{O}_D^\times$ given by $(G,\iota,\phi)^{(g,b)} = (G, \iota\circ b, \phi\circ g)$. These actions coalesce into an action of $\mathrm{GL}_n(\mathcal{O}_K) \times \mathcal{O}_D^\times$ on the projective system $\varprojlim_m \mathcal{M}_{G_0,m}$, which can be extended via Hecke correspondences to an action of the subgroup $(\mathrm{GL}_n(K)\times D^\times)^{(0)}$ of $\mathrm{GL}_2(K)\times D^\times$ consisting of pairs (g,b) with $v(\det g) + v(N(b)) = 0$. (Here $N\colon D^\times \to K^\times$ is the reduced norm.)

Let $\mathbb{C}$ be the completion of an algebraic closure of K, let $\overline{\eta} = \operatorname{Spf} \mathcal{O}_{\mathbb{C}}$, and let $\mathcal{M}^{\mathrm{ad}}_{m,\overline{\eta}}$ be the adic[1] geometric generic fiber of the formal scheme $\mathcal{M}_m$. Let

$$H^i_c = \varinjlim_m H^i_c\left(\mathcal{M}^{\mathrm{ad}}_{m,\overline{\eta}}, \overline{\mathbb{Q}}_\ell\right).$$

Then H^i_c admits an action of a subgroup of $\mathrm{GL}_n(K) \times D^\times \times W_K$ of index $\mathbb{Z}$. *Nonabelian Lubin-Tate theory* refers to the realization of Langlands functoriality by the H^i_c. We spell out the details in the case that $n = 2$.

Let $\pi \mapsto \mathrm{LLC}(\pi)$ be the bijection between irreducible admissible representations of $\mathrm{GL}_2(K)$ (with complex coefficients) and two-dimensional Frobenius-semisimple Weil-Deligne representations of F afforded by the local Langlands correspondence. Write $\mathcal{H}(\pi) = \mathrm{LLC}(\pi \otimes |\det|^{-1/2})$; then $\pi \mapsto \mathcal{H}(\pi)$ is compatible under automorphisms of the complex field; it may therefore be extended unambiguously to representations with coefficients in any algebraically closed field of characteristic zero.

Theorem 2.4.1. *Let Π be a smooth admissible irreducible representation of $\mathrm{GL}_2(K)$, with coefficients in $\overline{\mathbb{Q}}_\ell$.*

1. *If Π is supercuspidal, then*

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}(H^1_c, \Pi) \cong \mathrm{JL}(\Pi) \otimes \mathcal{H}(\check{\Pi}).$$

2. *If $\Pi \cong \mathrm{St} \otimes (\chi \circ \det)$, where St is the Steinberg representation and χ is a character of $K^\times$, then*

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}(H^1_c, \Pi) \cong \mathrm{JL}(\Pi) \otimes (\chi^{-1} \circ \mathrm{Art}_K^{-1}).$$

3. *If Π belongs to the principal series, then* $\mathrm{Hom}_{\mathrm{GL}_2(K)}(H^1_c, \Pi) = 0$.

As a consequence we can compute the dimension of the cohomology of each space $\mathcal{M}_{m,\overline{\eta}}$ at finite level. Let $\mathcal{M}^{\circ,\mathrm{ad}}_{m,\overline{\eta}}$ be a connected component; the set of connected components is a principal homogenous space for $(\mathcal{O}_K/\pi^m)^\times$. Then

$$\dim H^1_c(\mathcal{M}^{\circ,\mathrm{ad}}_{m,\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = \dim \mathrm{St}^{\Gamma(\pi^m)} + 2\sum_{\Pi} \dim \Pi^{\Gamma(\pi^m)} \dim \mathrm{JL}(\Pi), \quad (2.4.1)$$

where $\Gamma(\pi^m)$ is the congruence subgroup $1 + \pi^m M_2(\mathcal{O}_K)$, and Π runs over *twist classes* of supercuspidal representations of $\mathrm{GL}_2(K)$.

[1]In much of this paper we work with adic spaces rather than rigid spaces, because presently we will be using adic spaces which do not come from rigid spaces. There is a fully faithful functor (see [Hub96]) from the category of rigid analytic varieties to the category of adic spaces, which identifies admissible opens with opens, and admissible open covers with open covers. A separated adic space arises lies in the image of this functor if it is locally topologically of finite type.

2.5 Formal vector spaces

Suppose A is a topological ring which is separated and complete for the topology induced by an ideal of definition I. For such a ring we write $\mathrm{Nil}(A)$ for the set of topologically nilpotent elements of A, which is to say that $\mathrm{Nil}(A)$ is the radical of I. Of course we allow for the trivial case in which $I = 0$ and A is discrete, in which case $\mathrm{Nil}(A)$ is the set of nilpotent elements of A. Let $\mathbf{Alg}_A$ be the category of topological A-algebras R which are separated and complete for the topology induced by an ideal J (which may be assumed to contain the image of I). Also let $\mathbf{Mod}_{\mathcal{O}_K}$ be the category of $\mathcal{O}_K$-modules, and let $\mathbf{Vect}_K$ be the category of K-vector spaces.

Recall that G_0 is a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$ of dimension 1 and height n. G_0 induces a functor $\mathbf{Alg}_{\overline{\mathbb{F}}_q} \to \mathbf{Mod}_{\mathcal{O}_K}$ whose value on an object R is $\mathrm{Nil}(R)$ with the $\mathcal{O}_K$-module structure afforded by G_0. This functor is representable by a formal scheme which we will simply call G_0. A choice of coordinate on G_0 is equivalent to a choice of isomorphism $G_0 \cong \operatorname{Spf} \overline{\mathbb{F}}_q[\![X]\!]$. Now consider the functor $\tilde{G}_0 \colon \mathbf{Alg}_{\overline{\mathbb{F}}_q} \to \mathbf{Vect}_K$ defined by

$$\tilde{G}_0(R) = \varprojlim G_0(R),$$

where the transition map is multiplication by a uniformizer π. Let us call $\tilde{G}$ the *formal K-vector space* associated to G.

Proposition 2.5.1. *$\tilde{G}_0$ is representable by an affine formal scheme isomorphic to $\overline{\mathbb{F}}_q[\![X^{1/q^\infty}]\!]$.*

See Prop. 3.1.2 of [Wei12] for a proof (in that context K has positive characteristic, but it makes no difference).

Now let A be an object of $\mathbf{Alg}_{\mathcal{O}_{K_0}}$ with ideal of definition I.

Proposition 2.5.2. *Let G be a one-dimensional formal $\mathcal{O}_K$-module over A, and define a functor $\tilde{G} \colon \mathbf{Alg}_A \to \mathbf{Vect}_K$ by $\tilde{G}(R) = \varprojlim G(R)$ (inverse limit along multiplication by π).*

1. *The natural reduction map $\tilde{G}(R) \to \tilde{G}(R/I)$ is an isomorphism.*
2. *If A/I is a perfect field, and if $G \otimes A/I$ has finite height, then $\tilde{G}$ is representable by $A[\![X^{1/q^\infty}]\!]$.*

Proof. Choose a coordinate on G, so that the $G(R)$ may be identified with $\mathrm{Nil}(R)$ of R. Let I_R be the extension of I to R, so that I_R is nilpotent. If $(x_1, x_2, \dots) \in \tilde{G}(R)$ lies in the kernel of $\tilde{G}(R) \to \tilde{G}_0(R/I)$, then each x_i lies in I_R. But the power series giving multiplication by π in G has $\pi \in I$ as its linear terms, so it carries I_R^m onto I_R^{m+1}. It follows that each x_i lies in $\cap_{m\geq 1} I_R^m = 0$.

We show that $\tilde{G}(R) \to \tilde{G}_0(R/I)$ is surjective using the standard "Teichmüller lift". Suppose $(x_1, x_2, \dots) \in \tilde{G}_0(R/I)$. Since I is nilpotent in R, we may lift each x_i to an element $y_i \in G(R)$. Then the sequence y_i,πy_{i+1}, $\pi^2 y_{i+2}, \dots$ must converge to an element $z_i \in G(R)$. Then $(z_1, z_2, \dots) \in \tilde{G}(R)$ lifts $(x_1, x_2, \dots) \in \tilde{G}(R/I)$. This settles part (1).

For part (2), let $G_0 = G \otimes_A A/I$. By Lemma 2.5.1, the functor $\tilde{G}_0 \colon \mathbf{Alg}_{R/I} \to \mathbf{Vect}_K$ is representable by a formal scheme isomorphic to $\operatorname{Spf}(A/I)[\![X^{1/q^\infty}]\!]$. Thus if R is an A/I-algebra, then $\tilde{G}_0(R)$ may be identified with $\varprojlim \operatorname{Nil}(R)$ (limit taken with respect to $x \mapsto x^q$). Now suppose R is an object of $\mathbf{Alg}_A$; then by part (1) we have

$$\begin{aligned} \tilde{G}(R) &\cong \tilde{G}(R/I) \\ &= \tilde{G}_0(R/I) \\ &\cong \varprojlim_{x \mapsto x^q} \operatorname{Nil}(R/I) \\ &\cong \varprojlim_{x \mapsto x^q} \operatorname{Nil}(R). \end{aligned}$$

In the last step, we have used the standard Teichmüller lift procedure. This functor is representable by $\operatorname{Spf} A[\![X^{1/q^\infty}]\!]$. □

Remark 2.5.3. The first part of the proposition shows that the functor $\tilde{G}$ only depends on $G_0 = G \otimes_A A/I$, in a functorial sense. That is, there is a functor

$$\begin{aligned} \{\text{Formal } \mathcal{O}_K\text{-modules over } A/I\} &\to \{\text{Formal schemes over } \operatorname{Spf} A\} \\ G_0 &\mapsto \tilde{G}, \end{aligned}$$

where $\tilde{G}$ represents the functor $R \mapsto \tilde{G}_0(R/\pi)$ for any object R of $\mathbf{Alg}_A$. Then if G' is a lift of G_0 to a formal $\mathcal{O}_K$-module over R, then we have a canonical isomorphism of functors $\tilde{G}' \cong \tilde{G}$.

Remark 2.5.4. In the situation of the second part of the proposition, where A/I is a perfect field and G is a formal $\mathcal{O}_K$-module over A, we will often use boldface letters, such as $\mathbf{X}$, to denote elements of the K-vector space $\tilde{G}(R)$, where R is an object of $\mathbf{Alg}_A$. Such an element corresponds to a compatible sequence $(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots)$ in the inverse limit $\varprojlim G(R)$. Assume that a coordinate on G has been chosen, so that $G(R)$ may be identified as a set with $\operatorname{Nil}(R)$. Then the proposition shows that $\mathbf{X}$ corresponds to a topologically nilpotent element $X \in R$ admitting arbitrary qth roots, which will simply be written X^{1/q^∞}.

Let us record the relationship between $\mathbf{X}$ and X. The formal module G_0 has height n, so $[\pi]_{G_0}(T)$ is a power series in T^{q^n}. If A/I is algebraically closed, we may even perform a change of coordinate so that $[\pi]_{G_0}(T) = T^{q^n}$. Then

$$X = \lim_{m\to\infty} \left(\mathbf{X}^{(m)}\right)^{q^{(m-1)n}}.$$

This pattern (boldface for elements of $\tilde{G}(R)$, Roman for elements of R) will be useful later on.

2.6 Determinants of formal modules

Assume for the moment that K has characteristic 0. Let $\wedge G_0$ be the formal group whose (covariant) Dieudonné module is the top exterior power of the Dieudonné module of G_0. Then $\wedge G_0$ has height one and dimension one; *i.e.* it is the Lubin-Tate formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$. Therefore $\wedge G_0$ admits a unique deformation $\wedge G$ to any $R \in \mathcal{C}$.

Now let (G, ι) be a deformation of G_0 to $R \in \mathcal{C}$.

Theorem 2.6.1. *There exists an alternating and multilinear map*

$$\lambda\colon G^n \to \wedge G$$

of formal $\mathcal{O}_K$-module schemes over R, which is universal in the sense that any alternating and multilinear map from G^n into another π-divisible $\mathcal{O}_K$-module must factor through λ.

Proof. This is a special case of the main theorem of Hadi Hedayatzadeh's thesis, [Hed10], Thm. 9.2.36. There the author constructs arbitrary exterior powers of arbitrary π-divisible $\mathcal{O}_K$-modules G over arbitary locally Noetherian $\mathcal{O}_K$-schemes, so long as $\dim G \leq 1$. Hedayatzadeh describes the height and dimension of his exterior powers; for instance, the top exterior power of G has height one and dimension one. Thus in our case (by the uniqueness of the Lubin-Tate formal module and its lift to R) the top exterior power of G must be isomorphic to what we have called $\wedge G$. □

In the context of the above theorem we will write $\lambda(x_1, \dots, x_n)$ as $x_1 \wedge \cdots \wedge x_n$.

Proposition 2.6.2. *Let $R \in \mathcal{C}$, let (G, ι) be a deformation of G_0 to R, and let $x_1, \dots, x_n$ be a Drinfeld level π^m level structure on G. Then $x_1 \wedge \cdots \wedge x_n$ is a Drinfeld level π^m structure on $\wedge G$.*

Proof. It suffices to treat the universal case, where $R = A_m$, G is the universal deformation, and $X_1, \dots, X_n \in G[\pi^m](A_m)$ is the universal level structure. Let $X = X_1 \wedge \cdots \wedge X_n$. Certainly X actually belongs to $\wedge G[\pi^m](A_m)$, because λ is $\mathcal{O}_K$-multilinear. By Remark 2.2.1, we can reduce to the case that $m = 1$. It suffices to show that X is a primitive element of $\wedge G[\pi](A)$. Now we appeal to the fact that A_1 is a domain: if X isn't a primitive element, then it must be 0. But then we would have $\lambda = \pi\mu$ for another alternating multilinear map $\mu \colon G^n \to \wedge G$, which could not factor through λ, contradiction. □

Thm 2.6.1 and Prop. 2.6.2 are true in the case that K has positive characteristic; see [Wei12], Prop. 4.4.1.

From Prop. 2.6.2 we deduce the existence of a morphism of formal schemes $\mathcal{M}_{G_0,m} \to \mathcal{M}_{\wedge G_0,m}$. Recall from §2.3 that $\mathcal{M}_{\wedge G_0,m}$ has dimension 0: after passing to $\mathcal{O}_{\mathbb{C}}$ it breaks up as the union of $q^{m-1}(q-1)$ points. The fibers of $\mathcal{M}_{G_0,m,\overline{\eta}}$ over each of these points is connected; this is the main result of [Str08].

2.7 Determinants of formal vector spaces

Thm. 2.6.1 applied to $R = \overline{\mathbb{F}}_q$ and $G = G_0$ shows that there exists an alternating multilinear map

$$G_0[\pi^m]^n \to \wedge G_0[\pi^m]$$

of $\mathcal{O}_K$-module schemes over $\overline{\mathbb{F}}_q$. Taking inverse limits gives an alternating multilinear map

$$\lambda_0 \colon \tilde{G}_0^n \to \widetilde{\wedge G_0}$$

of formal K-vector spaces over $\overline{\mathbb{F}}_q$.

After choosing a coordinates on G_0 and $\wedge G_0$, we get isomorphisms $\tilde{G}_0 \cong \operatorname{Spf} \overline{\mathbb{F}}_q[\![X^{1/q^\infty}]\!]$ and $\wedge G_0 \cong \operatorname{Spf}[\![T^{1/q^\infty}]\!]$. The morphism λ_0 above amounts to having an element

$$\delta_0(X_1, \dots, X_n) \in \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$$

which comes equipped with a distinguished family of q^mth power roots for $m = 1, 2, \dots$. These will simply be written $\delta_0(X_1, \dots, X_n)^{1/q^m}$.

Let $\operatorname{Moore}(X_1, \dots, X_n)$ be the determinant of the "Moore matrix" $\det(X_i^{q^j})$, $0 \le j \le n-1$, $1 \le i \le n$.

Proposition 2.7.1. *Possibly after replacing δ_0 with $[\alpha]_{\wedge G_0}(\delta_0)$ for some $\alpha \in K^\times$, the congruence*

$$\delta_0(X_1, \dots, X_n) \equiv \mathrm{Moore}(X_1, \dots, X_n)$$

holds modulo higher-degree terms in $\overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$.

Proof. Let d be the least degree of any term appearing in $\delta_0(X_1, \dots, X_n)$, and let T be the homogeneous part of δ_0 of degree d. Since δ_0 is $\mathcal{O}_K$-alternating with respect to the operations $+_{G_0}$ and $[\alpha]_{G_0}$ (and similarly for $\wedge G_0$), and since these operations are simply addition and multiplication modulo quadratic terms, we see that T is an $\mathbb{F}_q$-alternating form, which is to say it is of the form

$$T = \sum c_{a_1,\dots,a_n} X_1^{q^{a_1}} X_2^{q^{a_2}} \dots X_n^{q^{a_n}}$$

where $(a_1, \dots, a_n)$ runs over n-tuples of integers with $\sum_i q^{a_i} = d$, and $c_{a_1,\dots,a_n}$ is an alternating $\overline{\mathbb{F}}_q$-valued function of $(a_1, \dots, a_n)$. After applying $[\pi^m]_{\wedge G_0}$ for some m we may assume that $a_1 = 0$ for some nonzero term in the expression for T above. For any nonzero term in T, the exponents $(a_1, \dots, a_n)$ must be distinct. We claim that up to permutation we have $(a_1, \dots, a_n) = (0, 1, \dots, n-1)$, which would establish the proposition. Otherwise, T contains a term of the form $X_1 X_2^{q^{a_2}} \cdots X_n^{q^{a_n}}$, with $0, a_2, \dots, a_n$ a strictly increasing sequence, and $a_n > n-1$. Since

$$\delta_0([\pi]_{G_0}(X_1), X_2, \dots, X_n) = [\pi]_{\wedge G_0}(\delta_0(X_1, \dots, X_n)),$$

we find that δ_0^q contains a term of the form $X_1^{q^n} X_2^{q^{a_2}} \cdots X_n^{q^{a_n}}$, implying that δ_0 contains a term of the form $X_1^{q^{n-1}} X_2^{q^{a_2-1}} \cdots X_n^{q^{a_n-1}}$. Since T was assumed to contain only the terms of least degree, we have

$$1 + q^{a_2} + q^{a_3} + \dots + q^{a_n} \le q^{n-1} + q^{a_2-1} + q^{a_3-1} + \dots + q^{a_n-1},$$

and this is only possible if $(0, a_2, \dots, a_n) = (0, 1, \dots, n-1)$. $\square$

By Prop. 2.5.2, $\tilde{G}_0$ and $\widetilde{\wedge G_0}$ lift canonically to formal K-vector spaces over $\tilde{G}$ and $\widetilde{\wedge G}$ over $\operatorname{Spf} \mathcal{O}_{K_0}$. The K-alternating map $\lambda_0 \colon \tilde{G}_0^n \to \widetilde{\wedge G_0}$ lifts to a K-alternating map

$$\lambda \colon \tilde{G}^n \to \widetilde{\wedge G}$$

over $\operatorname{Spf} \mathcal{O}_{K_0}$. This morphism corresponds to an element

$$\delta(X_1, \dots, X_n) \in \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$$

which comes equipped with a distinguished family of qth power roots. If δ_m is any lift of δ_0^{1/q^m} to $\mathcal{O}_{K_0}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]$, then

$$\delta = \lim_{m\to\infty} \delta_m^{q^m}.$$

From this and from Prop. 2.7.1 we find

Proposition 2.7.2. *Possibly after replacing δ with $[\alpha]_{\wedge G}(\delta)$ for some $\alpha \in K^\times$, the congruence*

$$\delta(X_1,\dots,X_n) \equiv \mathrm{Moore}(X_1,\dots,X_n)$$

holds modulo the ideal of $\mathcal{O}_{K_0}[\![X_1^{1/q^\infty},\dots,X_n^{1/q^\infty}]\!]$ generated by π and by elements of degree greater than $1+q+\cdots+q^{n-1}$.

2.8 The structure of the Lubin-Tate moduli problem at infinite level

Recall that $\mathcal{M}_{G_0,m} = \mathrm{Spf}\, A_m$ for a complete local ring A_m. Let A be the completion of $\varinjlim A_m$ with respect to the topology induced by the maximal ideal of A_0 (or any A_m, it doesn't matter), and let $\mathcal{M}_{G_0} = \mathrm{Spf}\, A$. Similarly, by §2.3 we have $\mathcal{M}_{\wedge G_0,m} = \mathrm{Spf}\,\mathcal{O}_{K_m}$, where K_m/K_0 is the totally ramified abelian extension of degree $q^{m-1}(q-1)$. Let $\hat{K}_\infty$ be the π-adic completion of $K_\infty = \bigcup_m K_m$, and let $\mathcal{M}_{\wedge G_0} = \mathrm{Spf}\,\mathcal{O}_{\hat{K}_\infty}$. In §2.6 we constructed a morphism $\mathcal{M}_{G_0,m} \to \mathcal{M}_{\wedge G_0,m}$. Taking inverse limits with respect to m, we get a morphism $\mathcal{M}_{G_0} \to \mathcal{M}_{\wedge G_0}$.

Let G^{univ} be the universal deformation of G_0 to A_0. Then over A_m, we have a universal Drinfeld basis $X_1^{(m)},\dots,X_n^{(m)} \in G^{\mathrm{univ}}[\pi^m](A_m)$. In the limit, we get n distinguished elements

$$\mathbf{X}_1,\dots,\mathbf{X}_n \in \tilde{G}^{\mathrm{univ}}(A). \tag{2.8.1}$$

Now suppose that G is an arbitrary lift of G_0 to $\mathcal{O}_{K_0}$. Let $I_0 \subset A_0$ be the maximal ideal, so that $A_0/I_0 = \overline{\mathbb{F}}_q$. We have $G\otimes_{\mathcal{O}_{K_0}}\overline{\mathbb{F}}_q = G^{\mathrm{univ}}\otimes_{A_0} A_0/I_0 = G_0$. Twice applying part (1) of Prop. 2.5.2, we get isomorphisms

$$\tilde{G}^{\mathrm{univ}}(A) \cong \tilde{G}^{\mathrm{univ}}(A/I_0) = \tilde{G}_0(A/I_0) = \tilde{G}(A/I_0) \cong \tilde{G}(A).$$

Let $\mathbf{Z}_i$ be the image of $\mathbf{X}_i$ under the above isomorphism. By unwinding the proof of Lemma 2.5.2, we can say what these are explicitly. Choose coordinates on G and G^{univ}, so that $G(A)$ and $G^{\mathrm{univ}}(A)$ may be identified with $\mathrm{Nil}(A)$. Then $\mathbf{Z}_i = (Z_i^{(1)}, Z_i^{(2)}, \dots) \in \varprojlim G(A)$, where

$$Z_i^{(m)} = \lim_{r\to\infty} [\pi^r]_G \left(X_i^{(r+m)} \right). \tag{2.8.2}$$

The tuple $(\mathbf{Z}_1, \dots, \mathbf{Z}_n)$ represents an A-point of $\tilde{G}$, which is to say a morphism of formal schemes $\mathcal{M}_{G_0} \to \tilde{G}^n$ over $\mathrm{Spf}\, \mathcal{O}_{K_0}$. Recall by Prop. 2.5.2, $\tilde{G}$ is representable by a formal scheme isomorphic to $\mathrm{Spf}\, \mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!]$. Thus in fact we have a continuous $\mathcal{O}_{K_0}$-algebra homomorphism $\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!] \to A$ which sends X_i^{1/q^m} to Z_i^{1/q^m}.

Applying the same constructions to $\wedge G_0$, we have a morphism of formal schemes $\mathcal{M}_{\wedge G_0} \to \widetilde{\wedge G}$. By the naturality of the determinant morphism, the diagram

$$\begin{CD} \mathcal{M}_{G_0} @>>> \mathcal{M}_{\wedge G_0} \\ @VVV @VVV \\ \tilde{G}^n @>>{\delta}> \widetilde{\wedge G} \end{CD} \tag{2.8.3}$$

commutes.

Theorem 2.8.1. *The above diagram is Cartesian. That is, $\mathcal{M}_{G_0}$ is isomorphic to the fiber product of $\tilde{G}^n$ and $\mathcal{M}_{\wedge G_0}$ over $\widetilde{\wedge G}$.*

2.9 Proof of Thm. 2.8.1

The fiber product of $\tilde{G}^n$ and $\mathcal{M}_{\wedge G_0}$ is an affine formal scheme, say $\mathrm{Spf}\, B$, where B is a complete local ring. We have a homomorphism of local rings $\phi\colon A \to B$ which we claim is an isomorphism. We need a few lemmas.

Lemma 2.9.1. *Let R be an object of $\mathcal{C}$ in which $\pi = 0$. Any n-tuple of elements in $G_0[\pi^{m-1}](R)$ constitutes a Drinfeld basis for $G_0[\pi^m](R)$. Similarly, any element in $\wedge G_0[\pi^{m-1}](R)$ constitutes a Drinfeld basis for $\wedge G_0[\pi^m](R)$.*

Proof. The claim for G_0 is equivalent to the assertion that the n elements $0, \dots, 0$ constitute a Drinfeld basis for $G_0[\pi](R)$. This in turn is equivalent to the assertion that T^{q^n} be divisible by $[\pi]_{G_0}(T)$ in $R[\![T]\!]$. But $[\pi]_{G_0}(T)$ equals T^{q^n} times a unit in $\overline{\mathbb{F}}_q[\![T]\!]$, because G_0 has height n. The claim for $\wedge G_0$ is proved similarly. ☐

Recall the parameters $X_1^{(m)}, \dots, X_n^{(m)} \in A_m$, which represent the universal Drinfeld basis for the $G^{\mathrm{univ}}[\pi^m](A_m)$. Let $I \subset A_1$ be the ideal generated by $(X_1^{(1)}, \ldots, X_n^{(1)})$, which is to say that I is the maximal ideal of A_1. We will often be considering the extension of I to the rings A_m and A, and we will abuse notation in calling these ideals I as well. Note that $I \subset A_1$ is the maximal ideal of A_1, so that $A_1/I = \overline{\mathbb{F}}_q$. In particular $\pi \in I$.

Recall that I_0 is the maximal ideal of A_0. Thus $I_0 \subset I$. In fact:

Lemma 2.9.2. *$I_0 \subset I^2$.*

Proof. $X_1^{(1)}, \ldots, X_n^{(1)}$ is a Drinfeld basis for $G^{\text{univ}}[\pi](A_1)$. Thus the polynomial

$$\prod_{(a_1,\ldots,a_n)\in k^n} \left(T - ([a_1]_{G_{\text{univ}}}(X_1^{(1)}) +_{G^{\text{univ}}} \cdots +_{G^{\text{univ}}} [a_n]_{G^{\text{univ}}}(X_n^{(1)})\right)$$

is divisible by $[\pi]_{G^{\text{univ}}}(T)$ in $A_1[\![T]\!]$. This product is congruent to T^{q^n} modulo I^2. Now we apply Lemma 2.1.1. Since the coefficient of T^{q^n} in $[\pi]_{G^{\text{univ}}}(T)$ is a unit, we find that the the coefficients of $T, T^2, \ldots, T^{q^n-1}$ in $[\pi]_{G^{\text{univ}}}(T)$ must lie in I^2. But these coefficients generate I_0, whence the lemma. $\square$

Lemma 2.9.3. *In $A_1[\![T]\!]$, The congruence $[\pi]_{G^{\text{univ}}}(T) \equiv [\pi]_G(T)$ holds modulo $I^2[\![T]\!]$.*

Proof. Indeed, both sides of the congruence lie in $A_0[\![T]\!]$ and are both congruent to $[\pi]_{G_0}(T)$ modulo $I_0[\![T]\!]$, so this follows from Lemma 2.9.2. $\square$

The next lemma describes the closed subscheme $\operatorname{Spec} A_m/I$ of the formal scheme $\mathcal{M}_{G_0,m} = \operatorname{Spf} A_m$.

Lemma 2.9.4. *There is an isomorphism of affine k-schemes $\operatorname{Spec} A_m/I \to G_0[\pi^{m-1}]^n$.*

Proof. For an object R of $\mathcal{C}$ in which $\pi = 0$, we have that $\operatorname{Hom}_{\mathcal{C}}(A_m/I, R)$ is the set of deformations G' of G_0 to R equipped with a Drinfeld basis $x_1, \ldots, x_n$ for $G'[\pi^m](R)$ which satisfy $\pi^{m-1}x_i = 0$, $i = 1, \ldots, n$. For such a deformation we have

$$G' = G^{\text{univ}} \otimes_{A_0} R = (G^{\text{univ}} \otimes_{A_0} A_0/I_0) \otimes_{A_0/I_0} R = G_0 \otimes_{\overline{\mathbb{F}}_q} R$$

Thus $\operatorname{Hom}_{\mathcal{C}}(A_m/I, R)$ is the set of Drinfeld bases $x_1, \ldots, x_n$ for $G_0[\pi^m](R)$ which satisfy $\pi^{m-1}x_i = 0$, $i = 1, \ldots, n$; that is, $x_1, \ldots, x_n \in G_0[\pi^{m-1}](R)$. But by Lemma 2.9.1, any such n-tuple is automatically a Drinfeld basis. Thus $\operatorname{Hom}_{\mathcal{C}}(A_m/I, R)$ is simply the set of n-tuples of elements of $G_0[\pi^{m-1}](R)$. This identifies $\operatorname{Spec} A_m/I$ with $G_0[\pi^{m-1}]^n$. $\square$

We now turn to B, which by definition is the coordinate ring of the affine formal scheme $\tilde{G}_0^n \times_{\widetilde{\wedge G}} \mathcal{M}_{\wedge G_0}$.

Lemma 2.9.5. *The qth power Frobenius map is surjective on B/π.*

Proof. We have the following presentation of B:

$$B \approx \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] \hat{\otimes}_{\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty} \tag{2.9.1}$$

Since the Frobenius map is surjective on $\mathcal{O}_{\hat{K}_\infty}/\pi$ (Lemma 2.3.2) and on $\mathcal{O}_{K_0}/\pi = \overline{\mathbb{F}}_q$, it is surjective on B/π. □

For an object R of $\mathbf{Alg}_{\mathcal{O}_{K_0}}$, $\mathrm{Hom}(B, R)$ is in bijection with the set of n-tuples $x_1, \dots, x_n \in \tilde{G}(R)$ such that $x_1 \wedge \cdots \wedge x_n$, a priori just an element of $\widetilde{\wedge G}(R)$, actually lies in $T(\wedge G)(R) = \varprojlim_m \wedge G[\pi^m](R)$, and constitutes a Drinfeld basis for each $\wedge G[\pi^m](R)$. The identity homomorphism $\mathrm{Hom}(B, B)$ corresponds to an n-tuple of universal elements $Y_1, \dots, Y_n \in \tilde{G}(B)$. For $i = 1, \dots, n$, let us write $Y_i = (Y_i^{(1)}, Y_i^{(2)}, \dots)$. After choosing a coordinate on G, we can identify $Y_i^{(m)}$ with a (topologically nilpotent) element of B.

Let $J \subset B$ be the ideal generated by π and by $Y_1^{(1)}, \dots, Y_n^{(1)}$.

Lemma 2.9.6. *$\phi(J) \subset I$, and ϕ descends to an isomorphism $B/J \to A/I$.*

Proof. For an $\mathcal{O}_{K_0}$-algebra R in which $\pi = 0$, $\mathrm{Hom}(B/J, R)$ is in bijection with the set of n-tuples $x_1, \dots, x_n \in \tilde{G}_0(R)$ such that (a) $x_i^{(1)} = 0$ for $i = 1, \dots, n$ and such that (b) $x_1 \wedge \cdots \wedge x_n$ constitutes a compatible family of Drinfeld bases $(x_1 \wedge \cdots \wedge x_n)^{(m)}$ for $\wedge G_0[\pi^m](R)$. However, if condition (a) is satisfied, then

$$(x_1 \wedge \cdots \wedge x_n)^{(1)} = x_1^{(1)} \wedge \dots x_n^{(1)} = 0,$$

and 0 is always a Drinfeld basis for $\wedge G[\pi](R)$ by Lemma 2.9.1, so that condition (b) is satisfied as well.

Therefore $\operatorname{Spec} B/J = \varprojlim_m G[\pi^{(m-1)}]^n$. By Lemma 2.9.4 this is isomorphic to $\varprojlim_m \operatorname{Spec} A_m/I = \operatorname{Spec} A/I$. □

Lemma 2.9.7. *$\phi(J)A$ is a dense subset of I.*

Proof. By Eq. (2.8.2) we have

$$\phi(Y_i^{(r)}) = \lim_{m\to\infty} [\pi^m]_G \left(X_i^{(m+r)} \right).$$

Since the limit converges I-adically, $\phi\left(Y_i^{(1)}\right) \equiv [\pi^m]_{G_0}\left(X_i^{(m+1)}\right) \pmod{I^2}$ some sufficiently large m. By Lemma 2.9.3 we have

$$X_i^{(1)} = [\pi^{m-1}]_{G^{\text{univ}}}\left(X_i^{(m)}\right) \equiv [\pi^{m-1}]_G\left(X_i^{(m)}\right) \equiv \phi\left(Y_i^{(1)}\right) \pmod{I^2},$$

and therefore $X_i^{(1)} \in \phi(J)A + I^2$. Since the $X_i^{(1)}$ generate I, we have $I \subset \phi(J)A + I^2$, which when iterated yields $I \subset \phi(J)A + I^m$ for all $m \geq 1$. Since I generates the topology on A, the closure of $\phi(J)A$ must equal I. □

Lemma 2.9.8. *The qth power Frobenius map on A/π has dense image.*

Proof. Let $\overline{A} = A/\pi$, and let $\overline{I}$ be the image of I in $\overline{A}$, so that $\overline{I}$ generates the topology on $\overline{A}$. Similarly define $\overline{B}$ and $\overline{J}$. By Lemma 2.9.6 we have $\overline{A}/\overline{I} \cong \overline{B}/\overline{J}$, so the qth power Frobenius map is also surjective on $\overline{A}/\overline{I}$. Thus $\overline{A} = \overline{A}^q + \overline{I}$.

We will prove by induction that for all $m \geq 1$, $\overline{A} = \overline{A}^q + \overline{I}^m$ and $\overline{I} = (\overline{A}^q \cap \overline{I}) + \overline{I}^m$. The first claim proves the lemma, since $\overline{I}$ generates the topology on $\overline{A}$. As for the base case $m = 1$, the first claim is discussed above, and the second claim is vacuous. For the induction step, assume both claims for m. By Lemma 2.9.7, $\phi(\overline{J})\overline{A}$ is dense in $\overline{I}$, so that $\overline{I} = \phi(\overline{J})\overline{A} + \overline{I}^{m+1}$. Since Frobenius is surjective on $\overline{B}$ (Lemma 2.9.5), we have $\phi(\overline{J}) \subset \overline{A}^q \cap \overline{I}$. Thus

$$\begin{aligned} \overline{I} &\subset (\overline{A}^q \cap \overline{I})\overline{A} + \overline{I}^{m+1} \\ &= (\overline{A}^q \cap \overline{I})(\overline{A}^q + \overline{I}^m) + \overline{I}^{m+1} \\ &\subset (\overline{A}^q \cap \overline{I}) + \overline{I}^{m+1}. \end{aligned}$$

The reverse containment is obvious, so that $\overline{I} = (\overline{A}^q \cap \overline{I}) + \overline{I}^{m+1}$, thus establishing the second claim for $m+1$. Inserting this into $\overline{A} = \overline{A}^q + \overline{I}$ gives $\overline{A} = \overline{A}^q + \overline{I}^{m+1}$, which establishes the first claim for $m+1$. □

Lemma 2.9.9. *The qth power Frobenius map on A/π is surjective.*

Proof. Once again let $\overline{A} = A/\pi$ and let $\overline{I}$ be the image of I in $\overline{A}$. The ideal $\overline{I}$ is finitely generated; let $f_1, \ldots, f_n$ be a set of generators (*e.g.*, the images of the elements $X_i^{(1)}$, $i = 1, \ldots, n$). Recall that for $m \geq 0$, $\overline{I}^{[q^m]}$ is the ideal generated by the q^mth powers of elements of $\overline{I}$, so that $\overline{I}^{[q^m]}$ is generated by the $f_i^{q^m}$, $i = 1, \ldots, n$. Obviously have $\overline{I}^{[q^m]} \subset \overline{I}^{q^m}$. But also we have $\overline{I}^{q^N} \subset \overline{I}^{[q^m]}$ for N large enough. Thus the sequence of ideals $\overline{I}^{[q^m]}$ generates the topology on A.

Let $a \in \overline{A}$. By Lemma 2.9.8 there exists $b \in \overline{A}$ such that $a - b^q \in \overline{I}^{[q]}$. Let us write

$$a = b^q + \sum_i a_i f_i^q, \ a_i \in \overline{A}.$$

For each i we may also find $b_i \in \overline{A}$ with $a_i - b_i^q \in \overline{I}^{[q]}$; write $a_i = b_i^q + \sum_j a_{ij} f_j^q$, $a_{ij} \in A$. Thus

$$a = b^q + \sum_i b_i^q f_i^q + \sum_{i,j} a_{ij} f_i^q f_j^q.$$

Continuing this process, we find a qth root of a in $\overline{A}$, namely

$$b + \sum_i b_i f_i + \sum_{i,j} b_{ij} f_i f_j + \dots.$$

This completes the proof. □

Lemma 2.9.10. *The induced map $B/\pi \to A/\pi$ is surjective.*

Proof. Let us write $\overline{\phi}\colon \overline{B} \to \overline{A}$ for the induced map. Let $a \in \overline{A}$. By Lemma 2.9.6, there exists $b_0 \in \overline{B}$ and $a_0 \in \overline{I}$ with $a = \phi(b_0) + a_0$. By Lemma 2.9.9, a_0 has a qth root in $\overline{A}$, call it $a_0^{1/q}$. Apply Lemma 2.9.6 to write $a_0^{1/q} = \phi(b_1) + a_1$, with $b_1 \in \overline{B}$, $a_1 \in \overline{I}$. Then $\phi(b_1^q) = a_0 - a_1^q \in \overline{I}$, so that (by Lemma 2.9.6) $b_1^q \in \overline{J}$. Similarly write $a_1^{1/q} = \phi(b_2) + a_2$, and so on. We have $b_m^q \in J$ for all $m \geq 1$. Therefore the series $b_0 + b_1^q + b_2^{q^2} + \dots$ converges to an element b with $\phi(b) = a$. □

We are now ready to show that that $\phi\colon B \to A$ is an isomorphism. We will first show it is surjective. If $a \in A$, use Lemma 2.9.10 to find $b \in B$ with $a = \phi(b_0) + \pi a_1$, $a_1 \in A$. Write $a_1 = \phi(b_1) + \pi a_2$, and so on. Then $b = b_0 + \pi b_1 + \pi^2 b_2 + \dots$ satisfies $\phi(b) = a$.

We now turn to injectivity. First observe that since the Lubin-Tate extension K_∞/K_0 is totally ramified with Galois group $\mathcal{O}_K^\times$, the ring $\mathcal{O}_{\hat{K}_\infty}$ contains ideal-theoretic q^mth roots of π for all $m \geq 1$. Let $\eta_m \in \mathcal{O}_{\hat{K}_\infty}$ be an element with $\eta_m^{q^m} \mathcal{O}_{\hat{K}_\infty} = \pi \mathcal{O}_{\hat{K}_\infty}$. The existence of the determinant morphism $\mathcal{M}_{G_0} \to \mathcal{M}_{\wedge G_0}$ combined with the fact that A is $\mathcal{O}_K$-flat shows that A contains $\mathcal{O}_{\hat{K}_\infty}$ as a subring.

Suppose that $b \in B$ is an element with $\phi(b) = 0$. Let $m \geq 1$ Since Frobenius is surjective on B/π (Lemma 2.9.5), we may write $b \equiv c^{q^m} \pmod{\pi B}$, with $c \in B$. Then $\phi(c)^{q^m} \in \pi A$. Since A is integrally closed in $A[1/\pi]$, $\phi(c) \in \eta_m A$. Since ϕ is an isomorphism $B/I \to A/J$ (Lemma 2.9.6), we have $c \in \eta_m B + J$, and therefore $b \in \pi B + J^{[q^m]}$ for all $m \geq 1$. Since B is J-adically complete, this implies that b lies in the closure of πB, which is πB itself. But then $b/\pi \in \ker \phi$, so that in fact $b \in \pi^2 B$. Inductively, we find $b \in \pi^m B$ for all $m \geq 1$. Since B is π-adically separated, $b = 0$. This completes the proof of Theorem 2.8.1.

Corollary 2.9.11. *If K has positive characteristic, then*

$$A \cong \overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!].$$

If K has characteristic 0, then put $A^\flat = \varprojlim A/\pi$, where the limit is taken with respect to the qth power Frobenius map. Then

$$A^\flat \cong \overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$$

Proof. By Thm. 2.8.1, we have an isomorphism of complete local $\mathcal{O}_{\hat{K}_\infty}$-algebras

$$\mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] \hat{\otimes}_{\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!]} \mathcal{O}_{\hat{K}_\infty} \to A \tag{2.9.2}$$

In the tensor product appearing in Eq. (2.9.2), the image of $X \in \mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!]$ is $\delta(X_1, \dots, X_n)$ in the left factor and t in the right. First assume that K has positive characteristic. Then the map $\mathcal{O}_{K_0}[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{\hat{K}_\infty}$ is surjective with kernel generated by $\pi - g(X)$ for some fractional power series $g(X) \in \overline{\mathbb{F}}_q[\![X^{1/q^\infty}]\!]$ without constant term. Recalling that $\mathcal{O}_{K_0} = \overline{\mathbb{F}}_q[\![\pi]\!]$, we have

$$\begin{aligned} A &\cong \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]/(\pi - g(\delta(X_1, \dots, X_n))) \\ &= \overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]. \end{aligned}$$

Now assume K has characteristic 0. We have

$$A/\pi \cong \overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] \hat{\otimes}_{\overline{\mathbb{F}}_q[\![X^{1/q^\infty}]\!]} \mathcal{O}_{K_\infty}/\pi.$$

Now take the inverse limit along the qth power Frobenius maps. In doing so, the surjection $\overline{\mathbb{F}}_q[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{K_\infty}/\pi$ becomes an isomorphism $\overline{\mathbb{F}}_q[\![X^{1/q^\infty}]\!] \to \mathcal{O}_{K_\infty^\flat}$. Thus $A^\flat = \overline{\mathbb{F}}_q[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$ as required. □

2.10 Geometrically connected components

Each A_m is an algebra over the ring of integers $\mathcal{O}_{K_m}$ in the Lubin-Tate extension K_m/K_0. Let $\mathbb{C}$ be the completion of an algebraic closure of K, and write $\overline{\eta} = \operatorname{Spec} \mathcal{O}_{\mathbb{C}}$. Fix an embedding $\hat{K}_\infty \hookrightarrow \mathbb{C}$. This is tantamount to choosing a point $\mathbf{t} \in \widetilde{\wedge G}(\mathcal{O}_C)$, which in turn corresponds (see Rmk. 2.5.4) to a topologically nilpotent element $t \in \mathcal{O}_C$ together with a compatible system of qth power roots t^{1/q^m} for $m \geq 1$.

Let

$$A_m^\circ = A_m \hat{\otimes}_{\mathcal{O}_{K_m}} \mathcal{O}_{\mathbb{C}},$$

and let A° be the completion of $\varinjlim_m A_m^\circ$. Let $\mathcal{M}^\circ = \operatorname{Spf} A^\circ$. From the description of $\mathcal{M}$ in Thm. 2.8.1, we may identify $\mathcal{M}^\circ$ with the fiber of $\delta\colon \tilde{G}^n_{\mathcal{O}_C} \to \widetilde{G}_{\mathcal{O}_C}$ over the single point $\mathbf{t} \in \widetilde{\wedge G}(\mathcal{O}_C)$. Thus

$$A^\circ \cong \mathcal{O}_{\mathbb{C}}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!] / \left(\delta(X_1, \dots, X_n)^{1/q^m} - t^{1/q^m} \right). \qquad (2.10.1)$$

The $\mathcal{O}_C$-algebra A° admits an action of the group $(\mathrm{GL}_n(K) \times D^\times)^{\det = 1}$ consisting of pairs (g, b) with $\det g = N(b)$ (where N is the reduced norm).

2.11 The Lubin-Tate perfectoid space

Let $\mathcal{M}^{\mathrm{ad}}$ be the adic space associated to the formal scheme $\mathcal{M} = \operatorname{Spf} A$. That is, $\mathcal{M}^{\mathrm{ad}}$ is the set of continuous valuations on A, as in [Hub94]. The existence of the map $\mathcal{O}_{\hat{K}_\infty} \to A$ shows that $\mathcal{M}^{\mathrm{ad}}$ is fibered over the two-point space $\operatorname{Spa}(\mathcal{O}_{\hat{K}_\infty}, \mathcal{O}_{\hat{K}_\infty})$. Let $\eta = \operatorname{Spa}(\hat{K}_\infty, \mathcal{O}_{\hat{K}_\infty})$ be the generic point. Let $\mathcal{M}^{\mathrm{ad}}_\eta$ be the fiber of $\mathcal{M}^{\mathrm{ad}}$ over η. Thus $\mathcal{M}^{\mathrm{ad}}_\eta$ is the set of continuous valuations $|\;|$ on A for which $|\pi| \neq 0$.

The field $\hat{K}_\infty$ is a complete nonarchimedean field whose valuation is rank 1 and non-discrete. Further, the Frobenius map is a surjection on $\mathcal{O}_{\hat{K}_\infty}/\pi$. Thus $\hat{K}_\infty$ is a *perfectoid field*, cf. [Sch11], Def. 3.1.

Lemma 2.11.1. *$\mathcal{M}^{\mathrm{ad}}_\eta$ is a perfectoid space.*

Proof. This is a consequence of the fact that A is a reduced complete flat adic $\mathcal{O}_{\hat{K}_\infty}$-algebra admitting a finitely generated ideal of definition I containing π, such that the Frobenius map is surjective on A/π.

Indeed, let $f_1, \dots, f_n$ be a set of generators for I (such as the elements $X_i^{(1)}$). Then any valuation $|\;|$ belonging to $\mathcal{M}^{\mathrm{ad}}_\eta$ must satisfy $|f_i| < 1$ for $i = 1, \dots, n$. Since $|\pi| \neq 0$, there exists $r \geq 1$ for which $|f_i^r| \leq |\pi|$. Let $R_r = A\langle f_i^r/\pi \rangle [1/\pi]$, and let $R_r^+ \subset R_r$ be the integral closure of $A\langle f_i^r/\pi \rangle$ in R_r. The Frobenius map is surjective on R_r^+/π (this follows quickly from the corresponding property of A), and thus (R_r, R_r^+) is a perfectoid $\mathcal{O}_{\hat{K}_\infty}$-algebra, cf. [Sch11], Def. 6.1. The valuation $|\;|$ extends uniquely to R_m and satisfies $|R_r^+| \leq 1$. We have thus shown that $\mathcal{M}^{\mathrm{ad}}_\eta$ is the union of perfectoid affinoids $\operatorname{Spa}(R_r, R_r^+)$. $\square$

We can also pass to geometrically connected components. Let $\mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad}}$ be the adic generic fiber of $\operatorname{Spf} A^\circ$. Then $\mathcal{M}^{\circ,\mathrm{ad}}$ is a perfectoid space over $\mathbb{C}$.

2.12 CM points

Let $n = dm$ be a factorization of n with $d > 1$. Let L/K be an extension of degree d. Let H_0 be a formal $\mathcal{O}_L$-module of height m over k^{ac}. Let $\mathcal{O}_E = \operatorname{End} H_0$ be its endomorphism algebra, so that E/L is a central simple division algebra with invariant $1/m$. Then H_0, when considered instead as a formal $\mathcal{O}_K$-module, has height n. As such it is isomorphic to G_0. Choose an isomorphism $f'\colon G_0 \to H_0$. Then for every deformation (H, ι) of H_0, the pair $(H, \iota \circ f')$ is a deformation of G_0. Now suppose an $\mathcal{O}_K$-linear isomorphism $f\colon \mathcal{O}_K^n \to \mathcal{O}_L^m$ is given. Then a Drinfeld level structure on (H, ι) can be transported via f to a Drinfeld level structure on $(H, \iota \circ f')$. We find that for each pair (f, f'), there exists a morphism of formal schemes $\rho_{f,f'}\colon \mathcal{M}_{H_0} \to \mathcal{M}_{G_0} \hat{\otimes} \mathcal{O}_{L_0}$. Points in the image of this morphism correspond to deformations of G_0 with endomorphisms by $\mathcal{O}_L$. The set of such pairs (f, f') is a principal homogeneous space for the product $\mathrm{GL}_n(\mathcal{O}_K) \times \mathcal{O}_D^\times$.

Note that f determines an $\mathcal{O}_K$-linear embedding of $M_m(\mathcal{O}_L)$ into $M_n(\mathcal{O}_K)$, and that f' determines an $\mathcal{O}_K$-linear embedding of $\mathcal{O}_E$ into $\mathcal{O}_D$. Suppose that $(\alpha, \alpha') \in \mathrm{GL}_m(\mathcal{O}_L) \times \mathcal{O}_E^\times$, which we view as an element in $\mathrm{GL}_n(\mathcal{O}_K) \times \mathcal{O}_D^\times$ via these embeddings. Then $\rho_{f\circ\alpha, f'\circ\alpha'}(x) = \rho_{f,f'}(x)^{(\alpha,\alpha')}$.

Slightly more generally, one could start with a pair (f, f') and an element $(\alpha, \alpha') \in (\mathrm{GL}_n(K) \times D^\times)^{(0)}$, and put $g = f \circ \alpha$, $g' = f' \circ \alpha'$. Then g is an isomorphism $K^n \to L^m$ and $g'\colon G_0 \to H_0$ is a quasi-isogeny. We may put $\rho_{g,g'}(x) = \rho_{f,f'}(x)^{(\alpha,\alpha')}$. Points in the image of this morphism correspond to deformations of G_0 with endomorphisms by an order in L.

Points in $\mathcal{M}_{G_0}(\mathcal{O}_C)$ lying in the image of some $\rho_{g,g'}$ will be called *CM points*, or points with CM by L. Let $\mathcal{M}_{G_0,\overline{\eta}}^{\mathrm{ad,non\text{-}CM}}$ be the complement in $\mathcal{M}_{G_0,\overline{\eta}}^{\mathrm{ad}}$ of the set of CM points, and similarly for $\mathcal{M}_{G_0,\overline{\eta}}^{\circ,\mathrm{ad,non\text{-}CM}}$.

3 Bushnell-Kutzko types for $\mathrm{GL}_2(K)$ and its inner twist

In this section we review the constructions of Bushnell-Kutzko [BK93] regarding the classification of admissible representations of $\mathrm{GL}_n(K)$ via "strata". A stratum is (more or less) a character of a certain compact subgroup of $\mathrm{GL}_n(K)$; admissible representations may be distinguished according to which strata they contain. We borrow the notational conventions from §4 in the book [BH06], which is well-suited to the study of *supercuspidal* representations of $\mathrm{GL}_2(K)$. There is a parallel study of strata for the quaternionic unit group $D^\times$, which we also review. Finally we present results from [Wei10], wherein for each "simple" stratum S we construct a "linking orders" $\mathcal{L} \subset M_2(K)\times D$ and a finite-dimensional character ρ_S of $\mathcal{L}^\times$. Loosely speaking, when ρ is induced up to $\mathrm{GL}_2(K)\times D^\times$, the result is a direct sum of representations of the form $\Pi\otimes \mathrm{JL}(\check{\Pi})$, where Π ranges over those supercuspidal representations of $\mathrm{GL}_2(K)$ which contain S. As S varies, one sees all of the wild supercuspidal representations Π. This observation is going to be crucial in our construction of a semistable covering of the Lubin-Tate tower.

3.1 Chain orders and strata

A *lattice chain* is an K-stable family of lattices $\Lambda = \{L_i\}$, with each $L_i \subset K\oplus K$ an $\mathcal{O}_K$-lattice and $L_{i+1}\subset L_i$ for all $i\in\mathbb{Z}$. There is a unique integer $e(\Lambda)\in\{1,2\}$ for which $\pi L_i = L_{i+e(\Lambda)}$. Let $\mathfrak{A}_\Lambda$ be the stabilizer in $M_2(K)$ of Λ. Up to conjugacy by $\mathrm{GL}_2(K)$ we have

$$\mathfrak{A}_\Lambda = \begin{cases} M_2(\mathcal{O}_K), & e_\Lambda = 1,\\ \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathfrak{p}_K & \mathcal{O}_K\end{pmatrix}, & e_\Lambda = 2\end{cases}$$

Definition 3.1.1. A *chain order* in $M_2(K)$ is an $\mathcal{O}_K$-order $\mathfrak{A}\subset M_2(K)$ which is equal to $\mathfrak{A}_\Lambda$ for some lattice chain Λ. We say $\mathfrak{A}$ is unramified or ramified as e_Λ is 1 or 2, respectively.

Suppose $\mathfrak{A}$ is a chain order in $M_2(K)$; let $\mathfrak{P}$ be its Jacobson radical. Then $\mathfrak{P} = \pi\mathfrak{A}$ if $\mathfrak{A}$ is unramified and $\mathfrak{P} = \begin{pmatrix} \mathfrak{p}_K & \mathcal{O}_K \\ \mathfrak{p}_K & \mathfrak{p}_K\end{pmatrix}$ in the case that $\mathfrak{A} = \begin{pmatrix} \mathcal{O}_K & \mathcal{O}_K \\ \mathfrak{p}_F & \mathcal{O}_K\end{pmatrix}$. We have a filtration of $\mathfrak{A}^\times$ by subgroups $U_{\mathfrak{A}}^n = 1+\mathfrak{P}^n$, $n\geq 1$.

These constructions have obvious analogues in the quaternion algebra D: If $\mathfrak{A} = \mathcal{O}_D$ is the maximal order in B, then the Jacobson radical $\mathfrak{P}$ is the unique maximal two-sided ideal of $\mathfrak{A}$, generated by a prime element ϖ of D. We let $U_{\mathfrak{A}}^n = 1+\mathfrak{P}^n$.

3.2 Characters and Bushnell-Kutzko types

In the following discussion, $\psi\colon K\to\mathbb{C}^\times$ is a fixed additive character. We assume that ψ is of level one, which means that $\psi(\mathfrak{p}_K)$ is trivial but $\psi(\mathcal{O}_K)$

is not. (The choice of level of ψ is essentially arbitrary, but it has become customary to use characters of level one.) Write $\psi_{M_2(K)}$ for the (additive) character of $M_2(K)$ defined by $\psi_{M_2(K)}(x) = \psi(\operatorname{tr} x)$. Similarly define $\psi_D\colon D \to \mathbb{C}$ by $\psi_D(x) = \psi(\operatorname{tr}_{D/K}(x))$, where $\operatorname{tr}_{D/K}$ is the reduced trace.

Now let A be either $M_2(F)$ or D. Let $\mathfrak{A} \subset A$ be an $\mathcal{O}_K$-order which equals a chain order (if $A = M_2(K)$) or the maximal order in D (if $A = D$). Let $n \geq 1$. We have a character ψ_α of $U_{\mathfrak{A}}^m$ defined by

$$\begin{array}{rcl} U_{\mathfrak{A}}^m / U_{\mathfrak{A}}^{m+1} & \to & \mathbb{C}^\times \\ 1 + x & \mapsto & \psi(\alpha x) \end{array}$$

If π is an admissible irreducible representation of $\mathrm{GL}_2(K)$, one may ask for which α is the character ψ contained in $\pi|_{U_{\mathfrak{A}}^m}$. This is the basis for the classification of representations by Bushnell-Kutzko types, c.f. [BK93].

Definition 3.2.1. A *stratum* in A is a triple of the form $S = (\mathfrak{A}, m, \alpha)$, where $m \geq 1$ and $\alpha \in \mathfrak{P}_{\mathfrak{A}}^{-m}$. Two strata $(\mathfrak{A}, m, \alpha)$ and $(\mathfrak{A}, m, \alpha')$ are equivalent if $\alpha \equiv \alpha' \pmod{\mathfrak{P}^{1-m}}$.

Definition 3.2.2. Let $S = (\mathfrak{A}, m, \alpha)$ be a stratum.

1. S is *ramified simple* if $L = K(\alpha)$ is a ramified quadratic extension field of K, m is odd, and $\alpha \in E$ has valuation exactly $-m$.

2. S is *unramified simple* if $L = K(\alpha)$ is an unramified quadratic extension field of K, $\alpha \in E$ has valuation exactly $-m$, and the minimal polynomial of $\pi^m\alpha$ over F is irreducible mod π.

3. S is *simple* if it is ramified simple or unramified simple.

If $S = (\mathfrak{A}, m, \alpha)$ is a stratum in $M_2(K)$ (resp., D) and Π is an admissible representation of $\mathrm{GL}_2(K)$ (resp., smooth representation of $D^\times$), we say that Π *contains the stratum* S if $\pi|_{U_{\mathfrak{A}}^m}$ contains the character ψ_α.

We call Π *minimal* if its conductor cannot be lowered by twisting by one-dimensional characters of $F^\times$.

From [BH06] we have the following classification of supercuspidal representations of $\mathrm{GL}_2(K)$:

Theorem 3.2.3. *A minimal irreducible admissible representation* Π *of* $\mathrm{GL}_2(K)$ *is supercuspidal if and only if one of the following holds:*

1. Π *contains the trivial character of* $U^1_{M_2(\mathcal{O}_F)}$ *(i.e.* Π *has "depth zero").*

2. Π *contains a simple stratum.*

The analogous statement for D is:

Theorem 3.2.4. *A minimal irreducible representation Π of $D^\times$ of dimension greater than 1 satisfies exactly one of the following properties:*

1. Π *contains the trivial character of $U^1_{\mathcal{O}_D}$ (i.e. Π has "depth zero").*

2. Π *contains a simple stratum.*

3.3 Linking orders

Let L/K be a separable extension of degree n. We write $\mathcal{O}_L$ for its ring of integers and $\mathfrak{p}_L = \pi_L\mathcal{O}_L$ for the maximal ideal of $\mathcal{O}_L$. Let $\mathbf{x} \in \mathcal{M}(\mathcal{O}_C)$ be a point with CM by L. Then $\mathbf{x}$ determines embeddings of L into $M_n(K)$ and D, respectively. Let $\Delta\colon L \to M_n(K) \times D$ be the diagonal map. Finally, let $m \geq 0$ be an integer. In this section we define a $\Delta(\mathcal{O}_L)$-order $\mathcal{L} = \mathcal{L}_{\mathbf{x},m}$ inside of $M_n(K) \times D$ which play an important role in our analysis of the Lubin-Tate tower.

The data of $\mathbf{x}$ is tantamount to the data of a deformation G of G_0 to $\mathcal{O}_C$, and a basis for the free $\mathcal{O}_K$-module $TG = \varprojlim G[\pi^n](\mathcal{O}_C)$. We may then identify $M_n(K)$ with the algebra of K-linear endomorphisms of $VG = TG \otimes_{\mathcal{O}_K} K$. Let $\mathfrak{A}_{\mathbf{x}} = \mathfrak{A} \subset M_n(K)$ be the chain order corresponding to the lattice chain $\left\{\mathfrak{p}_L^i TG\right\}$. That is, $\mathfrak{A}$ is the subalgebra of elements which send $\mathfrak{p}_L^m TG$ into $\mathfrak{p}_L^m TG$ for each $m \in \mathbb{Z}$. Certainly $\mathfrak{A}$ contains $\mathcal{O}_L$. Let $\mathfrak{P} \subset \mathfrak{A}$ be the subset of elements which send $\mathfrak{p}_L^m TG$ into $\mathfrak{p}_L^{m+1} TG$ for all $m \geq 1$; then $\mathfrak{P}$ is the double-sided ideal generated by π_L.

We have a K-linear pairing $M_n(K) \times M_n(K) \to K$ given by $(a, b) \mapsto \operatorname{tr}(ab)$. With respect to this pairing we may write $M_n(K) = L \oplus C_1$, where C_1 is a L-vector space (left or right!) of dimension $n-1$. Let $\operatorname{pr}_1\colon M_n(K) \to L$ be the projection. Note that pr_1 is (L, L)-linear, meaning that $\operatorname{pr}_1(\alpha x \beta) = \alpha \operatorname{pr}_1(x)\beta$ for $\alpha, \beta \in L$ and $x \in M_n(K)$, and of course is the identity on L itself. These properties determine pr_1 uniquely:

Lemma 3.3.1. *Let $\lambda\colon M_n(K) \to L$ be an (L, L)-linear map which is the identity on L. Then $\lambda = \operatorname{pr}_1$.*

Proof. Consider the map $M_n(K) \to \operatorname{Hom}_L(M_n(K), L)$ (meaning maps which are left L-linear) given by $x \mapsto (y \mapsto \operatorname{pr}_1(yx))$. If x lies in the kernel of this map, then so does the left ideal $I = M_n(K)xL$. This means that $I \subset C_1$. Since I is a left ideal in a matrix algebra, it must equal the annihilator of some K-subspace $V \subset K^n$. Conversely V is exactly the subspace of K^n annihilated by I. Since I is stable under right multiplication by L, the subspace

V must be L-stable; *i.e.* it is an L-vector space. If $V = 0$, then $I = M_n(K)$, which contradicts $I \subset C_1$. Thus $V \neq 0$, which implies it must be all of K^n, and $I = 0$. Thus $x = 0$ and the map $M_n(K) \to \mathrm{Hom}_L(M_n(K), L)$ is injective, thus (comparing dimensions) an isomorphism.

Now suppose $\lambda\colon M_n(K) \to L$ is (L, L)-linear. By the above observation there exists $x \in M_n(K)$ such that $\lambda(y) = \mathrm{pr}_1(yx)$ for all $y \in M_n(K)$. Since λ is right L-linear, we have $\mathrm{pr}_1(y\alpha x) = \mathrm{pr}_1(yx\alpha)$ for all $y \in M_n(K)$ and $\alpha \in L$. Thus for all $\alpha \in L$ we have $M_n(K)(\alpha x - x\alpha)L \subset C_1$. By the same argument as the previous paragraph, $\alpha x - x\alpha = 0$. Thus x lies in the centralizer of L in $M_n(K)$, which is L itself (because L is a maximal subfield in $M_n(K)$). We have $\lambda(y) = \mathrm{pr}_1(yx) = \mathrm{pr}_1(y)x$. Since λ is the identity on L we have $x = 1$. ☐

Let r be the largest integer such that $\mathrm{pr}_1(\mathfrak{P}^{2r}) \subset \mathfrak{p}_L^m$. Now let

$$P_{1,m} = \left\{ a \in \mathfrak{P}^r \;\middle|\; \mathrm{pr}_1(a) \in \mathfrak{p}_L^m \right\},$$

an $\mathcal{O}_L$-submodule of $\mathcal{A}$. By our definition of the integer r, we have $P_{m,M_n(K)}^2 \subset P_{1,m}$. Therefore $J = 1 + P_{m,M_n(K)}$ is a finite-index subgroup of $\mathcal{A}^\times$ which contains $1 + \mathfrak{p}_L^m$.

We define similar structures for the division algebra D. The CM point $\mathbf{x}$ determines an embedding of $\mathcal{O}_L$ into $\mathcal{O}_D$. We have an K-linear pairing $D \times D \to K$ given by the reduced trace. With respect to this pairing we may write $D = L \oplus C_2$, where C_2 is an L-vector space of dimension $n - 1$. Let $\mathrm{pr}_2\colon D \to L$ be the projection.

Lemma 3.3.2. *Let $\lambda\colon M_n(K) \to L$ be an (L, L)-linear map which is the identity on L. Then $\lambda = \mathrm{pr}_2$.*

Proof. This is proved the same way as Lemma 3.3.1, and in fact is easier because D has no nonzero left ideals other than itself. It is still the case that L is a maximal subfield of D. ☐

Let $\mathfrak{P}_D \subset \mathcal{O}_D$ be the maximal double-sided ideal. Let r' be the largest integer such that $\mathrm{pr}_D \mathfrak{P}_D^{2r'} \subset \mathfrak{p}_L^m$, and let

$$P_{2,m} = \left\{ b \in \mathfrak{P}_D^{r'} \;\middle|\; \mathrm{pr}_2(b) \in \mathfrak{p}_L^m \right\}.$$

Then $P_{2,m}^2 \subset P_{2,m}$, and $J_D = 1 + P_{2,m}$ is an open subgroup of $\mathcal{O}_D^\times$ which contains $1 + \mathfrak{p}_L^m$.

Now we consider structures within the product $M_n(K)\times D$. Let $\mathrm{pr}\colon M_n(K)\times D\to L$ be $\mathrm{pr}(a,b)=\mathrm{pr}_1(a)-\mathrm{pr}_2(b)$. Let

$$\mathcal{L}=\mathcal{L}_{\mathbf{x},m}=\Delta(\mathcal{O}_L)+(P_{1,m}\times P_{2,m}).$$

Then $\mathcal{L}$ is a $\Delta(\mathcal{O}_L)$-order in $M_n(K)\times D$. We also define a double-sided ideal $\mathcal{P}\subset\mathcal{L}$ by

$$\mathcal{P}=\Delta(\mathfrak{p}_L)+(P_{1,m+1}\times P_{2,m+1})$$

Then $\mathcal{R}=\mathcal{L}/\mathcal{P}$ is a finite-dimensional algebra over the residue field $\mathcal{O}_L/\mathfrak{p}_L$.

3.4 The Jacquet-Langlands correspondence for $\mathrm{GL}_2(K)$

Here we assume $n=2$. Let $\mathbf{x}\in\mathcal{M}(G_0)$ be a point with CM by a separable quadratic extension L/K, and let $m\geq 0$. Let $\mathcal{L}=\mathcal{L}_{\mathbf{x},m}$ be the linking order, defined in §3.3. Then $\Delta(L)^\times$ normalizes $\mathcal{L}^\times$, and we may set $\mathcal{K}=\mathcal{K}_{\mathbf{x},m}=\Delta(L)^\times\mathcal{L}^\times$.

In [Wei10], linking orders are implicated in the Jacquet-Langlands correspondence for $\mathrm{GL}_n(K)$. To wit, for each simple stratum $S=(\mathcal{A},m,\alpha)$, there exists a finite-dimensional representation ρ_S of $\mathcal{K}$ having the property that (modulo some fussing with the center) the induced representation of ρ_S up to $\mathrm{GL}_2(K)\times D^\times$ decomposes into representations of the form $\Pi\otimes\mathrm{JL}(\check{\Pi})$, where Π runs over exactly those representations which contain S. The restriction of ρ_S to $\mathcal{L}^\times$ is inflated from a representation of the finite group $\mathcal{R}^\times$, where $\mathcal{R}=\mathcal{L}/\mathcal{P}$. The construction of ρ_S is broken into cases, depending on whether L/K is ramified. We refer to [Wei10] for the details.

If L/K is unramified and $m\geq 1$, then

$$\mathcal{R}\cong\left\{\begin{pmatrix}a & b & c\\ & a^q & b^q\\ & & a\end{pmatrix}\;\middle|\; a,b,c\in\mathbb{F}_{q^2}\right\}\tag{3.4.1}$$

Let $\mathcal{R}^\times$ act on the nonsingular projective curve Z over $\overline{\mathbb{F}}_q$ with affine model $Y^{q^2}-Y=X^{q^2+q}-X^{q+1}$. Then the action of $\mathcal{R}^\times$ on the first cohomology of Z splits into irreducible representations of degree q; these are our representations ρ_S. They can be extended to $\mathcal{K}$ by having $\Delta(\pi)$ act trivially.

If L/K is ramified and $m\geq 1$ is odd, then $\mathcal{R}\cong\mathbb{F}_q[\varepsilon]/\varepsilon^2$. The representations ρ_S are the nontrivial characters of $\mathcal{R}^\times/\mathbb{F}_q^\times$, extended to $\mathcal{K}$ by having $\Delta(\pi_L)$ act as the scalar -1.

The following paraphrases Thm. 6.0.1 of [Wei10]:

Theorem 3.4.1. *Let Π (resp., Π') be a smooth admissible representation of $\mathrm{GL}_2(K)$ (resp., a smooth representation of $B^\times$). Assume that the central characters of Π and Π' are contragredient to one another. The following are equivalent:*

1. $\dim \mathrm{Hom}_{\mathcal{K}}(\rho_S, \Pi \otimes \Pi') \neq 0$.

2. Π *contains* S, *and* $\Pi' = \mathrm{JL}(\check{\Pi})$.

There is an auxiliary theorem for the representations of depth zero. For L/K unramified and $m = 0$, we have $\mathcal{R} = M_2(\mathbb{F}_q) \times \mathbb{F}_{q^2}$. For a character χ of $\mathbb{F}_{q^2}$ which is unequal to its $\mathbb{F}_q$-conjugate. Let π_χ be the corresponding cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$, and put ρ_χ be the inflation of $\pi_\chi \otimes \chi^{-1}$ to $\mathcal{L}^\times$, which we extend to $\mathcal{K}$ by having $\Delta(\pi)$ act trivially.

Theorem 3.4.2. *Under the same hypotheses as in Thm. 3.4.1, the following are equivalent:*

1. $\dim \mathrm{Hom}_{\mathcal{K}}(\rho_\chi, \Pi \otimes \Pi') \neq 0$.

2. Π *is a supercuspidal representation which contains* π_χ, *and* $\Pi' = \mathrm{JL}(\check{\Pi})$.

4 Special affinoids in the Lubin-Tate perfectoid space

4.1 Special affinoids: overview

Let Π be a discrete series representation of $\mathrm{GL}_n(K)$. Then Π appears in the cohomology of the Lubin-Tate tower. In this section we address the question of whether the presence of Π can be traced to an open affinoid $\mathscr{Z}$ of the Lubin-Tate tower. We answer this question affirmatively in the special case that σ is a two-dimensional irreducible Weil parameter which is induced from a one-dimensional character of a tamely ramified degree quadratic extension L/K.

The construction of the affinoid $\mathscr{Z}$ fits the following pattern. It will depend on a tamely ramified extension L/K and an integer $m \geq 1$. Let $\mathbf{x}$ be a point of $\mathcal{M}(\mathcal{O}_C)$ with CM by L. The image of $\mathbf{x}$ under the morphism $\mathcal{M} \to \tilde{G}^n$ is a tuple $(\mathbf{x}_1, \dots, \mathbf{x}_n)$, with $\mathbf{x}_i \in \tilde{G}(\mathcal{O}_{\mathbb{C}})$. We will first construct an affinoid $\mathscr{Y} \subset \tilde{G}^{n,\mathrm{ad}}_{\overline{\eta}}$, and then define $\mathscr{Z} = \mathscr{Z}_{\mathbf{x},m} = \mathscr{Y} \cap \mathcal{M}^{\circ,\mathrm{ad}}_{\overline{\eta}}$.

The affinoid $\mathscr{Y}$ is essentially a "parallelepiped" centered around the CM point $\mathbf{x}$. Let B be the coordinate ring of $\tilde{G}^n$, so that

$$B \cong \mathcal{O}_{K_0}[\![X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]\!]$$

$\tilde{G}^n = \operatorname{Spf} B$. We have n distinguished elements $\mathbf{X}_1, \ldots, \mathbf{X}_n \in \tilde{G}(B)$, representing the n projections in $\operatorname{Hom}(\tilde{G}^n, \tilde{G})$. We define elements $\mathbf{Y}_1, \ldots, \mathbf{Y}_n \in \tilde{G}(B \hat{\otimes} \mathcal{O}_C)$ through an affine change of variables

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{x}_1 + a_{11}\mathbf{Y}_1 + \cdots + a_{1n}\mathbf{Y}_n \\
\mathbf{X}_2 &= \mathbf{x}_2 + a_{21}\mathbf{Y}_1 + \cdots + a_{2n}\mathbf{Y}_n \\
&\vdots \\
\mathbf{X}_n &= \mathbf{x}_n + a_{n1}\mathbf{Y}_1 + \cdots + a_{nn}\mathbf{Y}_n.
\end{aligned}$$

Here (a_{ij}) is a certain invertible matrix with entries in $\mathcal{O}_D$. This matrix depends on $\mathbf{x}$ and m.

By Rmk. 2.5.4, each $\mathbf{Y}_i$ corresponds to a topologically nilpotent element $Y_i \in B \hat{\otimes} \mathcal{O}_C$ admitting arbitrary qth power roots. The Y_i generate an ideal of definition for $B \hat{\otimes} \mathcal{O}_C$, since the X_i do. The affinoid $\mathscr{Y}$ is then defined by inequalities $|Y_i| \leq |z_i|$ for certain elements $z_1, \ldots, z_n \in \mathcal{O}_C$ which depend on m.

Let $Z_i = Y_i / z_i$. Then $\mathscr{Y} = \operatorname{Spa}(R, R^+)$, where

$$R^+ = \mathcal{O}_{\mathbb{C}} \left\langle Z_1^{1/q^\infty}, \ldots, Z_n^{1/q^\infty} \right\rangle.$$

We then calculate an approximation for $\delta(X_1, \ldots, X_n)$ in terms of the variables $Z_1, \ldots, Z_n$:

$$\delta(X_1, \ldots, X_n) \equiv t + \beta N(Z_1, \ldots, Z_n) \pmod{\beta t^\varepsilon R^+} \tag{4.1.1}$$

for some polynomial $N(Z_1, \ldots, Z_n) \in \overline{\mathbb{F}}_q[Z_1, \ldots, Z_n]$, some $\beta \in \mathcal{O}_C$ with $|\beta| < 1$ and some $\varepsilon > 0$.

Recall that $\mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad}}$ is the adic generic fiber of $\operatorname{Spf} A^\circ$ where A° is the ring described by Eq. (2.10.1). Let $\mathscr{X} = \mathscr{Y} \cap \mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad}}$. Then $\mathscr{X} = \operatorname{Spa}(S, S^+)$, where S is the quotient of R by the ideal generated by $\delta(X_1, \ldots, X_n)^{1/q^r} - t^{1/q^r}$ $(r \geq 0)$ and S^+ is the integral closure of the image of R^+ in S. By the congruence above we have

$$S^+ \otimes_{\mathcal{O}_{\mathbb{C}}} \overline{\mathbb{F}}_q = \overline{\mathbb{F}}_q[Z_1^{1/q^\infty}, \ldots, Z_n^{1/q^\infty}] / (N(Z_1, \ldots, Z_n)^{1/q^r})_{r \geq 0}.$$

We will refer to the scheme $\operatorname{Spec} S^+ \otimes_{\mathcal{O}_{\mathbb{C}}} \overline{\mathbb{F}}_q$ as the *reduction* of $\mathscr{X}$, and denote it $\overline{\mathscr{X}}$.

In the following sections we will give a recipe for the affinoid $\mathscr{Y}$ for all pairs $(\mathbf{x}, m)$, where $\mathbf{x}$ is a CM point and $m \geq 0$. It is a subtle matter to determine the scheme $\overline{\mathscr{X}}$ as a function of $(\mathbf{x}, m)$, but we give a description in a number of cases, including all possibilities when $n = 2$. At least in that case, the $\overline{\mathscr{X}}$ is the perfection of a nonsingular variety over $\overline{\mathbb{F}}_q$, and its H^1_c realizes all supercuspidal representations of $\mathrm{GL}_2(K)$ containing a simple stratum of the form $(\mathfrak{A}, m, \alpha)$ (or, if $m = 0$, all supercuspidal representations of depth zero).

4.2 Definition of the affinoid $\mathscr{X}_m$

There exists $x \in \tilde{G}(\mathcal{O}_C)$ and a basis $\alpha_1, \ldots, \alpha_n$ for $\mathcal{O}_L/\mathcal{O}_K$ for which $x_i = \alpha_i x$.

For a row vector $\beta = (\beta_1, \ldots, \beta_n) \in D^n$, there is the obvious operator $\beta\colon \tilde{G}^n \to \tilde{G}$ which sends $\mathbf{T} = (\mathbf{T}_1, \ldots, \mathbf{T}_n)$ to $\sum_i \beta_i \mathbf{T}_i$; we write this as $\beta \cdot \mathbf{T}$. If $a \in M_n(K)$ we can write βa for the matrix product $(\beta_1, \ldots, \beta_n)a$. Similarly if $b \in B$ we can write βb for $(\beta_1 b, \ldots, \beta_n b)$. Finally, if $\zeta = (a, b) \in M_n(K) \times D$, write $\beta\zeta$ for $\beta a - \beta b$.

Lemma 4.2.1. *The set of vectors $\beta = (\beta_1, \ldots, \beta_n) \in L^n$ satisfying $\beta\Delta(\alpha) = 0$ for all $\alpha \in L$ is an L-vector space of dimension one. For any such β we have*

$$(\beta\zeta) \cdot \mathbf{x} = \mathrm{pr}(\zeta)\beta \cdot \mathbf{x}$$

for all $\zeta \in M_n(K) \times D$.

Proof. Let α generate L over K, and let A be the image of α in $M_n(K)$. The condition $\beta\Delta(\alpha) = 0$ means that β is a left eigenvector for A with eigenvalue α, and this is unique up to scalar.

There exists a basis $\alpha_1, \ldots, \alpha_n$ of L/K and a point $\mathbf{y} \in \tilde{G}(\mathcal{O}_C)$ for which $\mathbf{x}_i = \alpha_i \mathbf{y}$, $i = 1, \ldots, n$. Then $\beta \cdot \mathbf{x} = \sum_i \beta_i \alpha_i \mathbf{y} \neq 0$. For any $a \in M_n(K)$ we have $(\beta a) \cdot x = \lambda(a)\beta \cdot x$ for some $\lambda(a) \in L$. Observe that $\lambda\colon M_n(K) \to L$ is (L, L)-linear, and that λ is the identity on L, so that by Lemma 3.3.1 we have $\lambda = \mathrm{pr}_1$. The argument for D is similar. □

Choose a nonzero vector β in the space described by Lemma 4.2.1. Let us adopt the following convention regarding continuous valuations $|\ |$ in $\tilde{G}^n_{\overline{\eta}}$. For sections $\mathbf{v}, \mathbf{w}$ in $\tilde{G}^n(B \hat{\otimes} \mathcal{O}_C)$, the inequality $|\mathbf{v}| \leq |\mathbf{w}|$ shall mean that $|v| \leq |w|$, where v and w are the corresponding topologically nilpotent elements of S.

Define an affinoid $\mathscr{Y}$ by the inequalities

$$|\beta \cdot (\mathbf{X} - \mathbf{x})| \leq |\pi_L^m \beta \cdot \mathbf{x}|, \tag{4.2.1}$$

and

$$|(\beta\zeta) \cdot \mathbf{X}| \leq |\pi_L^m \beta \cdot \mathbf{x}| \tag{4.2.2}$$

for all ζ lying in $\mathcal{P}$, and finally

$$|(\beta\zeta)\cdot\mathbf{X}| \le |\pi_L^m \beta\cdot\mathbf{x}|^{\frac{q^f+1}{2}} \tag{4.2.3}$$

for those ζ lying in $\mathcal{P}$ which satisfy $\mathrm{pr}(\zeta\mathcal{L}) \subset \mathfrak{p}_L^{m+1}$. Here f is the residue degree of L/K.

Lemma 4.2.2. *The affinoid $\mathscr{Y}$ contains the CM point $\mathbf{x}$, and is preserved by the subgroup $\Delta(L^\times)\mathcal{L}^\times$ of $\mathrm{GL}_n(K)\times D^\times$.*

Proof. For the first claim, we must check Eqs. (4.2.1)-(4.2.3) for $\mathbf{X}=\mathbf{x}$. Eq. (4.2.1) is trivial. For Eq. (4.2.2), let $\zeta\in\mathcal{P}$; then $(\beta\zeta)\cdot\mathbf{x} = \mathrm{pr}(\zeta)\cdot\beta\mathbf{x}$ by Lemma 4.2.1, and this is bounded by $|\pi_L^m\mathbf{x}|$ because $\mathrm{pr}(\mathcal{P})\subset\mathfrak{p}_L^m$. The argument for Eq. (4.2.3) is similar.

For the second claim, we observe that $\Delta(L^\times)\mathcal{L}^\times$ is generated by $\Delta(L^\times)$, $1+P_{1,m}$ and $1+P_{2,m}$, so it suffices to check the claim for these groups separately. Let $|\,|$ be a valuation belonging to $\mathscr{Y}$, so that Eqs. (4.2.1), (4.2.2) and (4.2.3) are satisfied. Let $a\in P_{1,m}$. We must check that the same equations hold when $\mathbf{X}$ is replaced by $\mathbf{X}+a^T\mathbf{X}$. We have

$$\beta\cdot(\mathbf{X}+a^T\mathbf{X}-\mathbf{x}) = \beta\cdot(\mathbf{X}-\mathbf{x}) + \beta a\cdot\mathbf{X},$$

and these two terms are bounded by $|\pi_L^m\beta\cdot\mathbf{x}|$ by Eqs. (4.2.1) and (4.2.2), respectively. Thus Eq. (4.2.1) is preserved. By Lemma 4.2.1, $\beta a\cdot\mathbf{x} = \mathrm{pr}_1(a)\beta\cdot x$. Since $\mathrm{pr}_1(a)\in\mathfrak{p}_L^m$, we have $|\beta a\cdot\mathbf{x}|\le|\pi_L\beta\cdot\mathbf{x}|$. Thus Eq. (4.2.1) is satisfied for $\mathbf{X}+a^T\mathbf{X}$. The argument for Eqs. (4.2.2) and (4.2.3), and for the group $1+P_{2,m}$, is similar.

Now suppose $\alpha\in L^\times$. Let $\Delta(\alpha)=(a,b)$. The vector β was chosen so that $\beta a=\beta b$. The action of $\Delta(\alpha)$ on A effects the substitution $\mathbf{X}\mapsto b^{-1}a^T\mathbf{X}$. We have $\beta\cdot b^{-1}a^T\mathbf{X} = b^{-1}\beta a\cdot\mathbf{X} = \beta\cdot\mathbf{X}$. Thus this substitution preserves Eqs. (4.2.1), (4.2.2) and (4.2.3).

□

4.3 Case: L/K unramified, $m\ge 1$ odd

Here L/K is an unramified extension of degree n.

Let L/K be the unramified extension of degree n. Let $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)\in\mathcal{M}^\circ(\mathcal{O}_C)$ be a point with CM by L. Up to replacing $\mathbf{x}$ by a translate by $\mathrm{GL}_n(K)$, we may assume that $\mathbf{x}$ corresponds to a deformation of G_0 to $\mathcal{O}_\mathbb{C}$ with endomorphisms by $\mathcal{O}_L$ (rather than a smaller order). Without loss of generality, we may take G to be the unique deformation for which

$[\pi]_G(T) = \pi T + T^{q^n}$. Then if $x_1, \ldots, x_n \in \mathcal{O}_{\mathbb{C}}$ are the associated topologically nilpotent elements (Rmk. 2.5.4), we have $|x_i|^{q^n-1} = |\pi|$ for $i = 1, \ldots, n$.

We have $\mathbf{x}_i = \alpha_i \mathbf{y}$ for a basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_L/\mathcal{O}_K$. Let $\beta_1, \ldots, \beta_n \in \mathcal{O}_K$ be elements such that

$$\sum_i \beta_i \alpha_i^{\sigma^j} = \delta_{j0}$$

for $j = 0, 1, \ldots, n-1$. Then β is an element of the sort described in Lemma 4.2.1, and $\beta \cdot x = \mathbf{y}$. Let σ be the Frobenius automorphism of L/K. Then D is generated over K by L and an element ϖ which satisfies $\varpi^n = \pi$ and $\varpi\alpha = \alpha^\sigma \varpi$. We have $\mathfrak{A} = M_n(\mathcal{O}_K)$ and $\mathfrak{P} = \pi M_n(\mathcal{O}_K)$. Now let $m \geq 1$ be odd. We have $r = (m+1)/2$ and $r' = n(m-1)/2 + 1$. Thus P_1 is spanned over $\mathcal{O}_L$ by $\pi^{(m+1)/2}\sigma, \ldots, \pi^{(m+1)/2}\sigma^{n-1}$, while P_2 is spanned over $\mathcal{O}_L$ by $\pi^{(m-1)/2}\varpi, \ldots, \pi^{(m-1)/2}\varpi^{n-1}$. The linking order is therefore

$$\mathcal{L} = \Delta(\mathcal{O}_L) + \mathfrak{p}_L^m \times \mathfrak{p}_L^m + \left(\bigoplus_{i=1}^{n-1} \mathfrak{p}_L^{\frac{m+1}{2}} \sigma^i\right) \times \left(\bigoplus_{i=1}^{n-1} \mathfrak{p}_L^{\frac{m+1}{2}} \varpi^i\right).$$

Define elements $\mathbf{Y}_1, \ldots, \mathbf{Y}_n \in \tilde{G}(A_{\mathbb{C}})$ by

$$\mathbf{Y}_n = \pi^{-m}\beta \cdot (\mathbf{X} - \mathbf{x})$$

and

$$\mathbf{Y}_i = \pi^{(m-1)/2}\beta\varpi^{n-i} \cdot \mathbf{X}$$

for $i = 1, \ldots, n-1$. Then the inequality $|Y_i| \leq |x|$ holds on $\mathscr{Y}$ for $i = 1, \ldots, n$. We have

$$\mathbf{X}_i = \mathbf{x}_i + \pi^{\frac{m-1}{2}}\varpi\alpha_i\mathbf{Y}_1 + \pi^{\frac{m-1}{2}}\varpi^2\alpha_i\mathbf{Y}_2 + \cdots + \pi^{\frac{m-1}{2}}\varpi^{n-1}\alpha_i\mathbf{Y}_{n-1} + \pi^m\alpha_i Y_n$$

From this presentation one can check that $\mathscr{Y}$ equals the affinoid defined by $|Y_i| \leq |y|$ (the condition in Eq. (4.2.3) being superfluous in this case).

We now compute an approximation for $\delta(X_1, \ldots, X_n)$ as in Eq. (4.1.1), *under the assumption that* $n = 2$. We begin with the system

$$\begin{aligned} \mathbf{X}_1 &= \alpha_1\mathbf{y} + \pi^{\frac{m-1}{2}}\varpi\alpha_1\mathbf{Y}_1 + \pi^m\alpha_1\mathbf{Y}_2 \\ \mathbf{X}_2 &= \alpha_2\mathbf{y} + \pi^{\frac{m-1}{2}}\varpi\alpha_2\mathbf{Y}_1 + \pi^m\alpha_2\mathbf{Y}_2, \end{aligned}$$

and consider the wedge product $\mathbf{X}_1 \wedge \mathbf{X}_2 \in \widetilde{\wedge G}(R^+)$. We have $\alpha_1\mathbf{y} \wedge \alpha_2\mathbf{y} = \mathbf{t}$, which by Prop. 2.7.2 implies that $\mathrm{Moore}(\alpha_1, \alpha_2)y^{q+1} = t$ plus smaller terms. Further, we have

$$\alpha_1\mathbf{y} \wedge \varpi\alpha_2\mathbf{Y}_1 = \alpha_1\mathbf{y} \wedge \alpha_2^\sigma\varpi\mathbf{Y}_1 = \alpha_1\alpha_2\mathbf{y} \wedge \varpi Y_1,$$

and similarly $\varpi\alpha_1 Y_1 \wedge \alpha_2 \mathbf{y} = \varpi \mathbf{Y}_1 \wedge \alpha_1\alpha_2\mathbf{y}$, so that $\alpha_1\mathbf{y}\wedge\varpi\alpha_2\mathbf{Y}_1 + \varpi\alpha_1 Y_1 \wedge \alpha_2\mathbf{y} = 0$. Turning to $\pi^{\frac{m-1}{2}}\varpi\alpha_1\mathbf{Y}_1 \wedge \pi^{\frac{m-1}{2}}\varpi\alpha_2\mathbf{Y}_1$, we find that its corresponding nilpotent element of R^+ equals $-\operatorname{Moore}(\alpha_1,\alpha_2)Y_1^{q^m(q+1)}$ plus smaller terms, by Prop. 2.7.2. Finally, $(\pi^m\alpha_1\mathbf{Y}_2\wedge\alpha_2\mathbf{y}) + (\alpha_1\mathbf{y}\wedge\pi^m\alpha_2\mathbf{Y}_2)$ corresponds to an element which is $\operatorname{Moore}(\alpha_1,\alpha_2)(Y_2^{q^{m+1}} - Y_2^{q^m})$ plus smaller terms. The other terms in $\mathbf{X}_1\wedge\mathbf{X}_2$ are smaller than the ones so far enumerated. Putting $Y_i = Z_i/y$, we get

$$\delta(X_1, X_2) = t + (Z_2^q - Z_2 - Z_1^{q+1})^{q^m} t^{q^m} + (\text{smaller}).$$

Therefore the reduction $\overline{\mathscr{X}}$ is the perfection of the affine curve with equation $Z_2^q + Z_2 = Z_1^{q+1}$.

4.4 Case: L/K unramified, $m \geq 2$ even

If m is even, then $r = m/2$ and $r' = nm/2$. The linking order is

$$\mathcal{L} = \Delta(\mathcal{O}_L) + \mathfrak{p}_L^m \times \mathfrak{p}_L^m + \pi^{m/2}C_1 \times \pi^{m/2}C_2$$

Then we define elements $\mathbf{Y}_1, \dots, \mathbf{Y}_n \in \tilde{G}(A_{\mathbb{C}})$ by

$$\mathbf{Y}_n = \pi^{-m}\beta\cdot(\mathbf{X} - \mathbf{x})$$

and

$$\mathbf{Y}_r = \pi^{-m/2}\beta\sigma^{-r}\cdot\mathbf{X}$$

for $r = 1, \dots, n-1$. Then

$$\mathbf{X}_i = \mathbf{x}_i + \pi^{\frac{m}{2}}\alpha_i^{\sigma}\mathbf{Y}_1 + \pi^{\frac{m}{2}}\alpha_i^{\sigma^2}\mathbf{Y}_2 + \dots + \pi^{\frac{m}{2}}\alpha_i^{\sigma^{n-1}}Y_{n-1} + \pi^m\mathbf{Y}_n.$$

Once again $\mathscr{Y}$ equals the affinoid defined by $|Y_i| \leq |x|$.

In the $n = 2$ case, a similar calculation as in the previous section shows that the reduction $\overline{\mathscr{X}}$ is the perfection of the curve $Z_2^q + Z_2 = Z_1^{q+1}$.

For the case of n general, we refer to [Wei12] for the calculation of $\overline{\mathscr{X}}$.

4.5 Case: L/K unramified, $m = 0$

In this case the affinoid $\mathscr{X}$ we construct is related to the semistable model of the Lubin-Tate space of level 1 described by Yoshida in [Yos10].

We have that $\mathcal{L} = M_n(\mathcal{O}_K)\times\mathcal{O}_D$, and $\mathscr{Y}$ is simply the affinoid described by the conditions $|X_i| \leq |y| = |\pi|^{1/(q^n-1)}$, $i = 1, \dots, n$. Let $z \in \mathcal{O}_C$ satisfy $z^{1+q+\dots+q^{n-1}} = t$, so that $|z| = |y|$. Let us write $Z_i = X_i/z$. Then $\mathscr{Y} =$

$\mathrm{Spa}(R, R^+)$, where $R^+ = \mathcal{O}_{\mathbb{C}}\left\langle Z_1^{1/q^\infty}, \dots, Z_n^{1/q^\infty}\right\rangle$. By Prop. 2.7.2 we have the following congruence in R^+:

$$\delta(X_1, \dots, X_n) \equiv t\,\mathrm{Moore}(Y_1, \dots, Y_n) \pmod{\beta^{q^n} R^+}.$$

Let $\mathscr{X}$ be the intersection of $\mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad}}$ with $\mathscr{Y}$.

Proposition 4.5.1. *The reduction $\overline{\mathscr{X}}$ is isomorphic to the perfection of the nonsingular variety over $\overline{\mathbb{F}}_q$ with equation* $\mathrm{Moore}(Y_1, \dots, Y_n) = 1$.

We remark that the variety $\mathrm{Moore}(Y_1, \dots, Y_n) = 1$ is a Deligne-Lusztig variety for the group GL_n over $\mathbb{F}_q$.

4.6 Case: L/K tamely ramified quadratic, m odd

When L/K has ramification, the calculation of the reduction of $\mathscr{X}$ becomes somewhat delicate. Let us work out the case that $n = 2$, L/K is a tamely ramified extension, and m is odd. Since L/K is tame, q must be odd, and we have $L = K(\sqrt{\pi})$, up to changing the uniformizer π of K. Write $\pi_L = \sqrt{\pi}$.

Let G be a deformation of G_0 to $\mathcal{O}_L$ admitting endomorphisms by $\mathcal{O}_L$. We may assume that $[\sqrt{\pi}]_G(T) = \sqrt{\pi}T + T^q$. Let $\mathbf{y} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots)$ be an element of $\tilde{G}(\mathcal{O}_{\mathbb{C}})$ with $\mathbf{y}^{(1)}$ a Drinfeld level π_L structure on $G[\pi_L]$. Then $|y| = |\pi|^{1/2(q-1)}$. Then $\mathbf{x} = (\mathbf{y}, \pi_L \mathbf{y}) \in \mathcal{M}(\mathcal{O}_C)$ has CM by L. We will assume that $\mathbf{x} \in \mathcal{M}^\circ(\mathcal{O}_C)$, so that $\mathbf{y} \wedge \pi_L \mathbf{y} = \mathbf{t}$.

We have $C_1 = L\sigma$, where $\sigma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ is the nontrivial element in $\mathrm{Gal}(L/K) \subset \mathrm{Aut}_{\mathcal{O}_K} \mathcal{O}_L \cong \mathrm{GL}_2(\mathcal{O}_L)$. We also have $C_2 = L\varepsilon$, where $\varepsilon^2 \in \mathcal{O}_K$ is a nonresidue modulo π. This can be chosen to satisfy $\varepsilon\sqrt{\pi} = -\sqrt{\pi}\varepsilon$. We have

$$\begin{aligned} \mathcal{L} &= \Delta(\mathcal{O}_L) + \mathfrak{p}_L^m \times \mathfrak{p}_L^m + \mathfrak{p}_L^{\frac{m+1}{2}} \sigma \times \mathfrak{p}_L^{\frac{m+1}{2}} \varepsilon \\ \mathcal{P} &= \Delta(\mathfrak{p}_L) + \mathfrak{p}_L^{m+1} \times \mathfrak{p}_L^{m+1} + \mathfrak{p}_L^{\frac{m+1}{2}} \sigma \times \mathfrak{p}_L^{\frac{m+1}{2}} \varepsilon. \end{aligned}$$

Thus $\mathcal{R} = \mathcal{L}/\mathcal{P}$ is isomorphic to $\mathbb{F}_q[e]/e^2$, where e corresponds to the image of $(\pi_L^m, 0)$.

Set $\beta = \frac{1}{2}(1, \sqrt{\pi}^{-1}) \in L^2$. Then β is as in Lemma 4.2.1. Define elements $\mathbf{Y}_1, \mathbf{Y}_2 \in \tilde{G}(A_{\mathbb{C}})$ by

$$\begin{aligned} \mathbf{Y}_2 &= \pi_L^{-m} \beta \cdot (\mathbf{X} - \mathbf{x}) \\ \mathbf{Y}_1 &= \pi_L^{-m} (\beta\sigma) \cdot \mathbf{X}. \end{aligned}$$

Then

$$\begin{aligned}\mathbf{X}_1 &= \mathbf{y} + \pi_L^{\frac{m-1}{2}}\mathbf{Y}_1 + \pi_L^m\mathbf{Y}_2\\ \mathbf{X}_2 &= \pi_L\mathbf{y} - \pi_L^{\frac{m+1}{2}}\mathbf{Y}_1 + \pi_L^{m+1}\mathbf{Y}_2.\end{aligned}$$

We have $\mathbf{y}\wedge\pi_L\mathbf{y} = \mathbf{t}$, and $\pi_L^{(m-1)/2}\mathbf{Y}_1\wedge\pi_L\mathbf{y} = -\pi_L^{(m+1)/2}\mathbf{Y}_1\wedge\mathbf{y}$, and similarly $\pi_L^m\mathbf{Y}_2\wedge\pi_L\mathbf{y} = \mathbf{y}\wedge\pi_L^{m+1}\mathbf{Y}_2 = \pm\pi_L^{(m+1)/2}\mathbf{Y}_2\wedge\pi_L^{(m+1)/2}\mathbf{y}$. Thus

$$\mathbf{X}_1\wedge\mathbf{X}_2 = \mathbf{t} - \pi_L^{\frac{m-1}{2}}\mathbf{Y}_1\wedge\pi_L^{\frac{m+1}{2}}\mathbf{Y}_1 \pm 2\pi_L^{\frac{m-1}{2}}\mathbf{Y}_2\wedge\pi_L^{\frac{m-1}{2}}\mathbf{y}$$

plus smaller terms. The affinoid $\mathscr{Y}$ is then defined by the conditions $|\mathbf{Y}_1| \leq |\mathbf{y}|^{(q+1)/2}$ and $|\mathbf{Y}_2| \leq |\mathbf{y}|$. Write $Z_1 = Y_1/2y^{(q+1)/2}$ and $Z_2 = \pm Y_2/y$, so that $R^+ = \mathcal{O}_C\left\langle Z_1^{1/q^\infty}, Z_2^{1/q^\infty}\right\rangle$. Prop 2.7.2 shows that we have the estimate

$$\delta(X_1,X_2) = t + y^{(q+1)q^{\frac{m+1}{2}}}(Z_2^q - Z_2 - Z_1^2)^{q^{\frac{m+1}{2}}} + (\text{smaller})$$

in R^+. The reduction of $\mathscr{X}$ is therefore isomorphic to the perfection of the nonsingular curve $Z_2^q - Z_2 = Z_1^2$ over $\overline{\mathbb{F}}_q$.

5 Semistable coverings for the Lubin-Tate tower of curves

5.1 Generalities on semistable coverings of wide open curves

The following notions are taken from [Col03]. We will assume in the following that the field of scalars is $\mathbb{C}$.

Definition 5.1.1. A *wide open* (curve) is an adic space conformal to $C - D$, where C is the adic space attached to a smooth complete curve and $D \subset C$ is a finite disjoint union of closed disks. Each connected component of C is required to contain at least one disk from D.

If W is a wide open, an *underlying affinoid* $Z \subset W$ is an open affinoid subset for which $W\backslash Z$ is a finite disjoint union of annuli U_i. It is required that no annulus U_i be contained in any open affinoid subset of W.

An *end* of W is an element of the inverse limit of the set of connected components of $W\backslash Z$, where Z ranges over open affinoid subsets of W.

Finally, W is *basic* if it has an underlying affinoid Z whose reduction $\overline{Z}$ is a semistable curve over $\overline{\mathbb{F}}_q$. (Recall that if $Z = \mathrm{Spa}(R, R^+)$, then $\overline{Z} = \mathrm{Spec}\, R^+ \otimes_{\mathcal{O}_\mathbb{C}} \overline{\mathbb{F}}_q$.)

For an affinoid X, there is a reduction map $\mathrm{red}\colon X \to \overline{X}$. The following is a special case of Thm. 2.29 of [CM10].

Theorem 5.1.2. *If X is a smooth one-dimensional affinoid, and x is a closed point of $\overline{X}$, then the residue region* $\mathrm{red}^{-1}(x)$ *is a wide open.*

In particular, $\mathcal{M}_{m,\eta}^{\circ,\mathrm{ad}}$ is a wide open, because it is the residue region $\mathrm{red}^{-1}(x)$ of a supersingular point x of the special fiber of an appropriate Shimura curve or Dinfeld modular curve. Alternatively, the theory of the canonical subgroup shows that the complement in $\mathcal{M}_{m,\eta}^{\circ,\mathrm{ad}}$ of a sufficiently large quasi-compact open is a disjoint union of annuli. Attaching open disks to the outer ends of those annuli, one arrives at a complete curve C equal to the disjoint union of $\mathcal{M}_{m,\eta}^{\circ}$ with finitely many closed discs.

We adapt the definition of semistable covering in [Col03], §2, which only applies to coverings of proper curves. Our intention is to construct semistable coverings of the spaces $\mathcal{M}_{m,\eta}^{\circ,\mathrm{ad}}$. Therefore we define:

Definition 5.1.3. Let W be a wide open curve. A *semistable covering* of W is a covering $\mathcal{D}$ of W by connected wide opens satisfying the following axioms:

1. If U, V are distinct wide opens in $\mathcal{D}$, then $U \cap V$ is a disjoint union of finitely many open annuli.
2. No three wide opens in $\mathcal{D}$ intersect simultaneously.
3. For each $U \in \mathcal{D}$, if
$$Z_U = U \backslash \left(\bigcup_{U \neq V \in \mathcal{D}} V \right),$$
then Z_U is a non-empty affinoid whose reduction is nonsingular.

In particular U is a basic wide open and Z_U is an underlying affinoid of U.

Suppose $\mathcal{D}$ is a semistable covering of a wide open W. For each $U \in \mathcal{D}$, let $A^0(U)$ be the ring of analytic functions on U of norm ≤ 1, and let $X_U = \mathrm{Spf}\, A^0(U)$. Similarly if $U, V \in \mathcal{D}$ are overlapping wide opens, similarly define $X_{U\cap V} = \mathrm{Spf}\, A^0(U \cap V)$. Let $\mathscr{W}$ denote the formal scheme over $\mathcal{O}_F$ obtained by gluing the X_U together along the maps
$$X_{U,V} \to X_U \coprod X_V.$$
Then $\mathscr{W}$ has generic fiber W. The special fiber $\mathscr{W}_s$ of $\mathscr{W}$ is a scheme whose geometrically connected components are exactly the nonsingular projective

curves $\overline{Z}_U^{\mathrm{cl}}$ with affine model $\overline{Z}_U$; the curves $\overline{Z}_U^{\mathrm{cl}}$ and $\overline{Z}_V^{\mathrm{cl}}$ intersect exactly when U and V do.

Example 5.1.4. Note that $\mathcal{D}$ will in general not be finite. Suppose W is the adic open disc over $\mathbb{C}$. We construct a semistable covering $\mathcal{D} = \{U_n\}$ of W indexed by integers $n \geq 0$. First let $U_0 = \left\{|z| < \left|\pi^{1/2}\right|\right\}$, and for $n \geq 1$ let $U_n = \left\{|\pi|^{1/n} < |z| < |\pi|^{1/(n+2)}\right\}$. Then Z_{U_0} is the closed disk $\{|z| \leq |\pi|\}$ and for $n \geq 1$, Z_{U_n} is the "circle" $\{|z| = 1/(n+1)\}$. The resulting formal scheme $\mathscr{W}$ has special fiber which is an infinite union of rational components; the dual graph Γ is a ray.

Let C be the adic space attached to a smooth complete curve, let $D \subset C$ be a disjoint union of closed discs, and let $W = C \backslash D$. A semistable covering of W yields a semistable covering of C (in the sense of [Col03]) by the following procedure. Let $\mathcal{D}$ be a semistable covering of W corresponding to the formal model $\mathscr{W}$. Let Γ be the dual graph attached to the special fiber of $\mathscr{W}$. There are bijections among the following three finite sets:

1. ends of W,
2. ends of Γ, and
3. connected components of D.

Suppose $v_1, v_2, \dots$ is a ray in Γ corresponding to the wide opens $U_1, U_2, \dots \subset W$. Then there exists $N > 0$ such that for all $i \geq N$, U_i is an open annulus. If $D_0 \subset D$ is the connected component corresponding to the ray $v_1, v_2, \dots$, then (possibly after enlarging N) $D_0 \cup \bigcup_{i \geq N} U_i$ is an open disc, which intersects U_{N-1} in an open annulus. Repeating this process for all ends of Γ yields a semistable covering $\mathcal{D}_0$ of C by finitely many wide opens. Let Γ_0 be the dual graph corresponding to $\mathcal{D}_0$.

In [CM10], §2.3, the genus $g(W)$ of a wide open curve W is defined. It is shown (Prop. 2.32) that in the above context that the genus of W equals the genus (in the usual sense) of the smooth complete curve whose rigidification is C. It is also shown (in the remark preceding Prop. 2.32) that if U is a basic wide open whose underlying affinoid Z_U has nonsingular reduction, then $g(U) = g(\overline{Z}_U^{\mathrm{cl}})$, where $\overline{Z}_U$ is the reduction of Z_U and $\overline{Z}_U^{\mathrm{cl}}$ is the unique nonsingular projective curve containing $\overline{Z}_U$. In Prop. 2.34 we find the formula

$$g(C) = \sum_{U \in \mathcal{D}_0} g(U) + \dim H^1(\Gamma_0, \mathbb{Q}). \tag{5.1.1}$$

Proposition 5.1.5. *Let W be a wide open curve, and let $\mathcal{D}$ be a semistable covering of W. Let Γ be the dual graph of the special fiber of the corresponding semistable model $\mathscr{W}$ of W, so that the irreducible components $\overline{Z}_v^{\mathrm{cl}}$ of the special fiber of $\mathscr{W}$ are indexed by the vertices of Γ. Then*

$$\dim H^1_c(W, \mathbb{Q}_\ell) = \sum_{v \in \Gamma} \dim H^1_c(\overline{Z}_v^{\mathrm{cl}}, \mathbb{Q}_\ell) + \dim H^1_c(\Gamma, \mathbb{Q}_\ell)$$

Proof. Part of the long exact sequence in compactly supported cohomology for the pair (C, D) reads

$$0 \to H^0(C, \mathbb{Q}_\ell) \to H^0(D, \mathbb{Q}_\ell) \to H^1_c(W, \mathbb{Q}_\ell) \to H^1(C, \mathbb{Q}_\ell) \to 0. \quad (5.1.2)$$

Note that $H^0_c(D, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell[\mathrm{Ends}(\Gamma)]$ is the space of $\mathbb{Q}_\ell$-valued functions on the set of ends of Γ. On the other hand, Γ_0 is up to homotopy the graph obtained by deleting the ends from Γ, so that

$$\dim H^1_c(\Gamma, \mathbb{Q}_\ell) = \dim H^1(\Gamma_0, \mathbb{Q}_\ell) + \#\,\mathrm{Ends}(\Gamma) - \dim H^0(\Gamma, \mathbb{Q}_\ell). \quad (5.1.3)$$

The result now follows from Eqs. (5.1.1), (5.1.2), and (5.1.3). □

5.2 Special affinoids in the Lubin-Tate tower of curves

§ From now on we assume $n = 2$, and q odd. Let G_0 be a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$ of height 2 and dimension 1, and let $\mathcal{M} = \mathcal{M}_{G_0}$ be the deformation space at infinite level. For every CM point $\mathbf{x}$ and every $m \geq 0$ we have associated an affinoid $\mathscr{Z}_{\mathbf{x},m} \subset \mathcal{M}^{\circ,\mathrm{ad}}_{\overline{\eta}}$.

We consider pairs $(\mathbf{x}, m)$, where $\mathbf{x} \in \mathcal{M}(\mathcal{O}_C)^\circ$ is a CM point and $m \geq 0$. We call $\mathbf{x}$ unramified or ramified as its CM field L is an unramified or ramified extension of K. Recall that $\mathfrak{A}_{\mathbf{x}}$ is the chain order associated to $\mathbf{x}$, so that $\mathfrak{A}_{\mathbf{x}}$ is conjugate either to $M_2(\mathcal{O}_K)$ or to its Iwahori subalgebra, depending on whether $\mathbf{x}$ is unramified or ramified, respectively. To such a pair we have an associated embedding $\Delta\colon L \to M_2(K) \times D$, a linking order $\mathcal{L}$ and an affinoid $\mathscr{Z}_{\mathbf{x},m} \subset \mathcal{M}^\circ_{\overline{\eta}}$. By Lemma 4.2.2, $\mathscr{Z}_{x,m}$ is stabilized by the group

$$\mathcal{K}^1_v = \left(\Delta(L^\times)\mathcal{L}^\times\right)^{\det=1},$$

this being the intersection of $\mathcal{K}_v = \Delta(L^\times)\mathcal{L}^\times$ with $(\mathrm{GL}_2(K) \times D^\times)^{\det=1}$.

Pairs $(\mathbf{x}, m)$ and $(\mathbf{y}, m)$ shall be *equivalent* if one of the following conditions holds:

1. $m = 0$, and $\mathfrak{A}_{\mathbf{x}} = \mathfrak{A}_{\mathbf{y}}$.

2. $m > 0$, and there exists $g \in \mathcal{K}^\times_{\mathbf{x},m}$ for which $\mathbf{x} = \mathbf{y}^g$.

Then the affinoid $\mathscr{Z}_{\mathbf{x},m}$ only depends on the equivalence class of $(\mathbf{x}, m)$.

Define a graph $\mathcal{T}$ as follows: the vertices are equivalence classes of pairs $(\mathbf{x}, m)$, and vertices $(\mathbf{x}, n)$ and $(\mathbf{y}, m)$ will be adjacent under the following circumstances:

1. $m = n = 0$, $\mathbf{x}$ is unramified, $\mathbf{y}$ is ramified, and $\mathfrak{A}_{\mathbf{y}} \subset \mathfrak{A}_{\mathbf{x}}$.

2. $(\mathbf{y}, m)$ is equivalent to $(\mathbf{x}, n + 1)$, or vice versa.

Then $\mathcal{T}$ admits an action of $(\mathrm{GL}_2(K) \times D^\times)^{\det=1}$.

Let us call a vertex $v = (\mathbf{x}, m)$ of $\mathcal{T}$ *imprimitive* if the CM field L of $\mathbf{x}$ is ramified and m is even. Otherwise we will call v *primitive*. For each primitive vertex $v = (\mathbf{x}, m)$, the scheme $\overline{\mathscr{Z}}_v$ is the perfection of a nonsingular affine curve $C_v/\overline{\mathbb{F}}_q$, which inherits an action of $\mathcal{K}^1_v$. Let us summarize the results of the previous section. If L is unramified and $m = 0$, then C_v is the curve $XY^q - Y^qX = 1$. If L is unramified and $m > 1$, then C_v is the curve $Y^q + Y = X^{q+1}$. Finally, if L is ramified and m is odd, then C_v is the curve $Y^q - Y = X^2$.

Let

$$H^1_{\mathrm{aff}} = \bigoplus_v H^1(C_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell)$$

where v ranges over primitive vertices of the graph Γ. Then H^1_{aff} admits an action of $(\mathrm{GL}_2(K) \times D^\times)^{\det=1}$. Let us write $\mathbf{G}$ for this group; then

$$H^1_{\mathrm{aff}} = \bigoplus_{[v] \in \mathcal{T}/\mathbf{G}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{K}^1_v} H^1(C_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell).$$

Proposition 5.2.1. *As a representation of* $\mathbf{G}$*,* H^1_{aff} *contains as a summand the direct sum*

$$\bigoplus_{\Pi} \left(\Pi \otimes \mathrm{JL}(\check{\Pi})\right)^{\oplus 2},$$

where Π *ranges over twist classes of supercuspidal representations of* $\mathrm{GL}_2(K)$*. (Or rather,* H^1_{aff} *contains the restriction of this direct sum to* $\mathbf{G}$*.)*

Proof. Let Π be a supercuspidal representation of $\mathrm{GL}_2(K)$, and let $\Pi' = \mathrm{JL}(\check{\Pi})$. After twisting Π, we may assume (Thm. 3.2.3) either that Π contains a simple stratum or else it has depth zero. First assume that Π contains a simple stratum $S = (\mathcal{A}, m, \alpha)$. Let $L = K(\alpha)$, so that L/K is a separable quadratic extension. We have the finite-dimensional representation ρ_S of $\mathcal{K}_v$, as in §3.3. By Thm. 3.4.1, $\mathrm{Hom}_{\mathcal{K}_v}(\rho_S, \Pi \otimes \Pi') \neq 0$.

Let v be the vertex of $\mathcal{T}$ associated to the equivalence class of a pair $(\mathbf{x}, m)$, where $\mathbf{x}$ is a point with CM by L. We now claim that the restriction of ρ_S to $\mathcal{K}_v^1$ is isomorphic to a subrepresentation of $\mathcal{K}_v^1$ on $H^1(C_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell)$. In case that L/K is unramified, this is manifest in the definition of ρ_S as an isotypic component of the cohomology of the curve $Y^q + Y = X^{q+1}$. In the case that L/K is ramified, this is a consequence of the fact that the first cohomology of the Artin-Schreier curve $Y^q - Y = X^2$ breaks up as a sum of the nontrivial characters of $\mathbb{F}_q$.

Therefore $\mathrm{Hom}_{\mathcal{K}_v^1}(H^1(\overline{\mathscr{X}}_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell)), \Pi \otimes \Pi') \neq 0$. By Frobenius reciprocity, $\mathrm{Hom}_{\mathbf{G}}\left(\mathrm{Ind}_{\mathcal{K}_v^1}^{\mathcal{G}} H^1(C_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell), \Pi \otimes \Pi'\right) \neq 0$. In fact the dimension of this space must be at least 2, because Π contains both $S = (\mathcal{A}, m, \beta)$ and its K-conjugate $S^\sigma = (\mathcal{A}, m, \beta^\sigma)$, and ρ_S and ρ_{S^σ} are distinct summands of $H^1(C_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell)$. Since Π is supercuspidal (and since $D^\times$ is compact modulo center), this means that $\Pi \otimes \Pi'$ is a direct summand of $\mathrm{Ind}_{\mathcal{K}_v^1}^{\mathbf{G}} H^1(C_v^{\mathrm{cl}}, \overline{\mathbb{Q}}_\ell)$.

The argument for depth zero representations is similar. Here one needs to observe that the first cohomology of the curve $xy^q - x^q y = 1$ decomposes into representations of the form $\tau_\chi \otimes \chi^{-1}$, where χ runs over characters of $\mathbb{F}_{q^2}^\times$ which do not factor through the norm map $\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$, and τ_χ is the associated cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$. $\square$

5.3 The fundamental domain

Let $\mathcal{F} \subset \mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad}}$ be the subset of valuations which satisfy

$$|\mathbf{X}_1| \geq |\mathbf{X}_2| \geq |\mathbf{X}_1|^q$$

Proposition 5.3.1. *The translates of $\mathcal{F}$ under $\mathbf{G} = (\mathrm{GL}_n(K) \times D^\times)^{\det=1}$ cover $\mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad}}$.*

Proof. Let $|\,|$ be a valuation on $A^\circ_{\mathcal{O}_\mathbb{C}}$ with $|t| \neq 0$. In A° we have the equation $\delta(X_1, X_2) = t$. Since $\delta(X_1, X_2)$ has no constant term, and is alternating in its variables, we have $|X_i| \neq 0$ for $i = 1, 2$. On the other hand each X_i is topologically nilpotent, so $|X_i| < 1$.

Certainly there exists $m \in \mathbb{Z}$ with $|X_1| \geq |X_2|^{q^m} > |X_1|^q$. Let $\varpi \in \mathcal{O}_D = \mathrm{End}\, G_0$ be an element with $N(\varpi) = \pi$, so that ϖ acts on the variables X_1, X_2 by the rule $X_i \mapsto aX_i^q$ plus smaller terms, with a a unit. If $m = 2k$ is even, then replacing $|\,|$ with its translate by the pair $(\pi^k I, \varpi^m)$ gives a new valuation with $|X_1| \geq |X_2| > |X_1|^q$. If $m = 2k+1$ is odd, then replacing $|\,|$ with its translate by the pair $(\pi^k I, \varpi^m)$ gives a new valuation with $|X_1| \geq |X_2|^q > |X_1|^q$, so that $|X_2| > |X_1| \geq |X_2|^q$; now apply the matrix $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. ☐

5.4 A covering of the Lubin-Tate perfectoid space

For each pair $(\mathbf{x}, m)$, let

$$\mathrm{red}_{\mathbf{x},m} \colon \mathscr{Z}_{\mathbf{x},m} \to \overline{\mathscr{Z}}_{\mathbf{x},m}$$

be the reduction map, and let $S_{\mathbf{x},m} \subset \overline{\mathscr{Z}}_{\mathbf{x},m}$ be the set of images of CM points.

We now define an open cover $\{\mathscr{W}_v\}$ of $\mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad},\text{non-CM}}$ indexed by vertices of $\mathcal{T}$. If $v = (\mathbf{x}, 0)$ for $\mathbf{x}$ unramified, assume that $\mathfrak{A}_{\mathbf{x}} = M_2(\mathcal{O}_K)$ and put

$$\mathscr{W}_v = \{|\mathbf{X}_1| \geq |\mathbf{X}_2| > |\mathbf{X}_1|^q\} \setminus \bigcup_{y \in S_{\mathbf{x},0}} \mathrm{red}_{\mathbf{x},0}^{-1}(y),$$

If $v = (\mathbf{x}, 0)$ for $\mathbf{x}$ ramified, assume that $\mathfrak{A}_{\mathbf{x}}$ is the standard Iwahori algebra, and put

$$\mathscr{W}_v = \{|\mathbf{X}_1| > |\mathbf{X}_2| \geq |\mathbf{X}_1|^q\} \setminus \bigcup_{y \in S_{\mathbf{x},0}} \mathrm{red}_{\mathbf{x},0}^{-1}(y).$$

If $v = (\mathbf{x}, m)$ for $m > 0$ we set

$$\mathscr{W}_v = \mathrm{red}_{\mathbf{x},m-1}^{-1}(\overline{\mathbf{x}}) \setminus \bigcup_{y \in S_{\mathbf{x},m}} \mathrm{red}_{\mathbf{x},m}^{-1}(y)$$

Proposition 5.4.1. *The $\mathscr{W}_v$ cover $\mathcal{M}_{\overline{\eta}}^{\circ,\mathrm{ad},\textit{non-CM}}$.*

Proof. By Prop. 5.3.1 it suffices to show that the $\mathscr{W}_v$ cover the set of non-CM points in $\mathcal{F} = \{|X_1| \geq |X_2| \geq |X_1|^q\}$. It is clear from the definitions of the $\mathscr{W}_v$ that any point in $\mathcal{F}$ not lying in one of the $\mathscr{W}_v$ must lie in $\cap_{m\geq 1} \mathscr{Z}_{x,m}$ for some CM point x. But $\cap_{m\geq 1} \mathscr{Z}_{x,m} = \{x\}$. ☐

The assignment $v \mapsto \mathscr{W}_v$ can be extended to all vertices $v \in \mathcal{T}$ in such a way that $\mathscr{W}_v^g = \mathscr{W}_{v^g}$ for all $g \in (\mathrm{GL}_2(K) \times D^\times)^{\det=1}$.

Let

$$\mathscr{Z}_v = W_v \setminus \bigcup_w W_w$$

where w runs over vertices adjacent to v. Then if $v = (\mathbf{x}, m)$:

$$\mathscr{Z}_v = \mathscr{Z}_{\mathbf{x},m} \setminus \bigcup_{y \in S_{(\mathbf{x},m)}} \mathrm{red}_{\mathbf{x},m}^{-1}(y)$$

is the complement in $\mathscr{Z}_{\mathbf{x},m}$ of finitely many residue regions.

5.5 A semistable covering of $\mathcal{M}^{\circ,\mathrm{ad}}_{m,\overline{\eta}}$.

In this paragraph we translate our results about $\mathcal{M}^{\mathrm{ad}}_{\overline{\eta}}$ into results about the Lubin-Tate spaces of finite level. Recall the tower of complete local rings A_m, with A defined as the completion of $\varinjlim A_m$. Passing to adic spaces, we have a morphism from $\mathcal{M}^{\mathrm{ad}}_{\eta}$ to the projective system $\varprojlim \mathcal{M}^{\mathrm{ad}}_{m,\eta}$.

Lemma 5.5.1. *For each m, the morphism $\mathcal{M}^{\mathrm{ad}}_{\eta} \to \mathcal{M}^{\mathrm{ad}}_{\eta,m}$ is surjective, and carries open affinoids onto open affinoids.*

Proof. The maps between the local rings A_r are finite. This implies that each continuous valuation on A_m can be extended to A_{m+r} for all $r \geq 0$, hence to A. This shows that $\mathcal{M}^{\mathrm{ad}}_{\eta} \to \mathcal{M}^{\mathrm{ad}}_{\eta,m}$ is surjective.

Now suppose $\mathscr{X} = \mathrm{Spa}(R, R^+)$ be an open affinoid in $\mathcal{M}^{\mathrm{ad}}_{\eta}$. Then $R = A\langle f_1/\varpi, \dots, f_n/\varpi\rangle$ for elements $f_1, \dots, f_n \in A$ generating an ideal of definition of A and an element $\varpi \in \mathcal{O}_K$ of positive valuation. Since $\varinjlim A_r$ is dense in A, we may assume that the elements f_i live in A_N for some sufficiently large $N \geq m$. Then the image of $\mathscr{X}$ in $\mathcal{M}^{\mathrm{ad}}_{\eta,N}$ is an affinoid $\mathscr{X}_N = \mathrm{Spa}(R_N, R_N^+)$, with $R_N = A_N\langle f_1/\varpi, \dots, f_n/\varpi\rangle$. Since $\mathcal{M}^{\mathrm{ad}}_{\eta,N} \to \mathcal{M}^{\mathrm{ad}}_{\eta,m}$ is an étale map of adic curves, the image of $\mathscr{X}_N$ in $\mathcal{M}^{\mathrm{ad}}_{\eta,m}$ is again affinoid. $\square$

Of course, Lemma 5.5.1 holds for the tower of geometrically connected components $\mathcal{M}^{\circ,\mathrm{ad}}_{m,\overline{\eta}}$ as well.

Fix $m \geq 0$. For each vertex v of $\mathcal{T}$, let $W_v = W_v^{(m)}$ be the image of $\mathscr{W}_v$ in $\mathcal{M}^{\circ}_{m,\overline{\eta}}$. Similarly define $Z_v = Z_v^{(m)}$ as the image of $\mathscr{Z}_v$. By repeatedly applying Lemmas 5.1.2 and 5.5.1, we deduce that W_v is a wide open.

Let $\Gamma^1(\pi^m) = \Gamma(\pi^m) \cap \mathrm{SL}_2(K)$. We have a map $\overline{\mathscr{Z}}_v \to \overline{Z}_v$. Since $\overline{\mathscr{Z}}_v$ is the spectrum of a perfect ring, this map extends to a map $\overline{\mathscr{Z}}_v \to \overline{Z}_v^{\mathrm{perf}}$, where $\overline{Z}^{\mathrm{perf}}$ is the perfection of $\overline{Z}_v$.

Lemma 5.5.2. *Assume $m \geq 1$. The map $\overline{\mathscr{Z}}_v \to \overline{Z}_v^{\mathrm{perf}}$ is a quotient by $\mathcal{K}_v^1 \cap \Gamma^1(\pi^m)$. That is, the coordinate ring of $\overline{\mathscr{Z}}_v^{\mathrm{perf}}$ is the ring of $\mathcal{K}_v^1 \cap \Gamma^1(\pi^m)$-invariants in the coordinate ring of $\overline{\mathscr{Z}}_v$.*

Proof. Let $H = \mathcal{K}_v^1 \cap \Gamma^1(\pi^m)$. Let S (resp. $\mathscr{S}$) be the integral coordinate ring of Z_v (resp. $\mathscr{Z}_v$), and let $\overline{S}$ (resp. $\overline{\mathscr{S}}$) be its reduction. It suffices to show that the map $\overline{S}^{\mathrm{perf}} \to \overline{\mathscr{S}}^H$ is surjective. Let $\overline{f} \in \overline{\mathscr{S}}$ be invariant by H, and let $f \in \mathscr{S}$ be any lift. We may assume that f is invariant by $\Gamma^1(\pi^M)$ for some sufficiently large M, for the set of such functions (as M varies) is dense in $\mathscr{S}$. Let $H' = \mathcal{K}_v^1 \cap \Gamma^1(\pi^M)$. Let g be the product of translates of f by a set of coset representatives for H/H', so that g is H-invariant and therefore belongs to S. Since $m \geq 1$, H is a p-group, so that $[H : H'] = p^n$ for some n. Since $\overline{f}$ is H-invariant, we have $\overline{g} = \overline{f}^{p^n}$. Thus $\overline{f}$ is the image of an element of the perfection of S, namely $\overline{g}^{1/p^n}$. □

We now use the wide opens W_v to construct an admissible covering of $\mathcal{M}_{m,\overline{\eta}}^{\circ,\mathrm{ad}}$. Let $\mathbf{x} \in \mathcal{M}^\circ(\mathcal{O}_\mathbb{C})$ be a CM point, and let $\mathbf{x}^{(m)}$ be the image of $\mathbf{x}$ in $\mathcal{M}_m^\circ(\mathcal{O}_C)$. Let $U_\mathbf{x}$ be a sufficiently small affinoid neighborhood of $\mathbf{x}^{(m)}$, so that $U_\mathbf{x}$ is a disc. Then $W_\mathbf{x}$ contains $\mathscr{Z}_{(\mathbf{x},m)}$ for m sufficiently large, say $m \geq N_\mathbf{x} + 1$. Now let T be the graph described by the following procedure: Start with $\mathcal{T}/\Gamma(\pi^m)$, but remove $(\mathbf{x}, m)$ whenever $m > N_\mathbf{x}$. Then we have a covering of $\mathcal{M}_{m,\overline{\eta}}^{\circ,\mathrm{ad}}$ by wide opens V_v indexed by the vertices of T, where we have put

$$V_{(\mathbf{x},m)} = \begin{cases} W_{(\mathbf{x},m)}, & m < N_x \\ W_{(\mathbf{x},m)} \cup U_x, & m = N_x. \end{cases}$$

Proposition 5.5.3. *We have*

$$\sum_{v\in T} \dim H^1(\overline{Z}_v^{\mathrm{cl}}, \mathbb{Q}_\ell) \geq 2 \sum_{\Pi} \dim \Pi^{\Gamma(\pi^m)} \dim \mathrm{JL}(\check{\Pi}),$$

where the sum ranges over twist classes of supercuspidal representations of $\mathrm{GL}_2(F)$*.*

Proof. Let C_v be a smooth affine curve with $C_v^{\mathrm{perf}} = \overline{\mathscr{Z}}_v$; then Lemma 5.5.2 shows that $H_c^1(\overline{\mathscr{Z}}_v, \mathbb{Q}_\ell) = H^1(C_v^{\mathrm{cl}}, \mathbb{Q}_\ell)^{\mathcal{K}_v^1 \cap \Gamma^1(\pi^m)}$. We have

$$\begin{aligned} \bigoplus_{v\in T} H_c^1(\overline{\mathscr{Z}}_v, \mathbb{Q}_\ell) &= \bigoplus_{[v]\in\mathcal{T}/\mathbf{G}} \bigoplus_{g\in\mathcal{K}_v^1\backslash\mathbf{G}/\Gamma^1(\pi^m)} H_c^1(C_{v^g}^{\mathrm{cl}}, \mathbb{Q}_\ell)^{g^{-1}\mathcal{K}_v^1 g\cap\Gamma^1(\pi^m)} \\ &= \bigoplus_{[v]\in\mathcal{T}/\mathbf{G}} \left(\mathrm{Ind}_{\mathcal{K}_v}^{\mathbf{G}} H_c^1(C_v^{\mathrm{cl}}, \mathbb{Q}_\ell)\right)^{\Gamma^1(\pi^m)} \end{aligned}$$

by Mackey's theorem. The result now follows from Prop. 5.2.1. □

Theorem 5.5.4. $\{V_v\}_{v\in T}$ *constitutes a semistable covering of* $\mathcal{M}_{m,\overline{\eta}}^{\circ,\mathrm{ad}}$*.*

Proof. $\mathcal{M}_{m,\overline{\eta}}^{\circ,\mathrm{ad}}$ admits *some* semistable covering, so suppose there is a graph T' and a collection of wide opens V_v' satisfying the criteria in Defn. 5.1.3. After refining the covering, we may assume that T' contains T as a subgraph, that $Z'_v \subset Z_v$ for all vertices $v \in T$, and that $\overline{Z}'_v \subset \overline{Z}_v$ is open, so that $\overline{Z}_v^{\mathrm{cl}} = (\overline{Z}'_v)^{\mathrm{cl}}$.

By Prop. 5.1.5 we have

$$\dim H^1_c(\mathcal{M}^{\circ,\mathrm{ad}}_{\overline{\eta}}, \mathbb{Q}_\ell) = \sum_{v \in T'} \dim H^1_c(\overline{Z}'_v, \mathbb{Q}_\ell) + \dim H^1_c(T', \mathbb{Q}_\ell).$$

On the other hand Eq. (2.4.1) gives

$$\dim H^1_c(\mathcal{M}^{\circ,\mathrm{ad}}_{\overline{\eta}}, \mathbb{Q}_\ell) = 2 \sum_{\Pi} \dim \Pi^{\Gamma(\pi^m)} \dim \mathrm{JL}(\check{\Pi}) + \dim \mathrm{St}^{\Gamma(\pi^m)}$$

We have $\dim \mathrm{St}^{\Gamma(\Pi^m)} = q^{m-1}(q-1)$. The dimension of $H^1_c(T', \mathbb{Q}_\ell)$ is at least this big, because the ends of T are in correspondence with $\mathbf{P}^1(\mathcal{O}_K/\pi^m)$. This has the following consequences:

1. The inequality in Prop. 5.5.3 is an equality,
2. For all imprimitive $v \in T$, and all $v \in T' \backslash T$, $H^1_c(\overline{Z}'_v, \mathbb{Q}_\ell) = 0$, so that $\left(\overline{Z}'_v\right)^{\mathrm{cl}} = \mathbf{P}^1$, and
3. $H^1_c(T', \mathbb{Q}_\ell) = H^1_c(T, \mathbb{Q}_\ell)$, so that T' is cycle-free and has no ends other than those of T.

These imply that $\{V_v\}_{v \in V}$ was a semistable covering to begin with, because otherwise its semistable refinement would have introduced new curves of positive genus in the special fiber, or else monodromy in the dual graph. ☐

6 Stable reduction of modular curves: Figures

In Figures 1-3, we draw the graph $\mathcal{T}$ constructed in §5.2. Each vertex v is labeled with its corresponding curve appearing on the list of four curves in Thm. 1.0.1. We sketch a procedure for calculating the dual graph corresponding to the special fiber of a stable model of the classical modular curve $X_m = X(\Gamma(p^m) \cap \Gamma_1(N))$, where $N \geq 5$. First one must calculate the quotient $\Gamma^1(p^m) \backslash \mathcal{T}$, where $\Gamma^1(p^m) = (1 + p^m M_2(\mathbb{Z}_p))) \cap \mathrm{SL}_2(\mathbb{Z}_p)$. The image of a vertex v in the quotient is labeled with the nonsingular projective curve constructed by quotienting Z_v by $\Gamma^1(p^m) \cap \mathcal{K}^1_v$. For almost every v, the quotient is rational. The quotient graph $\Gamma^1(p^m) \backslash \mathcal{T}$ has finitely many ends, and each end is (once one goes far enough) a ray consisting only of rational components. Erase all rational components lying on an end which

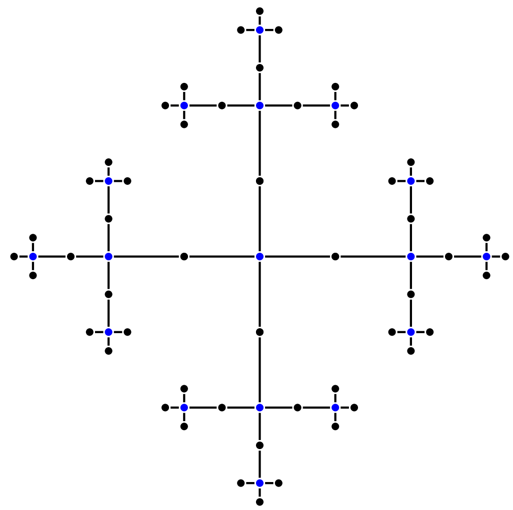

Figure 1: The "depth zero" subgraph of Γ, consisting of the vertices $v = (\mathbf{x}, 0)$. The blue vertices are unramified. Each represents a copy of the nonsingular projective curve with affine model $xy^q - x^q y = 1$. The stabilizer of any particular blue vertex in $\mathrm{SL}_2(K)$ is conjugate to $\mathrm{SL}_2(\mathcal{O}_K)$. The black vertices are imprimitive. Each represents a rational component. The stabilizer of any black vertex in $\mathrm{SL}_2(K)$ is an Iwahori subgroup.

corresponds to a CM point. The remaining ends correspond to the boundary of $\mathcal{M}^{\circ,\mathrm{ad}}_{m,\eta}$; these are in bijection with $\mathbf{P}^1(\mathbb{Z}/p^m\mathbb{Z})$. For each $b \in \mathbf{P}^1(\mathbb{Z}/p^m\mathbb{Z})$, erase all rational components lying on the end corresponding to b, and let v_b be the unique non-rational vertex which is adjacent to one of the vertices just erased. Call the resulting graph T_m.

Let $\mathrm{Ig}(p^m)$ denote the nonsingular projective model of the Igusa curve parameterizing elliptic curves over $\overline{\mathbb{F}}_p$ together with Igusa p^m structures and a point of order N. Draw $\mathbf{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ many vertices w_b, and label each with $\mathrm{Ig}(p^m)$. For each $b \in \mathbf{P}^1(\mathbb{Z}/p^m\mathbb{Z})$, and each supersingular point of $X_1(N)(\overline{\mathbb{F}}_p)$, attach a copy of T_m to w_b in such a way that the vertex w_b is adjacent to each v_b. Finally, blow down any superfluous rational components. The result is a finite graph representing the special fiber of a stable model of X_m.

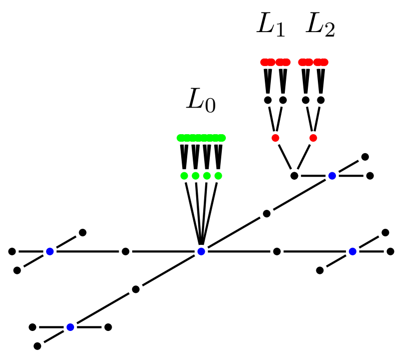

Figure 2: Here the depth zero subgraph of Γ is shown with several wild vertices to reveal structure. The green vertices are unramified; each represents a copy of the nonsingular (disconnected) projective curve with affine model $y^q + y = x^{q+1}$. The red vertices are ramified; each represents a copy of the nonsingular projective curve with affine model $y^q - y = x^2$. The wild vertices $(\mathbf{x}, m)$ labeled with an L_i are those for which $\mathbf{x}$ has CM by L_i. Here L_0, L_1, L_2 are the three quadratic extensions of L, with L_0/K unramified. The black vertices are imprimitive; each represents a rational component.

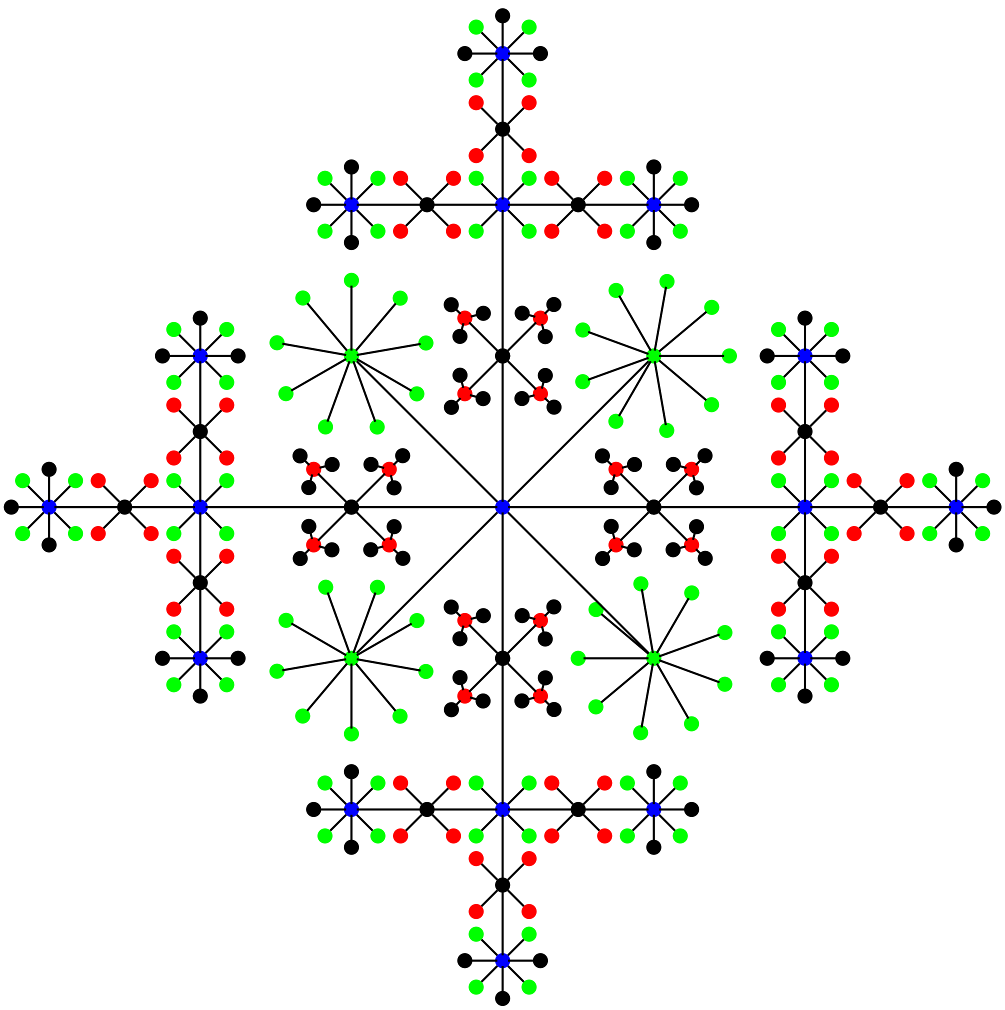

Figure 3: Dual graph of the special fiber of our semistable model of the tower of Lubin-Tate curves: complete picture.

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