MA 341 HW #11, due Friday, Apr. 21

1. Find Gaussian integers $q, r \in \mathbf{Z}[i]$ such that $7 + 11i = (1 + 3i)q + r$, where $N(r) < N(1 + 3i)$.

2. A complete set of representatives for $\mathbf{Z}[i]_3$ (this is Gaussian integers modulo 3) is $\{0, 1, 2, i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i\}$. Of these, the eight nonzero elements are units modulo 3. For each one, calculate the order of the element modulo 3. Is there a primitive root?

3. Let $\alpha, \beta \in \mathbf{Z}[i]$. Show that if the integers $N(\alpha)$ and $N(\beta)$ are relatively prime, then α and β are relatively prime as well.

4. Give a counterexample to the converse: find $\alpha, \beta \in \mathbf{Z}[i]$ which are relatively prime, but $(N(\alpha), N(\beta)) \neq 1$.

5. The Gaussian integers 100 and $1 + 4i$ are relatively prime. Find a solution to Bezout's equation $100x + (1 + 4i)y = 1$, with $x, y \in \mathbf{Z}[i]$.

6. Let $p \in \mathbf{Z}$ be a prime number. Show that p is *not* prime in $\mathbf{Z}[i]$ if and only if $p = x^2 + y^2$ for integers $x, y \in \mathbf{Z}$.

7. Let $p \in \mathbf{Z}$ be a prime number. Suppose $p \equiv 1 \pmod 4$. By HW #9, there exists $n \in \mathbf{Z}$ such that $p \mid n^2+1$. Note that $n^2+1 = (n+i)(n-i)$. Show that p cannot be prime in $\mathbf{Z}[i]$. Therefore $p = x^2 + y^2$ for integers $x, y \in \mathbf{Z}$.