MA442 Midterm Solutions

1. Decide if $(0, -4, -14, -40)$ is in the span of the vectors $(1, 1, 1, 1)$, $(1, 2, 4, 8)$ and $(1, 3, 9, 27)$ in $\mathbb{R}^4$.

   Solution. It is.

2. Let $P_3(\mathbb{R})$ be the vector space of polynomials with real coefficients of degree at most 3. Let $W \subset P_3(\mathbb{R})$ denote the subspace consisting of polynomials f which satisfy the equations

$$\begin{aligned} f(0) &= f'(0) \\ f(1) &= f'(1). \end{aligned}$$

   Find a basis for W.

   Solution. Any polynomial in $P_3(\mathbb{R})$ is of the form $f(x) = a+bx+cx^2+dx^3$. The given equations imply that

$$\begin{aligned} a &= b \\ a+b+c+d &= b+2c+3d, \end{aligned}$$

   a system of two linear equations is four unknowns. Eliminating b from the second equation gives $a = c + 2d$. Thus we may take c and d to be our independent variables (parameters), and a and b depend on these. A basis for W is given by setting (c, d) as $(1, 0)$ and $(0, 1)$, respectively, giving the basis $\left\{1 + x + x^2, 2 + 2x + x^3\right\}$.

3. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let $T\colon V \to V$ be the function defined by $(Tf)(x) = f(x^2)$. Thus for instance T takes the function $\sin(x)$ to $\sin(x^2)$. Is T a linear transformation? Briefly justify your answer.

   Solution. It is. The appearance of the x^2 term is just a red herring. To check that T is linear, we must show that for $f, g \in V$ and $c \in \mathbb{R}$, we have $T(f + g) = T(f) + T(g)$ and $T(cf) = cT(f)$. To check that $T(f+g) = T(f)+T(g)$, we can show that $T(f+g)(x) = (T(f)+T(g))(x)$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned} T(f+g)(x) &= (f+g)(x^2) \\ &= f(x^2) + g(x^2) \\ &= T(f)(x) + T(g)(x) \\ &= (T(f) + T(g))(x). \end{aligned}$$

A similar argument works for $T(cf) = cT(f)$. On the other hand, if $U\colon V \to V$ is the function defined by $(Uf)(x) = f(x)^2$, then U is not a linear transformation.

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation. Show that there exist scalars a, b and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$.

   Solution. Let

   $$\begin{aligned} a &= T(1,0,0) \\ b &= T(0,1,0) \\ c &= T(0,0,1) \end{aligned}$$

   Then for all $(x, y, z) \in \mathbb{R}^3$, we have

   $$\begin{aligned} T(x,y,z) &= T(x(1,0,0) + y(0,1,0) + z(0,0,1)) \\ &= xT(1,0,0) + yT(0,1,0) + z(0,0,1) \\ &= ax + by + cz \end{aligned}$$

5. Let $T\colon V \to W$ be a linear transformation between two vector spaces. Show that if $v_1, \ldots, v_n \in V$ are vectors and $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent, then so is $\{v_1, \ldots, v_n\}$.

   Solution. Suppose that there are scalars $a_1, \ldots, a_n$ with

   $$a_1 v_1 + \cdots + a_n v_n = 0.$$

   Applying T (and using the fact that T is linear) gives

   $$a_1 T(v_1) + \cdots + a_n T(v_n) = 0.$$

   Since $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent, all of the a_i are zero. This shows that $\{v_1, \ldots, v_n\}$ is linearly independent.

6. Let $T\colon V \to W$ be a linear transformation between two vector spaces, and let $U\colon W \to X$ be another. Recall that $UT\colon V \to X$ is the linear transformation defined by $UT(v) = U(T(v))$ for all $v \in V$. Show that if T and U are both one-to-one, then so is UT.

   Solution. A linear transformation is one-to-one if and only if its nullspace is 0. Let v be a vector in the nullspace of UT. Then $UT(v) = 0$, so that $U(T(v)) = 0$. This means that $T(v)$ lies in the nullspace of U, but since U is one-to-one, $T(v) = 0$. Thus v is in the nullspace of T, but since T is one-to-one, $v = 0$. We have shown that the only vector in the nullspace of UT is the zero vector, so that UT is one-to-one.