LECTURE MARCH 16: THE CHANGE OF COORDINATE MATRIX

1. Coordinates: what were those again?

My slogan is that *coordinates* of a point P in space are a list of numbers $(x_1, \dots, x_n)$ which tell you exactly where P is, with no redundancy. A *coordinate system* is a way of assigning coordinates to each point P. There may be multiple coordinate systems that apply, so that P may have one set of coordinates in one system, and a different set in another. Think of a street corner in Manhattan: the pair (street, avenue) is one set of coordinates, and (latitude, longitude) is another. Today we'll talk about how to translate from one coordinate system to another, by means of the *change of basis matrix*.

The coordinate system we learn about is $\mathbf{R}^2$, the plane. We learned a long time ago that a point in the plane can be represented by a pair of numbers (x, y), so that this is a very simple coordinate system. Same for $\mathbf{R}^n$: a point v in $\mathbf{R}^n$ is represented by a list of numbers $(x_1, \dots, x_n)$, rather by definition. These are the "standard coordinates" of v.

In the context of finite-dimensional vector spaces V, there is a connection between ordered bases and coordinate systems. Let $B = \{v_1, \dots, v_n\}$ be an ordered basis for V. Then every vector v can be written *uniquely* as a linear combination of vectors in B:

$$v = x_1 v_1 + \cdots + x_n v_n.$$

The numbers $x_1, \dots, x_n$ tell you exactly what v is, and so we refer to the coordinate of v with respect to B as follows:

$$[v]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example 1.1 (The standard basis in $\mathbf{R}^n$).

The standard basis in $\mathbf{R}^n$ is

$$\begin{aligned} e_1 &= (1, 0, \dots, 0) \\ e_2 &= (0, 1, \dots, 0) \\ \vdots &= \vdots \\ e_n &= (0, 0, \dots, 1) \end{aligned}$$

If

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then

$$v = x_1 e_1 + \cdots + x_n e_n.$$

If B is the standard basis, then

$$[v]_B = v.$$

2. Change of basis in $\mathbf{R}^n$ from the standard basis to another basis

Let $B = \{e_1, \ldots, e_n\}$ be the standard basis in $\mathbf{R}^n$, and let $B' = \{v_1, \ldots, v_n\}$ be a different ordered basis. Given a vector $v \in R^n$, it's easy to find $[v]_B$, but how do we find $[v]_{B'}$?

Let's give names to the vectors in B':

$$v_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, v_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

Another way of stating this is:

$$\begin{aligned} v_1 &= a_{11} e_1 + \cdots + a_{n1} e_n \\ v_2 &= a_{12} e_2 + \cdots + a_{n2} e_n \\ \vdots &= \vdots \\ v_n &= a_{1n} e_2 + \cdots + a_{nn} e_n \end{aligned}$$

Now, let's gather the coefficients a_{ij} into an $n \times n$ matrix Q:

$$Q = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

So, each columns of Q is the coordinate vector for one of the basis vectors in B'. In our formalism for naming the matrix attached to a linear transformation:

$$Q = [I_{\mathbf{R}^n}]_{B'}^{B}.$$

It turns out this matrix Q is useful for converting standard coordinates into B'-coordinates.

Theorem 2.1. *The matrix Q is invertible. If $v \in \mathbf{R}^n$, then $v = Q[v]_{B'}$, and therefore*

$$[v]_{B'} = Q^{-1} v.$$

Example 2.2. *With* $\mathbf{R}^2$ *we have our standard basis* $B = \{e_1, e_2\}$. *Let* $B' = \{v_1, v_2\}$, *where*

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Given $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, *find* $[v]_{B'}$.

Question is the same as asking: express $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a linear combination of the vectors v_1 and v_2. The change of coordinate matrix from B' to B is:

$$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The inverse of this matrix is

$$Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Therefore by the theorem,

$$[v]_{B'} = Q^{-1}v = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1/2 \end{pmatrix}$$

As a check:

$$\frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

3. Change between any two bases

Now let's say $B = \{v_1, \ldots, v_n\}$ and $B' = \{v_1', \ldots, v_n'\}$ are any two bases for a finite-dimensional vector space V. Given a vector $v \in V$, we can form the vectors $[v]_B$ and $[v]_{B'}$. Same vector, but two different coordinate systems. How are they related?

Again, the difference between the two coordinate vectors is meditated by a matrix. Take the vectors in B':

$$v_1', \ldots, v_n'$$

and find the coordinate vectors for each of them with respect to B:

$$[v_1']_B, \ldots, [v_n']_B.$$

Collect these into a matrix and call it Q:

$$Q = \big([v_1']_B \cdots [v_n']_B\big)$$

This is the *change of basis matrix* $Q = [I_V]_{B'}^B$, which converts B'-coordinates into B-coordinates:

Theorem 3.1. *The matrix* Q *is invertible. For a vector* $v \in V$, *we have* $[v]_B = Q[v]_{B'}$.

Example 3.2. *Let* $P_2(\mathbf{R})$ *be the vector space of polynomials of degree* $\leqslant 2$. *Let* $B = \{1, x, x^2\}$, *and let* $B' = \{1, 1+x, (1+x)^2\}$. *Find the change of basis matrix from* B' *to* B, *and also the change of basis matrix from* B' *to* B.

Solution: we write each of the basis vectors in B' as a linear combination of vectors in B:

$$\begin{aligned} 1 &= 1\cdot 1 + 0x + 0x^2 \\ 1+x &= 1\cdot 1 + 1x + 0x^2 \\ (1+x)^2 &= 1\cdot 1 + 2x + 1x^2 \end{aligned}$$

The change of basis matrix from B' to B is:

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

To find the change of basis matrix from B to B', we have to write each of the basis vectors in B as a linear combination of vectors in B'.

$$\begin{aligned} 1 &= 1\cdot 1 + 0(1+x) + 0(1+x)^2 \\ x &= -1\cdot 1 + 1(1+x) + 0(1+x)^2 \\ x^2 &= 1\cdot 1 - 2(1+x) + (1+x)^2 \end{aligned}$$

The change of basis matrix from B to B' is therefore

$$R = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that

$$QR = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Thus $R = Q^{-1}$.

4. The matrix of a linear transformation $T\colon V \to V$

We have spent a lot of time talking about linear transformations $T\colon V \to W$. This section concerns linear transformations $T\colon V \to V$, where the domain and codomain are the same finite-dimensional vector space V. These are sometimes called *linear operators*.

We have seen that there is a strong connection between linear transformations and matrices. To make this connection, we need an ordered basis for V, say $B = \{v_1, \ldots, v_n\}$. The important observation is that T is completely determined by what it does to the basis vectors. For $i = 1, \ldots, n$, we have a vector $T(v_i) \in V$, which has a coordinate vector

$$[T(v_j)]_B = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj}. \end{pmatrix}$$

Gather these together as column vectors into a matrix:

$$A = [T]_\beta^\beta = [T]_\beta.$$

This is the *matrix of T with respect to B*.

Example 4.1. *We have seen that the differentiation map*

$$T\colon P_2(\mathbf{R}) \to P_2(\mathbf{R})$$

which sends $f(x)$ to $f'(x)$ is a linear operator. Find its matrix with respect to $\{1, x, x^2\}$, and also with respect to $\{1, 1+x, (1+x)^2\}$.

First we'll do the case where the basis is $B = \{1, x, x^2\}$. We apply the linear operator T to all of the basis vectors:

$$\begin{aligned} T(1) = 0 &= 0 \cdot 1 + 0x + 0x^2 \\ T(x) = 1 &= 1 \cdot 1 + 0x + 0x^2 \\ T(x^2) = 2x &= 0 \cdot 1 + 2x + 0x^2 \end{aligned}$$

Each of the coordinate vectors gives a column in our matrix:

$$[T]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The reason we might be interested in $[T]_B$ is for the following reason:

Theorem 4.2. *Let V be a finite-dimensional vector space, with ordered basis B. Let $T\colon V \to V$ be a linear operator. Then for every $v \in V$ we have*

$$[T(v)]_B = [T]_B[v]_B$$

As an example, let $v = 3x^2 - 13x + 7$. Using our computation of $[T]_B$ from before, we're going to take its derivative. We have

$$[T(v)]_B = [T]_B[v]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 7 \\ -13 \\ 3 \end{pmatrix} = \begin{pmatrix} -13 \\ 6 \\ 0 \end{pmatrix}$$

Meanwhile, $T(v) = 6x - 13$. So we confirm that $[T(v)]_B$ is as the theorem predicts.

5. An analogy

I'd like to start this section with an analogy. Let's say you're on a trip to Mexico, and someone asks you,

(v): *¿Cuál es la fecha de hoy?*

You don't know Spanish, and they don't know English. So what do you do? How can you systematically answer questions in another language?

Of course, you could translate the question into English, answer it, and then translate it back into Spanish. Let's represent these operations using symbols.

Let's say:

- Q is a dictionary from Spanish to English.
- A is a procedure for answering English questions with English answers.
- A' is a procedure for answering Spanish questions with Spanish answer.

What we want but do not have is A'. How do we obtain it in terms of Q, A?

$$A' = Q^{-1}AQ$$

Let's see this in action:

$$\begin{aligned} A'(v) &= Q^{-1}AQ(v) \\ &= Q^{-1}A(\text{What is today's date?}) \\ &= Q^{-1}(\text{It is the 18th of March}). \\ &= \text{Es el 18 de marzo.} \end{aligned}$$

6. Similar matrices

We'll now return to linear algebra.

Say $T\colon V \to V$ is a linear operator on an n-dimensional vector space, and say that B and B' are bases for V. We have $n \times n$ matrices

$$\begin{aligned} A &= [T]_B \\ A' &= [T]_{B'} \end{aligned}$$

If you like, you can think of B as the "familiar" basis, and B' as the "foreign" basis. Also you can think of A as the "answering in English" procedure, and A' as the "answering Spanish questions in Spanish" procedure. We have already seen that there is a change of basis matrix Q from B' to B, the "translator".

(If you are philosophically inclined, you can think of V as a collection of ideas that are pre-linguistic, without any language, and T as a procedure which answers questions, also without any language.)

Theorem 6.1. *The matrices $[T]_{B'}$ and $[T]_B$ are related by*

$$[T]_{B'} = Q^{-1}[T]_B Q.$$

If A and A' are $n \times n$ matrices, and there exists an invertible $n \times n$ matrix Q for which $A' = Q^{-1}AQ$, then A and A' are called *similar matrices.*

Example 6.2. *Let $B = \{e_1, e_2\}$ be the standard basis for $\mathbf{R}^2$, and let $B' = \{v_1, v_2\}$ be the same basis rotated 45° counterclockwise:*

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection about the x-axis. Find $[T]_B$, find $[T]_{B'}$, find the change of basis matrix Q from B' to B, and verify the theorem: $[T]_{B'} = Q^{-1}[T]_B Q$.

To find $[T]_B$, we take each of our basis vectors e_1, e_2, and apply T:

$$\begin{aligned} T(e_1) &= 1e_1 + 0e_2 \\ T(e_2) &= 0e_1 + (-1)e_2 \end{aligned}$$

I use each vector here as a column in my matrix

$$[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

The change of basis matrix Q is

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

(This is the rotation matrix for the angle of $\pi/4$.) Therefore

$$[T]_{B'} = Q^{-1}[T]_B Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

this is

$$[T]_{B'} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

This is also a reflection matrix, through the line $y = -x$. Thus our two reflection matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (reflection through $y = 0$) and $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (reflection through $y = -x$) are similar. It is no coincidence that $y = -x$ is rotated from $y = 0$ by an angle of $\pi/4$!.