LECTURE APRIL 3, 6, 8: EIGENVECTORS AND EIGENVALUES

1. Review of the definitions

Let A be an $n \times n$ matrix, or a linear operator on a vector space. In this lecture we consider the equation

$$Av = \lambda v,$$

where v is a *nonzero* vector and λ is a scalar. In this situation we say that v is an *eigenvector* of A, with eigenvalue λ.

The easiest example of this occurs when A is a diagonal matrix, such as

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

If e_1, e_2 represent the standard basis for $\mathbf{R}^2$, then

$$Ae_1 = 3e_1, \; Ae_2 = 2e_2.$$

Thus e_1 is an eigenvector for A with eigenvalue 2, and e_2 is an eigenvector for A with eigenvalue 3.

2. An example of diagonalization

Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.$$

We might be interested in calculating its powers $A^2, A^3, \ldots$. This is easy to do for small powers, but if the power (or the matrix) gets large, it becomes cumbersome. (We also might be interested in the limiting properties of A^n as $n \to \infty$: this is important for graph theory.)

Let

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Then

$$Av_1 = -2v_1, \; Av_2 = 5v_2.$$

Then $B = \{v_1, v_2\}$ is a basis for $\mathbf{R}^2$. Let's write this in a different way:

$$\begin{aligned} Av_1 &= -2v_1 + 0v_2 \\ Av_2 &= 0v_1 + 5v_2 \end{aligned}$$

What this means is that the matrix of A with respect to B is

$$[A]_B = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix},$$

a diagonal matrix.

If we let

$$Q = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$$

be the change of basis matrix from B to the standard basis, then

$$[A]_B = Q^{-1}AQ.$$

Thus

$$Q^{-1}AQ = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}.$$

We took our original matrix A, and found an invertible matrix Q, such that $Q^{-1}AQ$ is a diagonal matrix. This is called *diagonalization*.

We can solve for A in this equation:

$$A = Q \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} Q^{-1}.$$

Then

$$\begin{aligned} A^2 &= Q \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} Q^{-1} Q \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} Q^{-1} \\ &= Q \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}^2 Q^{-1} \\ &= Q \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix} Q^{-1}. \end{aligned}$$

The pattern holds up for any power:

$$A^n = Q \begin{pmatrix} (-2)^n & 0 \\ 0 & 5^n \end{pmatrix} Q^{-1}.$$

3. How do we find eigenvectors and eigenvalues?

Remember that the equation is

$$Av = \lambda v.$$

If I is the identity matrix as usual, then λI is the scalar matrix. Observe that

$$\lambda I v = \lambda v.$$

So we're looking at

$$Av = \lambda I v.$$

We can rewrite this as

$$Av - \lambda I v = 0,$$

or

$$(A - \lambda I)v = 0.$$

Thus, v lies in the null space of $A - \lambda I$. If $v \neq 0$, it means that this null space is nontrivial. This means that $A - \lambda I$ is not invertible, and therefore

$$\det(A - \lambda I) = 0.$$

This equation is called the *characteristic equation* of A, and $\det(A - \lambda I)$ is the *characteristic polynomial* of the matrix A. The roots of $\det(A - \lambda I)$ are the eigenvalues of A.

Once again, let

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}.$$

How could we find its eigenvectors and eigenvalues? Let's compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix}.$$

The determinant of this matrix is

$$\det(A - \lambda I) = (1 - \lambda)(2 - \lambda) - 3(4) = \lambda^2 - 3\lambda - 10$$

To find the eigenvalues, we now have to solve

$$\lambda^2 - 3\lambda - 10 = 0$$

The polynomial factors as $(\lambda + 2)(\lambda - 5)$, so the roots are $\lambda = -2, 5$.

How do we find the eigenvectors? We proceed one eigenvalue at a time. So first, we'll look for the eigenvectors of $\lambda = -2$. This would be a vector v in the null space of $A - \lambda I$. Now,

$$A - \lambda I = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}.$$

We find that $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in the null space of this matrix. Therefore v_1 is an eigenvector of A with value $\lambda = -2$.

Similarly, for $\lambda = 5$, we have

$$A - \lambda I = \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix}.$$

A vector in the null space is $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Therefore v_2 is an eigenvector for A with eigenvalue 5.

Example 3.1. *Let* $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *What are the eigenvectors and eigenvalues of* A*?*

The only eigenvalue is 2. For any $v = \begin{pmatrix} a \\ b \end{pmatrix}$, we have

$$Av = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} = 2v$$

This matrix is $A = 2I$. All nonzero vectors in $\mathbf{R}^2$ are eigenvectors for A with eigenvalue 2.

4. How could a matrix not be diagonalizable?

Here's another example. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This matrix represents a rotation of $\pi/2$ in the plane. What are its eigenvalues and eigenvectors? The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1.$$

The roots of this polynomial are $\{i, -i\}$, which are not real numbers. So we might say that the matrix $A \in M_{2\times 2}(\mathbf{R})$ has no eigenvalues or eigenvectors, and it is not diagonalizable.

One more example of a different flavor. Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2.$$

The only eigenvalue is $\lambda = 2$. Let's look for eigenvectors. We have

$$A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The eigenvectors of A will be the nonzero vectors in the null space of this matrix. The null space of this matrix is one-dimensional, and it is spanned by $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This matrix is *not* diagonalizable, as you cannot find a basis for $\mathbf{R}^2$ consisting of eigenvectors for A.

5. Diagonalizability and Eigenspaces

To review: Let $T\colon V \to V$ be a linear operator. If $v \neq 0$ is a vector in V, and

$$Tv = \lambda v,$$

then we say that v is an eigenvector of T with eigenvalue λ.

If there is a basis for V consisting of eigenvectors, then we call T *diagonalizable.*

If V is a finite-dimensional vector space, and B is a basis for V consisting of eigenvectors, then

$$[T]_B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is a diagonal matrix.

If T happens to be the linear transformation associated to an $n \times n$ matrix A, then T being diagonalizable is the same as saying that there exists an invertible matrix Q such that

$$Q^{-1}AQ$$

is a diagonal matrix. Thus, a matrix being diagonalizable means that it is similar to a diagonal matrix.

Again in the context of an $n \times n$ matrix A, a scalar λ is an eigenvalue of A if and only if $(A - \lambda I)$ is not invertible. Then, eigenvectors of A with this value are exactly the nonzero vectors in the null space $N(A - \lambda I)$. We have that $A - \lambda I$ is not invertible if and only if $\det(A - \lambda I) = 0$, so that suggests looking at the *characteristic polynomial*

$$f(t) = \det(A - tI).$$

This is always a polynomial of degree n. Its roots are the eigenvalues of A.

Theorem 5.1. *Let $T\colon V \to V$ be a linear operator. Suppose that $v_1, \ldots, v_k$ are eigenvectors for T, with* distinct *eigenvalues. Then $\{v_1, \ldots, v_k\}$ is linearly independent.*

Note that $\{1, 3, 5, 7\}$ is a list of distinct numbers, but $\{1, 1, 2, 3\}$ is not.

Proof. Proof here is by induction on k. For $k = 1$, we only have one eigenvector v_1. By definition of eigenvector, $v_1 \neq 0$, and thus $\{v_1\}$ is linearly independent.

Suppose we have proved the theorem for all sets of vectors of size $< k$. Let $v_1, \ldots, v_k$ be a eigenvectors for T, with eigenvalues $\lambda_1, \ldots, \lambda_k$ which are distinct. Thus $Tv_i = \lambda_i v_i$ for $i = 1, \ldots, k$.

Assume that

$$a_1 v_1 + \cdots + a_k v_k = 0.$$

Apply T to both sides:

$$T(a_1 v_1 + \cdots + a_k v_k) = T(0).$$

We get

$$a_1 T(v_1) + \cdots + a_k T(v_k) = 0,$$

or

$$a_1\lambda_1 v_1 + \cdots + a_k\lambda_k v_k = 0.$$

In order to remove the final variable v_k, I take λ_k times the first equation, and subtract it from the last. We get

$$a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_1 - \lambda_k)v_2 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0.$$

We now have a linear combintation of the eigenvectors $v_1, \ldots, v_{k-1}$ which equals 0. By the inductive hypothesis, those vectors are linearly independent, and so

$$\begin{aligned} a_1(\lambda_1 - \lambda_k) &= 0 \\ a_2(\lambda_2 - \lambda_k) &= 0 \\ &\vdots \\ a_{k-1}(\lambda_{k-1} - \lambda_k) &= 0 \end{aligned}$$

Since the λs were assumed distinct, all of the $\lambda_i - \lambda_k$ are nonzero. Therefore, $a_1 = \cdots = a_{k-1} = 0$. We get $a_k v_k = 0$, and that implies $a_k = 0$ as well. □

Theorem 5.2. *Suppose V is an n-dimensional vector space, and that $T\colon V \to V$ is a linear operator with n distinct eigenvalues. Then T is diagonalizable.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the n distinct eigenvalues of T. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors. By the previous theorem, $v_1, \ldots, v_n$ are linearly independent. Since $\dim V = n$, these vectors form a basis for V. Therefore T is diagonalizable. □

So suppose A is an $n \times n$ matrix. Let $f(t) = \det(A - tI)$ be its characteristic polynomial. For A to have n distinct eigenvalues means the same as saying that

$$f(t) = \pm(t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$$

factors completely into linear factors, with distinct roots. If that is the case, then A is diagonalizable.

Let

$$A = \begin{pmatrix} 1 & 7 \\ 4 & 2 \end{pmatrix}$$

Is the matrix A diagonalizable? (I mean over the real numbers.) The characteristic polynomial is:

$$f(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 7 \\ 4 & 2-t \end{pmatrix} = (1-t)(2-t) - 28 = t^2 - 3t - 26$$

For 2×2 matrices, there's a shortcut. The characteristc polynomial of a 2×2 is always

$$f(t) = t^2 - (\mathrm{tr}A)t + \det A.$$

The trace is the sum along the diagonal entries: $1 + 2 = 3$. How do we find the roots of $f(t)$? Use the quadratic formula: the roots are:

$$\lambda_1, \lambda_2 = \frac{3 \pm \sqrt{113}}{2}.$$

Since λ_1 and λ_2 are distinct real numbers, the matrix A is diagonalizable.

The converse to this theorem is *false.* Consider the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

What are the eigenvalue(s)? Only 2. What are the eigenvector(s)? *All* nonzero vectors in $\mathbf{R}^2$ are eigenvectors for A. Is A diagonalizable? Yes, because there is a basis consisting of eigenvectors, namely $(1,0), (0,1)$.

Let's say you found your characteristic polynomial for a 5×5 matrix, and it was:

$$f(t) = -(t-2)^3(t-4)(t+7),$$

The roots of $f(t)$ are $2, 4, -7$. The eigenvalue 2 appears with *algebraic multiplicity* 3, whereas 4 and -7 appear with algebraic multiplicity 1.

6. What can happen with 2×2 matrices

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 matrix with real entries. Is A diagonalizable? This means, can I find an invertible matrix Q so that

$$Q^{-1}AQ = D$$

is a diagonal matrix? Another way of asking: can I find a basis for $\mathbf{R}^2$ consisting of eigenvectors for A?

Let

$$\begin{aligned} T &= a + d \\ D &= ad - bc \end{aligned}$$

be the trace and determinant of A, respectively. Then the characteristic polynomial of A is

$$f(t) = t^2 - Tt + D.$$

How do I know whether this polynomial factors? You look at the discriminant

$$Disc = T^2 - 4D.$$

If $Disc > 0$, then the polynomial has two distinct roots α, β:

$$f(t) = (t - \alpha)(t - \beta).$$

Then α and β are the eigenvalues. In this case A is automatically diagonalizable, and in fact it will be similar to the diagonal matrix

$$D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

If $Disc < 0$, it means the eigenvalues are complex. Then A is not diagonalizable, over the real numbers anyway. But if you allow yourself the use of complex numbers, then A is diagonalizable: it is similar to a diagonal matrix with *complex* entries.

If $Disc = 0$, then our characteristic polynomial factors like this:

$$f(t) = (t - \alpha)^2$$

So there is only one eigenvalue α. For A to be diagonalizable in this case, would mean that there exists a basis for $\mathbf{R}^2$, say v_1 and v_2, consisting of eigenvectors with value α: $Av_1 = \alpha v_1$ and $Av_2 = \alpha v_2$. If $x \in \mathbf{R}^2$ is any vector whatsoever, then we can write x as a linear combination of v_1 and v_2:

$$x = c_1 v_1 + c_2 v_2,$$

and then

$$Ax = A(c_1 v_1 + c_2 v_2) = c_1 A(v_1) + c_2 A(v_2) = c_1 \alpha v_1 + c_2 \alpha v_2 = \alpha x$$

Therefore the matrix must be

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha I$$

Thus: If our 2×2 matrix only has one eigenvalue (meaning its $Disc = 0$) but it is not a scalar matrix, then it is *not* diagonalizable.

7. Eigenspaces

Let $T: V \to V$ be a linear operator. If λ is a scalar, I let

$$E_\lambda = \left\{ v \in V \,\middle|\, Tv = \lambda v \right\}$$

Thus E_λ is the set of all eigenvectors with value λ, but I am also including the zero vector in this.

Theorem 7.1. *E_λ is a subspace of V.*

In fact $E_\lambda = N(T - \lambda I)$ is the nullspace of the operator $T - \lambda I$. Indeed if $Tv = \lambda v$, then

$$(T - \lambda I)v = Tv - \lambda I v = Tv - \lambda v = 0.$$

Assume that V has finite dimension, say $V = \mathbf{R}^n$, and that T is the linear operator associated to an $n \times n$ matrix A. How do I test for diagonalizability of A?

First thing would be to find the characteristic polynomial:

$$f(t) = \det(A - tI)$$

Then $f(t)$ is a polynomial of degree t. We say that $f(t)$ *splits* if we can factor it completely into linear factors:

$$f(t) = \pm(t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n).$$

For A to be diagonalizable, it is necessary that $f(t)$ splits. If all of the roots λ_i are distinct, then A is automatically diagonalizable. But generally there will multiplicity:

$$f(t) = \pm(t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}.$$

The number m_i is the *algebraic multiplicity* of λ_i. By the way, the sum of the m_is must equal the degree of $f(t)$, which is n.

The *geometric multiplicity* of λ_i is the dimension of E_{λ_i}.

Theorem 7.2. *Given an eigenvalue λ, the geometric multiplicity is always less than or equal to the algebraic multiplicity. If the two multiplicities are equal for all λ, then the matrix is diagonalizable.*

For instance, let

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

The characteristic polynomial is:

$$f(t) = t^2 - 4t + 4 = (t - 2)^2.$$

Therefore the algebraic multiplicity of 2 is 2. The geometric multiplicity is the dimension of $E_2 = N(A - 2I)$. We have

$$A - 2I = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The nullity of $A - 2I$ is 1, and therefore 1 is the geometric multiplicity. We have $1 < 2$, and so this matrix is not diagonalizable.

8. Transition matrices

In a certain population, 80% are healthy and 20% are sick. Each day, among the healthy people, 90% stay healthy, and 10% get sick. Also each day, among the sick people, 2% recover, and 98% stay sick. What is the behavior of the population's health as time goes on?

Let A be the transition matrix for this situation:

$$A = \begin{pmatrix} .9 & .02 \\ .1 & .98 \end{pmatrix}$$

The first column represents people who start the day healthy, and the second column represents people who start the day sick. The first row represents people who finish the day healthy, and the second row represents people who finish the day sick. Each entry of A represents the proportion of people who *transition* from one state to another.

(Check: the columns of a transition matrix must add to 1.) Let $v = \begin{pmatrix} .8 \\ .2 \end{pmatrix}$ be the vector determining the ratio of healthy to sick people on the first day. At the end of the first day, we look at :

$$Av = \begin{pmatrix} .9 & .02 \\ .1 & .98 \end{pmatrix} \begin{pmatrix} .8 \\ .2 \end{pmatrix} = \begin{pmatrix} (.9)(.8) + (.02)(.2) \\ (.1)(.8) + (.98)(.2) \end{pmatrix}.$$

This new vector tells us the proportion of healthy to sick people at the end of the first day. The sequence of vectors

$$v, Av, A^2v, A^3v, \ldots$$

tells us how the system evolves with time.

With diagonalization, we can predict what will happen. The characteristic polynomial of A is

$$f(t) = t^2 - 1.88t + .88 = (t-1)(t-.88).$$

Thus the eigenvalues of A are 1 and .88. These are distinct, and so we know that A is diagonalizable. The eigenvector for 1 is $v_1 = \begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}$, and the eigenvector fo .88 is $v_2 = \begin{pmatrix} -1/6 \\ 1/6 \end{pmatrix}$. The significance of $Av_1 = v_1$ is that a population with a 1:5 ratio of healthy to six people will stay that way forever: it is a *stable state.*

If we let $Q = \begin{pmatrix} 1/6 & -1/6 \\ 5/6 & 1/6 \end{pmatrix}$, then

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & .88 \end{pmatrix}.$$

Solving for A gives

$$A = Q \begin{pmatrix} 1 & 0 \\ 0 & .88 \end{pmatrix} Q^{-1}.$$

Then powers of A are

$$A^n = Q \begin{pmatrix} 1 & 0 \\ 0 & .88^n \end{pmatrix} Q^{-1}.$$

for large n, the number $.88^n$ is very close to 0. So the limiting behaviour of the system is modeled on the matrix

$$Q \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} = \begin{pmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{pmatrix}$$

This shows us that no matter what initial state $v = \begin{pmatrix} a \\ b \end{pmatrix}$ we start with (so long as $a + b = 1$ so that it makes sense as a ratio), the limiting state is

$$\lim_{n\to\infty} A^n v = \begin{pmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}.$$

Thus the model predicts that after a long time, 1/6 of the population will be healthy, and 5/6 will be sick.