1. Let $S$ be a linearly independent subset of a vector space $V$, and let $v \in V$. Show that $S \cup \{v\}$ is linearly dependent if and only if $v$ lies in the span of $S$.

**Solution.** Since this exact problem was on the practice final, I was very strict – I wanted to see every detail done correctly. Certainly it was necessary to do both the “if” and “only if” parts.

In one direction, suppose $S \cup \{v\}$ is linearly dependent. This means there exist scalars $a_0, a_1, \ldots, a_n$ (not all of which are zero) and vectors $v_1, \ldots, v_n \in S$ such that

$$a_0v + a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$$

We must have $a_0 \neq 0$, for otherwise the vectors $v_1, \ldots, v_n$ would be linearly dependent, which they are not. Thus we can divide this equation by $a_0$ and rearrange to get

$$v = -\frac{a_1}{a_0}v_1 - \cdots - \frac{a_n}{a_0}v_n,$$

which shows that $v$ lies in the span of $S$.

In the other direction, suppose $v$ lies in the span of $S$. This means there are scalars $a_1, \ldots, a_n$ and vectors $v_1, \ldots, v_n$ for which

$$v = a_1v_1 + \cdots + a_nv_n.$$

Rearranging, we get

$$-v + a_1v_1 + \cdots + a_nv_n = 0.$$

Note that $-1 \neq 0$. This equation tells us that $S \cup \{v\}$ has to be linearly dependent. (Note that $S$ is not required to be finite for this proof.)

2. For which real values of $a$ is the matrix

$$\begin{pmatrix} a & 2 - a \\ 2 + a & -a \end{pmatrix}$$

diagonalizable? (You don’t need to tell me any eigenvalues or eigenvectors.)

**Solution.** The characteristic polynomial is $t^2 - 4$, which has roots $2$ and $-2$. Since there are two distinct roots, the matrix is diagonalizable no matter what the value of $a$ is.

3. Let $V$ be the real vector space of functions $f : \mathbb{R} \rightarrow \mathbb{R}$. For vectors $f, g \in V$, let us define

$$\langle f, g \rangle = f(0)g(0).$$

Is $V$ an inner product space with respect to $\langle f, g \rangle$?

**Solution** This is not an inner product space, because if (for example) $f(x) = x$, we get $\langle f, f \rangle = 0$. An inner product space must satisfy $\langle f, f \rangle > 0$ whenever $f$ is nonzero.

4. Let $A$ be a real $n \times n$ matrix. Show that $AA^t$ is diagonalizable (where $A^t$ is the transpose of $A$).

**Solution** Let $B = AA^t$. Then

$$B^t = (AA^t)^t = (A^t)^tA^t = AA^t = B,$$

so that $B$ is symmetric. Every symmetric matrix is diagonalizable.
5. Let $A$ be a complex $n \times n$ matrix whose only eigenvalue is 1. Show that $(A - I_n)^n$ is the zero matrix.

**Solution** Since 1 is the only eigenvalue, the characteristic polynomial (which has degree $n$) has 1 as its only root. This means that (up to a sign) the characteristic polynomial is $(t-1)^n$. By Cayley-Hamilton, $(A - I_n)^n = 0$.

(Many of you tried to appeal to the theory of the Jordan canonical form, which meant doing a messy calculation.)

6. Find two orthogonal matrices of the form

$$
\begin{pmatrix}
\sqrt{3}/2 & * \\
1/2 & *
\end{pmatrix}
$$

(In other words, find two different ways to fill in the second column of this matrix in such a way that the result is an orthogonal matrix.)

**Solution.** The two matrices are

$$
\begin{pmatrix}
\sqrt{3}/2 & -1/2 \\
1/2 & \sqrt{3}/2
\end{pmatrix}
$$
and

$$
\begin{pmatrix}
\sqrt{3}/2 & 1/2 \\
1/2 & -\sqrt{3}/2
\end{pmatrix}.
$$

The first is a rotation and the second is a reflection.

7. Let $A$ be a $3 \times 3$ complex matrix for which $A^3$ is the zero matrix, but $A^2$ is not the zero matrix. Find the Jordan canonical form of $A$.

**Solution.** Let $J$ be the Jordan canonical form of $A$, so that $J = Q AQ^{-1}$ for some invertible matrix $Q$. Since $A^3 = 0$ it quickly follows that $J^3 = 0$. This means that the only eigenvalue of $J$ is 0. A priori there are three possibilities for $J$, but the fact that $A^2$ is nonzero means that all the cycles in $J$ have to have length at least 3. Since $J$ is $3 \times 3$ this means there is exactly one cycle of length exactly 3. We get

$$
J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
$$

8. Let $V$ be an inner product space with orthonormal basis $v_1, v_2, v_3$. Let $W = \{v_1\}^\perp$. Show that $W$ is the space spanned by $v_2$ and $v_3$.

**Solution.** Once again I was somewhat unforgiving. Here it was important to prove both containments: $W$ is contained in the span, and the span is contained in $W$.

Since $v_2$ and $v_3$ are both orthogonal to $v_1$ we have $v_2, v_3 \in W$, and therefore the span of $v_2$ and $v_3$ is contained in $W$.

For the other direction, suppose $w \in W$. Write $w = a_1 v_1 + a_2 v_2 + a_3 v_3$, where each $a_i \in \mathbb{F}$. Since $w$ is orthogonal to $v_1$ we have

$$
0 = \langle w, v_1 \rangle = a_1 \langle v_1, v_1 \rangle + a_2 \langle v_2, v_1 \rangle + a_3 \langle v_3, v_1 \rangle = a_1,
$$

so that in fact $w = a_2 v_2 + a_3 v_3$ lies in the span of $v_2$ and $v_3$.

9. Let $u$ and $v$ be vectors in a real inner product space. Show that $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $u$ and $v$ are orthogonal.

**Solution.** We have

$$
\|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + \|v\|^2 + 2 \langle u, v \rangle
$$

so that the desired equality holds if and only if $\langle u, v \rangle = 0$. 

2
10. Let \( A \) be a real \( n \times n \) matrix. Show that if \( A \) is orthogonally equivalent to a real diagonal matrix, then \( A \) is symmetric.

**Solution.** The hypothesis means that \( A = P^tDP \) for \( D \) a real diagonal matrix and \( P \) an orthogonal matrix. We find

\[
A^t = (P^tDP)^t = P^tD^tP = P^tDP = A,
\]

so that \( A \) is symmetric.