MA442 Final Exam

1. Let S be a linearly independent subset of a vector space V, and let $v \in V$. Show that $S \cup \{v\}$ is linearly dependent if and only if v lies in the span of S.

   Solution. Since this exact problem was on the practice final, I was very strict – I wanted to see every detail done correctly. Certainly it was necessary to do both the "if" and "only if" parts.

   In one direction, suppose $S \cup \{v\}$ is linearly dependent. This means there exist scalars $a_0, a_1, \ldots, a_n$ (not all of which are zero) and vectors $v_1, \ldots, v_n \in S$ such that

   $$a_0 v + a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

   We must have $a_0 \neq 0$, for otherwise the vectors $v_1, \ldots, v_n$ would be linearly dependent, which they are not. Thus we can divide this equation by a_0 and rearrange to get

   $$v = -\frac{a_1}{a_0} v_1 - \cdots - \frac{a_n}{a_0} v_n,$$

   which shows that v lies in the span of S.

   In the other direction, suppose v lies in the span of S. This means there are scalars $a_1, \ldots, a_n$ and vectors $v_1, \ldots, v_n$ for which

   $$v = a_1 v_1 + \cdots + a_n v_n.$$

   Rearranging, we get

   $$-v + a_1 v_1 + \cdots + a_n v_n = 0.$$

   Note that $-1 \neq 0$. This equation tells us that $S \cup \{v\}$ has to be linearly dependent. (Note that S is not required to be finite for this proof.)

2. For which real values of a is the matrix

   $$\begin{pmatrix} a & 2-a \\ 2+a & -a \end{pmatrix}$$

   diagonalizable? (You don't need to tell me any eigenvalues or eigenvectors.)

   Solution. The characteristic polynomial is $t^2 - 4$, which has roots 2 and -2. Since there are two distinct roots, the matrix is diagonalizable no matter what the value of a is.

3. Let V be the real vector space of functions $f\colon \mathbb{R} \to \mathbb{R}$. For vectors $f, g \in V$, let us define

   $$\langle f, g \rangle = f(0)g(0).$$

   Is V an inner product space with respect to $\langle f, g \rangle$?

   Solution This is not an inner product space, because if (for example) $f(x) = x$, we get $\langle f, f \rangle = 0$. An inner product space must satisfy $\langle f, f \rangle > 0$ whenever f is nonzero.

4. Let A be a real $n \times n$ matrix. Show that AA^t is diagonalizable (where A^t is the transpose of A).

   Solution Let $B = AA^t$. Then

   $$B^t = (AA^t)^t = (A^t)^t A^t = AA^t = B,$$

   so that B is symmetric. Every symmetric matrix is diagonalizable.

5. Let A be a complex $n \times n$ matrix whose only eigenvalue is 1. Show that $(A - I_n)^n$ is the zero matrix.

   Solution Since 1 is the only eigenvalue, the characteristic polynomial (which has degree n) has 1 as its only root. This means that (up to a sign) the characteristic polynomial is $(t-1)^n$. By Cayley-Hamilton, $(A - I_n)^n = 0$.

   (Many of you tried to appeal to the theory of the Jordan canonical form, which meant doing a messy calculation.)

6. Find *two* orthogonal matrices of the form

$$\begin{pmatrix} \sqrt{3}/2 & * \\ 1/2 & * \end{pmatrix}$$

   (In other words, find two different ways to fill in the second column of this matrix in such a way that the result is an orthogonal matrix.)

   Solution. The two matrices are

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}.$$

   The first is a rotation and the second is a reflection.

7. Let A be a 3×3 complex matrix for which A^3 is the zero matrix, but A^2 is not the zero matrix. Find the Jordan canonical form of A.

   Solution. Let J be the Jordan canonical form of A, so that $J = QAQ^{-1}$ for some invertible matrix Q. Since $A^3 = 0$ it quickly follows that $J^3 = 0$. This means that the only eigenvalue of J is 0. A priori there are three possibilities for J, but the fact that A^2 is nonzero means that all the cycles in J have to have length at least 3. Since J is 3×3 this means there is exactly one cycle of length exactly 3. We get

$$J = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$

8. Let V be an inner product space with orthonormal basis v_1, v_2, v_3. Let $W = \{v_1\}^\perp$. Show that W is the space spanned by v_2 and v_3.

   Solution. Once again I was somewhat unforgiving. Here it was important to prove both containments: W is contained in the span, and the span is contained in W.

   Since v_2 and v_3 are both orthogonal to v_1 we have $v_2, v_3 \in W$, and therefore the span of v_2 and v_3 is contained in W.

   For the other direction, suppose $w \in W$. Write $w = a_1v_1 + a_2v_2 + a_3v_3$, where each $a_i \in F$. Since w is orthogonal to v_1 we have

$$0 = \langle w, v_1 \rangle = a_1 \langle v_1, v_1 \rangle + a_2 \langle v_2, v_1 \rangle + a_3 \langle v_3, v_1 \rangle = a_1,$$

   so that in fact $w = a_2v_2 + a_3v_3$ lies in the span of v_2 and v_3.

9. Let u and v be vectors in a real inner product space. Show that $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ if and only if u and v are orthogonal.

   Solution. We have

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + \|v\|^2 + 2 \langle u, v \rangle$$

   so that the desired equality holds if and only if $\langle u, v \rangle = 0$.

10. Let A be a real $n \times n$ matrix. Show that if A is orthogonally equivalent to a real diagonal matrix, then A is symmetric.

Solution. The hypothesis means that $A = P^t DP$ for D a real diagonal matrix and P an orthogonal matrix. We find

$$A^t = (P^t DP)^t = P^t D^t P = P^t DP = A,$$

so that A is symmetric.