MA 225 PRACTICE FINAL SOLUTIONS

1. (12 points) Answer the following questions about 3D vector geometry.

(a) (3 pts) Find a vector which is normal to the plane $2x - 3z = 1$.

Solution: Copying the coefficients of x, y and z and pasting them into a vector gives $\langle 2, 0, -3\rangle$, which is a normal vector to this plane.

(b) (3 pts) Find the angle between the vectors $\langle 1, 0, 1\rangle$ and $\langle -1, 1, 0\rangle$.

Solution: The angle θ satisfies

$$\cos\theta = \frac{\langle 1,0,1\rangle \cdot \langle -1,1,0\rangle}{|\langle 1,0,1\rangle|\,|{-1},1,0\rangle|} = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2},$$

so that $\theta = 2\pi/3$.

(c) (3 pts) If $\mathbf{v}$ and $\mathbf{w}$ are any vectors in space, then $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ equals:

Solution: The cross product $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$, so their dot product must equal 0.

(d) (3 pts) Write down the interpretation of $|\mathbf{v} \times \mathbf{w}|$ as an area.
Solution: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram whose sides are v and w.

2. (6 points) Let $\mathbf{F} = \langle f, g \rangle$ be a vector field with continuous partial derivatives on a simply-connected region R in the plane. Which of the following statements does not mean the same as the others?

(1) The divergence of $\mathbf{F}$ is 0 everywhere on R.
(2) $\partial g/\partial x - \partial f/\partial y = 0$ everywhere on R.
(3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in R.
(4) $\mathbf{F} = \nabla \phi$ for a function ϕ defined on R.

Solution: The answer is (1). (2) means that the curl of $\mathbf{F}$ is 0, which means that $\mathbf{F}$ must be conservative (4). If $\mathbf{F}$ is conservative, then it has zero circulation around any closed curve (3). The statement (1) means that $\mathbf{F}$ is source-free, which means that $\mathbf{F}$ has zero flux through any closed curve, but that is something different.

3. (6 points) Suppose that x, y and z are positive numbers satisfying $x + 2y + z = 12$. Find the largest possible value of xyz.

Solution: Let's use Lagrange multipliers. We're trying to maximize $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + 2y + z = 12$. So there's going to be a scalar λ for which $\nabla f = \lambda \nabla g$. This means

$$\langle yz, xz, xy \rangle = \lambda \langle 1, 2, 1 \rangle \,,$$

from which we get equations

$$\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= \lambda \end{aligned}$$

Plugging $\lambda = yz$ into the second equation gives $xz = 2yz$. We can cancel the z (because it is positive) to get $x = 2y$. Doing the same with the third equation gives $x = z$. So, $x = z = 2y$. Finally we have the original equation $x + 2y + z = 12$, which means that $2y + 2y + 2y = 12$, or $y = 2$. Thus $(x, y, z) = (4, 2, 4)$, and the largest value of xyz is $4 \times 2 \times 4 = 32$.

4. (8 points) Let $\mathbf{F} = \langle xy, x+y \rangle$. Let C be the triangle in the plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$, oriented counterclockwise. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Solution: You can do three line integrals, but it's easier to use Green's theorem (the circulation form). This says that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \mathrm{curl}\mathbf{F}\, dA$, where R is the region enclosed by C. This is the region bounded by $y = 0$, $x = 1$ and $y = x$. The curl of $\mathbf{F}$ is $1 - x$. We get

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \mathrm{curl}\mathbf{F}\, dA \\
&= \int_{x=0}^{1} \int_{y=0}^{x} 1 - x \, dy\, dx \\
&= \int_{x=0}^{1} (1-x)y|_0^x \\
&= \int_{x=0}^{1} x - x^2 \, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\end{aligned}$$

5. (6 points) Find the Jacobian of the transformation $x = ve^u$, $y = ve^{-u}$.

Solution: The Jacobian is the determinant of the partial derivatives:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} ve^u & e^u \\ -ve^{-u} & e^{-u} \end{vmatrix} = v - (-v) = 2v.$$

6. (10 points) Let $z = f(x, y)$ be a differentiable function. Let r and θ be the usual variables from polar coordinates, so that $x = r\cos\theta$ and $y = r\sin\theta$. At the point $(r, \theta) = (1, \pi/4)$, find $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$.

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Solution: The chain rule says that

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} \\
&= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \\
&= \frac{\sqrt{2}}{2}\frac{\partial z}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial z}{\partial y}
\end{aligned}$$

It's similar with $\frac{\partial z}{\partial\theta}$.

7. (6 points) Sketch the graph of $z = \sqrt{x^2 + y^2}$.

Solution: This is a cone facing upward, with its vertex at the origin.

8. (10 points) Let S be the filled-in square in the xz-plane with vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$ and $(0,0,1)$. Orient S in the positive y-direction. Let $\mathbf{F}$ be the vector field $\langle -y, z, x\rangle$. Find the flux of $\mathbf{F}$ through S.

Solution: The flux is $\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma$. Since S is just a flat square in the xz-plane, the normal vector is just $\mathbf{n}=\mathbf{j}=\langle 0,1,0\rangle$. On S, the vector field is $\mathbf{F}=\langle 0,z,x\rangle$, because y is always 0. We get

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \int_0^1\int_0^1 \langle 0,z,x\rangle\cdot\langle 0,1,0\rangle\ dz\ dx = \int_0^1\int_0^1 z\ dz\ dx = \frac{1}{2}.$$

9. (10 points) Find $\iiint_D z^2 dV$, where D is the region in space defined by the inequalities $x^2+y^2\le 4$, $0\le z\le 1$.

Solution: D is part of a cylinder, so let's use cylindrical coordinates:

$$\iint_D z^2\ dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{1} z^2\ dz\ rdr\ d\theta = (2\pi)(2^2/2)(1/3) = 4\pi/3.$$

10. (10 points) Let $\phi(x, y, z) = x^3 y + z$. Find $\int_C \nabla\phi \cdot d\mathbf{r}$, where C is any curve starting at $(1, 0, 0)$ and ending at $(1, 1, 1)$.

By the fundamental theorem of line integrals,

$$\int_C \nabla\phi \cdot d\mathbf{r} = \phi(1, 1, 1) - \phi(1, 0, 0) = 2 - 0 = 2.$$