Resolution of singularities on the tower of modular curves

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IAS

Stanford Number Theory Seminar, Oct. 22, 2010

Semistable models: Definition

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Definition

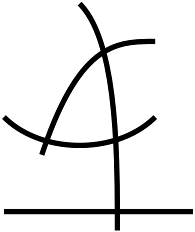
Let X/K be a smooth proper curve. A *semistable model* of X is a proper curve $\mathfrak{X}/R$ such that

- $\mathfrak{X} \otimes_R K = X$, and
- $\mathfrak{X} \otimes_R k$ has ordinary double points as singularities.

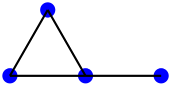
That is: The singularities of $\mathfrak{X}$ are as mild as possible.

Semistable models: Dual Graphs

If $\mathfrak{X} \otimes_R k$ looks like:



Then its *dual graph* Γ is:



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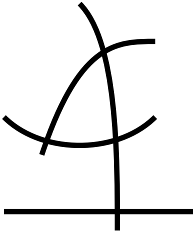

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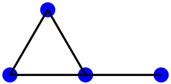

If a semistable model of X is known, then the cohomology of X can be computed readily.

Semistable models: Cohomological consequence

Suppose a smooth proper curve X/K has semistable model $\mathfrak{X}/R$ with dual graph Γ.

Theorem

We have an exact sequence of $\mathbf{Q}_\ell[\mathrm{Gal}(\overline{K}/K)]$*-modules*

$$0 \to H^1(\mathfrak{X} \otimes_R \overline{k}, \mathbf{Q}_\ell) \to H^1(X \otimes_K \overline{K}, \mathbf{Q}_\ell) \to H^1(\Gamma, \mathbf{Q}_\ell)(-1) \to 0$$

In particular, if Γ is a tree, then $H^1(\mathfrak{X} \otimes_R \overline{k}, \mathbf{Q}_\ell) \to H^1(X \otimes_K \overline{K}, \mathbf{Q}_\ell)$ is an isomorphism. (Consequence of Rapoport-Zink's computation of "vanishing cycles.")

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Semistable models make cohomology a combinatorial matter!

Semistable models: Existence

Given X/K, there might not exist a semistable model $\mathfrak{X}$. However:

Theorem (Deligne-Mumford)

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There exists a finite extension L/K for which X_L admits a semistable model $\mathfrak{X}_L$.

Theorem (Coleman)

Given a finite morphism $X \to Y$, there exists a finite extension L/K and a finite morphism $\mathfrak{X}_L \to \mathfrak{Y}_L$ of semistable models which extends $X \to Y$.

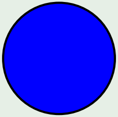
The above theorems have an interpretation in the language of *rigid analysis*.

Rigid analysis: Basic Concepts: Affinoids

In rigid analysis, the basic objects are the *affinoids*.

Example

The *affinoid unit disk* (over $\mathbf{Q}_p$) is:



$$= \operatorname{MaxSpec} \mathbf{Q}_p\langle T\rangle$$

$\mathbf{Q}_p\langle T\rangle$ is the ring of power series $\sum a_n T^n$ with $a_n \to 0$.
Generally, a one-dimensional *affinoid algebra* is a finite extension of $\mathbf{Q}_p\langle T\rangle$. This is always a Banach algebra under the sup norm.
A one-dimensional *affinoid* is the MaxSpec of an affinoid algebra.
A *rigid curve* is obtained by gluing affinoids.

Rigid analysis: Basic Concepts: The reduction of an affinoid

Let $Z = \operatorname{MaxSpec} \mathcal{A}$ be an affinoid. We have the following rings at our disposal:

- $\mathcal{A}^\circ = \{f \in \mathcal{A} \colon \ |f(z)| \leq 1, \text{ all } z \in Z\}$
- $\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} \colon \ |f(z)| < 1, \text{ all } z \in Z\}$
- $\overline{\mathcal{A}} = \mathcal{A}^\circ / \mathcal{A}^{\circ\circ}$.

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If Z is the affinoid unit disk, then $\mathcal{A} = \mathbf{Q}_p\langle T \rangle$, $\mathcal{A}^\circ = \mathbf{Z}_p\langle T \rangle$, and $\mathcal{A}^{\circ\circ} = p\mathcal{A}^\circ$. Thus, $\operatorname{red}(Z) = \operatorname{Spec} \mathbf{F}_p[T] = \mathbf{A}^1/\mathbf{F}_p$.

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If Z is the "circle" $\{z\colon\ |z| = 1\}$, then $\mathcal{A} = \mathbf{Q}_p\langle T, T^{-1}\rangle$ and $\mathrm{red}(Z) = \mathrm{Spec}\,\mathbf{F}_p[T, T^{-1}] = \mathbf{G}_m/\mathbf{F}_p$.

Rigid Analysis: Coleman's "wide opens"

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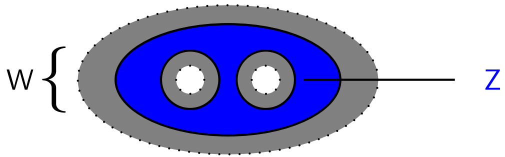
A *wide open* is a rigid-analytic curve W containing an affinoid Z such that $W \backslash Z$ is a disjoint union of open annuli:



Here, Z is an *underlying affinoid* of W.

Semistable models and semistable coverings

Let X be a rigid analytic curve. A *semistable covering* is a family $\{W_i\}$ of wide opens, such that

1. $X = \bigcup_i W_i$.
2. For $i \neq j$, $W_i \cap W_j$ is a disjoint union of open annuli.
3. No three of the W_i may mutually intersect.
4. For each i, the complement $Z_i = W_i \setminus \bigcup_{j \neq i} W_j$ is an underlying affinoid of W_i, with good reduction.

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If X/K is a smooth proper curve, then semistable models $\mathfrak{X}$ of X correspond to semistable coverings $\{U_i\}$ of X^{rig}. The components of $\mathfrak{X} \otimes_R k$ will be projective models of the reductions $\mathrm{red}(Z_i)$.

Semistable covering: Example

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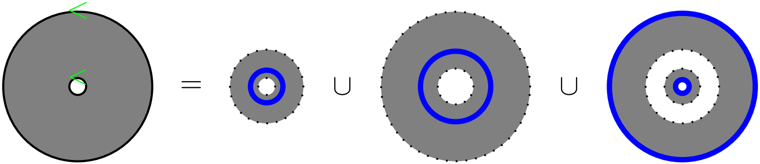

$|p|^3 \leq |z| \leq 1 \quad = \quad |p|^3 < |z| < |p| \quad \cup \quad |p|^2 < |z| < 1 \quad \cup \quad |p| < |z| < |p|^{-1}$

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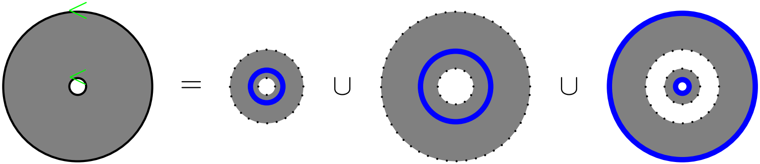

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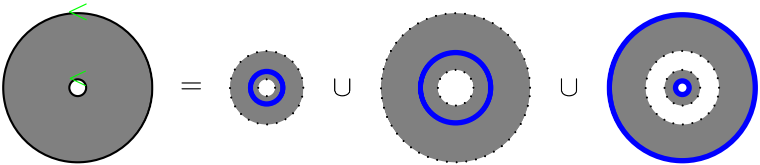

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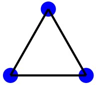
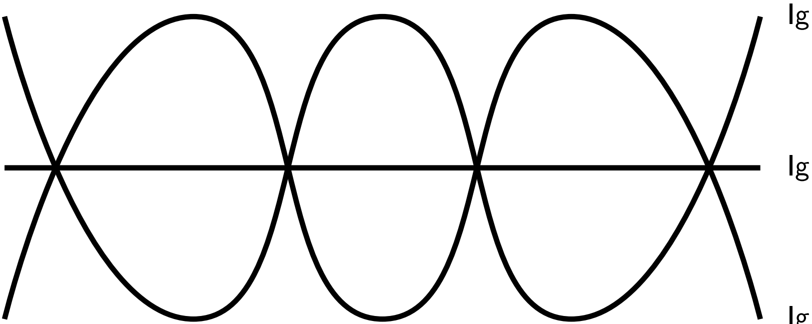
Each underlying affinoid has reduction $\mathbf{G}_m$.
We get a semistable model of X whose special fiber has dual graph



Each vertex is a $\mathbf{P}^1$.

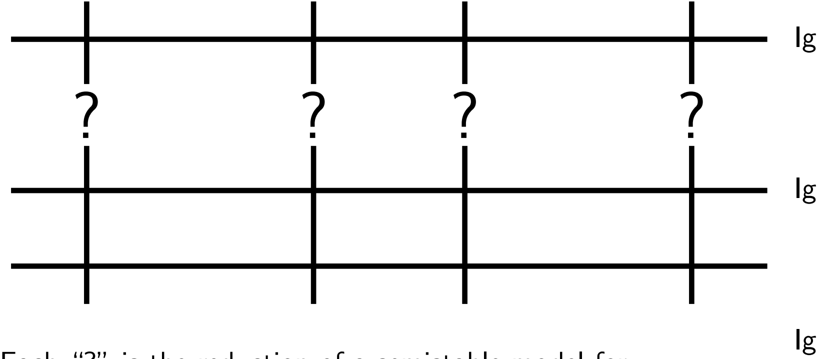
Semistable reduction of modular curves

Let $X_n = X(\Gamma(p^n) \cap \Gamma_1(N))$. The Katz-Mazur model of X_n over $\mathbf{Z}_p^{\mathrm{nr}}[\zeta_{p^n}]$ has reduction



After base extension, want to find a semistable model, whose reduction will look like...

Semistable reduction of modular curves



Each "?" is the reduction of a semistable model for $X_{n,x} = \operatorname{Spec} \hat{\mathcal{O}}_{X_{n,x}}$, where $x \in X_1(N)(\overline{\mathbf{F}}_p)$ is supersingular. The cohomology $H^1(?, \mathbf{Q}_\ell)$ encodes the local Langlands correspondence for irreducible representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$!

Semistable models of modular curves: previous work

1. The Deligne-Rapoport model of $X_0(p)$ is already semistable.
2. Edixhoven found a semistable model of $X_0(p^2)$.
3. Coleman-McMurdy found a semistable model of $X_0(p^3)$.
4. Wewers found equations for curves which should appear in the semistable reduction of $X(p^n)$.

Semistable models of modular curves

- Recall $\mathfrak{X}_{n,x} = \operatorname{Spec} \hat{\mathcal{O}}_{X_n,x}$ is the completion of X_n at a supersingular point x.

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- In the tower, $\cdots \to \mathfrak{X}_{n+1,x} \otimes \overline{\mathbf{F}}_p \to \mathfrak{X}_{n,x} \otimes \overline{\mathbf{F}}_p \to \cdots$, suppose $\cdots \to C_{n+1} \to C_n \to \cdots$ is a chain of irreducible components. Then the morphisms are *purely inseparable* for n large enough!

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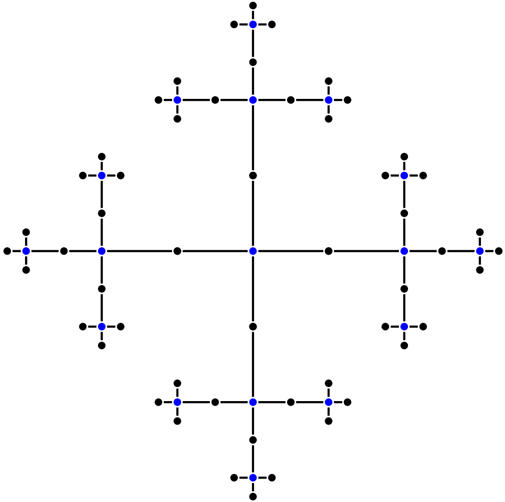
- We describe the inverse limit of the dual graphs of the $\mathfrak{X}_{n,x} \otimes \overline{\mathbf{F}}_p$. Each vertex corresponds to one of the chains $\cdots \to C_{n+1} \to C_n \to \cdots$ as above. Each vertex is labeled with the equation of C_n for n large enough.
- The stabilizer of any particular vertex is a group which arises in Bushnell-Kutzko's theory of *types* for classifying representations of $\mathrm{GL}_2(\mathbf{Q}_p)$.

Structure of the semistable reduction of the tower of modular curves: The "tame" part C^{tame}



Legend

- (blue) $xy^p - x^p y = 1$
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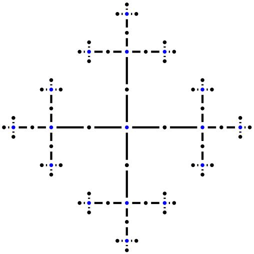

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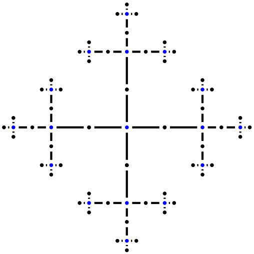

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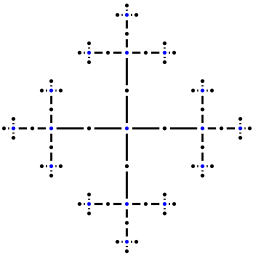

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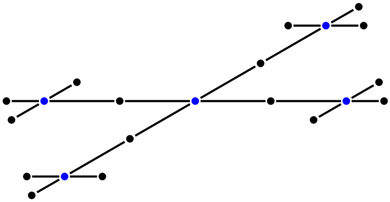
- $\mathrm{SL}_2(\mathbf{Z}_p)$ acts on X by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$
- $H^1(C^{\text{tame}}, \overline{\mathbf{Q}}_\ell)$ encodes LLC for supercuspidal π of conductor p^2.

Structure of the semistable reduction of the tower of modular curves: Wild components



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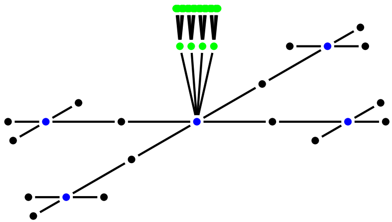

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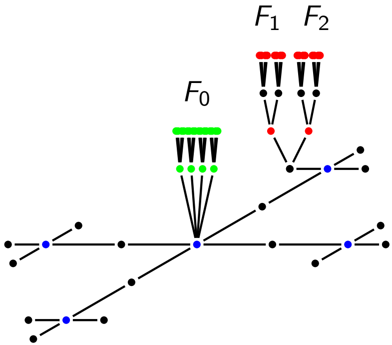

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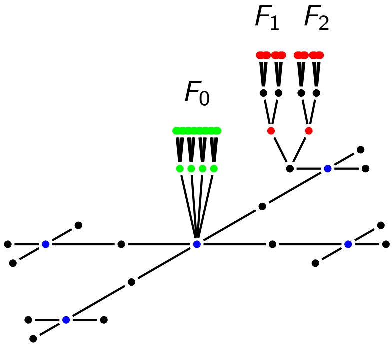

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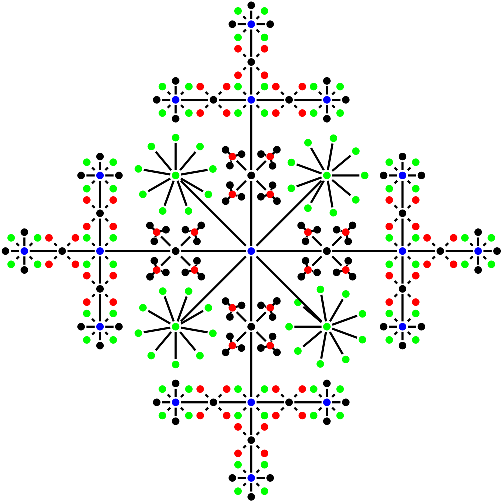
- $F_i/\mathbf{Q}_p$ are quad. extns.
- $F_0/\mathbf{Q}_p$ is unramified.
- A vertex labeled F_i $\leftrightarrow$ a Gal. rep. of the form $\mathrm{Ind}_{F_i/\mathbf{Q}_p}\theta$.

Structure of the semistable reduction of the tower of modular curves: Complete picture



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