MA442 Midterm

Due Mar. 1

The first three problems are computational/short answer format, so that you'll be graded more or less on the correctness of your answer (though it wouldn't hurt to clearly lay out the steps you use). In contrast, the last three problems require rigorous proofs. Therefore I will be unforgiving if your proof is confusing, vague, or uses variables in an imprecise way, even if I can tell you have the basic idea of the proof. Remember the proof techniques we covered in lecture: how to prove an if-then statement, how to use the word "let", how to proceed linearly from one statement to the next, and so on.

You may consult your lecture notes and the textbook, but not other books, internet sources, or anyone else.

1. Decide if $(0,-4,-14,-40)$ is in the span of the vectors $(1,1,1,1)$, $(1,2,4,8)$ and $(1,3,9,27)$ in $\mathbb{R}^4$.

2. Let $P_3(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most 3. Let $W \subset P_3(\mathbb{R})$ denote the subspace consisting of polynomials f which satisfy the equations

$$\begin{aligned} f(0) &= f'(0) \\ f(1) &= f'(1). \end{aligned}$$

   Find a basis for W.

3. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let $T\colon V \to V$ be the function defined by $(Tf)(x) = f(x^2)$. Thus for instance T takes the function $\sin(x)$ to $\sin(x^2)$. Is T a linear transformation? Briefly justify your answer.

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation. Show that there exist scalars a, b and c such that $T(x,y,z) = ax+by+cz$ for all $(x,y,z) \in \mathbb{R}^3$.

5. Let $T\colon V \to W$ be a linear transformation between two vector spaces. Show that if $v_1, \dots, v_n \in V$ are vectors and $\{T(v_1), \dots, T(v_n)\}$ is linearly independent, then so is $\{v_1, \dots, v_n\}$.

6. Let $T\colon V \to W$ be a linear transformation between two vector spaces, and let $U\colon W \to X$ be another. Recall that $UT\colon V \to X$ is the linear transformation defined by $UT(v) = U(T(v))$ for all $v \in V$. Show that if T and U are both one-to-one, then so is UT.