# LECTURES MARCH 20 AND MARCH 23: ROW OPERATIONS AND ELEMENTARY MATRICES

## 1. Our three row operations, and their relation to matrices

We have already seen how crucial it is to be able to perform *row operations* on matrices, as an aid to solving systems of linear equations. Row operations are combinations of *elementary row operations*, which the book numbers these relations this way:

(1) Switch row $i$ with row $j$.
(2) Multiply row $i$ by a nonzero scalar.
(3) Add a scalar multiple of row $i$ to row $j$.

There isn't anything special about rows versus columns. We can also talk about elementary column operations of type (1),(2),(3). Together, the six types of procedures are called *elementary operations*.

Today we'll learn a new interpretation of what is happening in terms of matrix multiplication. First, a definition:

**Definition 1.1.** *An* elementary matrix *is what you get when you perform an elementary row operation to the identity matrix.*

So we could start with the $3 \times 3$ identity matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and perform an operation of type (1):

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or an operation of type (2):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or an operation of type (3):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

The connection between elementary matrices and elementary operations is given by the following theorem.

**Theorem 1.2.** *Let $A$ be an $m \times n$ matrix. Doing an elementary row operation to $A$ returns $EA$, where $E$ is an $m \times m$ elementary matrix. In fact $E$ is the matrix obtained by doing the same row operation to $I_m$. Similarly, doing an elementary column operation to $A$ returns $AE$, where $E$ is an $n \times n$ elementary matrix, obtained similarly from $I_n$.*

Let's see this theorem in action with row operations. Here we have a typical $3 \times 4$ matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 4 & 1 \end{pmatrix}$$

In the course of row-reducing this matrix, we may want to subtract $-2$ times the first row and add it to the second. This row operation corresponds to the $3 \times 3$ matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -4 & -8 \\ 3 & 1 & 4 & 1 \end{pmatrix}$$

Thus, when we row reduce a matrix $A$ by applying successive elementary row operations, the result is a new matrix of the form

$$E_k \cdots E_3 E_2 E_1 A$$

where all the $E_i$ are elementary matrices. If I were to keep row reducing the matrix in the example, I would get a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{pmatrix}$$

This tell us already that the rank of the matrix is 3 and the dimension of the null space is 1. We can continue to change the matrix with *column operations* now, to get rid of the $x$,$y$ and $z$. For instance, multiplying the first column by $-x$ and adding it to the third gets rid of the $x$. The result it is that after taking $A$ and multiplying it on the left and the right by elementary matrices, the result is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

**Theorem 1.3.** *All elementary matrices are invertible.*

For instance, the inverse of the elementary matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

is

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

## 2. The rank of a matrix

I might remind you of a few definitions here.

**Definition 2.1.** *Let $V$ be a vector space, and let $v_1, \ldots, v_n$ be vectors. The* span *of $v_1, \ldots, v_n$ is the set of all linear combinations of these vectors.*

I'll remind you here that the span of $v_1, \ldots, v_n$ is always a subspace of $V$.

**Definition 2.2.** *Let $V$ and $W$ be finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. The* rank *of $T$ is the dimension of the range $R(T)$.*

In other words, let's suppose that $v_1, \ldots, v_n$ is a basis for $V$. Apply $T$ to all of these vectors to get $T(v_1), \ldots, T(v_n)$. Then the rank of $T$ is the dimension of the span of $T(v_1), \ldots, T(v_n)$.

Example: the rank of the zero transformation $T_0 : V \to W$ is always 0. Another example: the rank of the identity transformation $I_V : V \to V$ is always the same as the dimension of $V$ itself.

**Definition 2.3.** *The* rank *of an $m \times n$ matrix $A$ is the rank of the corresponding linear transformation $\mathbf{R}^n \to \mathbf{R}^m$.*

In fact, the rank of a matrix $A$ is the dimension of the span of its *columns*.

For example, the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank 2, because there are two linearly independent columns. We can think about $A$ as a linear transformation $\mathbf{R}^4 \to \mathbf{R}^3$. A basis for $\mathbf{R}^4$ consists of the standard basis vectors $e_1, e_2, e_3, e_4$. Calculation: $Ae_1$ is the first column, $Ae_2$ is the second column, etc. (!!!).

$$Ae_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So once again, the rank of a matrix is the dimension of the span of its columns. Another example:

$$\begin{pmatrix} 100 & 70 & 170 \\ 50 & 30 & 80 \\ 25 & 10 & 35 \\ 2 & 1 & 3 \end{pmatrix}$$

The rank of this matrix is 2, because the first two columns are linearly independent, but the third column is the sum of the first two.

**Theorem 2.4.** *Let $A$ be an $m \times n$ matrix. Performing elementary operations to $A$ does not change its rank.*

Sketch of proof: Performing an elementary operation to $A$ is like replacing $A$ with $EA$ or $AE$, where $E$ is an elementary matrix. Since $E$ is invertible, multiplying by $E$ doesn't change the rank.

**Theorem 2.5.** *Let $A$ be any matrix. After performing elementary operations to $A$ (row or column), we can obtain a matrix with some number of 1s going down the diagonal, and 0s in every other entry. The number of 1s is the rank of $A$.*

For instance, if we had started out with a $3 \times 4$ matrix, we would end up with exactly one of the following possibilities:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Or, if we had started with a $4 \times 3$ matrix, we would end up with exactly one of the following possibilities:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The number of 1s you see is the same as the rank.

Given an $m \times n$ matrix, what is the largest possible rank? The answer is the smaller of the two numbers $m$ and $n$.

## 3. Some more facts about the rank of a matrix

Remember from last time: the rank of an $m \times n$ matrix $A$ is the rank of the span of its columns. Another way of saying this is, the rank of $A$ is the maximum number of linearly independent columns.

We also learned some facts about row and column operations:

(1) Doing a row operation to $A$ changes $A$ to $EA$, where $E$ is an elementary matrix. Doing a column operation changes $A$ to $AE$, where $E$ is an elementary matrix. All elementary matrices $E$ are invertible: there exists another matrix $E^{-1}$ such that $EE^{-1} = E^{-1}E =$ the identity matrix.
(2) Doing successive row and column operations can bring $A$ into a really nice form: 1s going down the diagonal, and 0s everywhere else. The number of 1s in this new matrix is the rank of $A$.

Putting these facts together, we see that for any matrix $A$, there exist elementary matrices $E_1, \ldots, E_p$ and $G_1, \ldots, G_q$ such that

$$D = E_p \cdots E_1 A G_1 \cdots G_q$$

has 1s down the diagonal and 0s everywhere else. This fact has some really nice corollaries. To explore them, we'll review some basic facts.

**Theorem 3.1.** *Let $A$ and $B$ be invertible $n \times n$ matrices. Then $AB$ is also invertible, and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$.*

*Proof.* Let $A$ and $B$ be invertible matrices. Then there exist matrices $A^{-1}$ and $B^{-1}$ such that $AA^{-1} = BB^{-1} = I$. Then

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Similarly, $(AB)(B^{-1}A^{-1}) = I$. □

By induction, we see that if $A_1, \ldots, A_r$ are all invertible, then so is $A_1 \cdots A_r$, and

$$(A_1 \cdots A_r)^{-1} = A_r^{-1} \cdots A_1^{-1}.$$

There's another operation on matrices that works this way, which we haven't yet discussed.

**Definition 3.2.** *Let $A$ be an $m \times n$ matrix. The* transpose *of $A$ is the $n \times m$ matrix $A^t$ whose rows are the columns of $A$ and vice versa.*

Thus if

$$A = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 3 & 1 & 4 & 1 \\ 2 & 7 & 1 & 8 \end{pmatrix}$$

then

$$A^t = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 1 & 7 \\ 2 & 4 & 1 \\ 0 & 1 & 8 \end{pmatrix}$$

**Theorem 3.3.** *Let $A$ be an $m \times n$ matrix, and let $B$ be an $n \times p$ matrix, so that $AB$ is defined. We have*

$$(AB)^t = B^t A^t.$$

I won't prove this, but we can at least observe that $B^t$ is $p \times n$, and $A^t$ is $n \times m$, so that $B^t A^t$ is defined.

**Theorem 3.4.** *Let $A$ be any matrix. The rank of $A$ is the same as the rank of $A^t$. Thus, the rank of $A$ is also equal to the maximum number of linearly independent rows.*

*Proof.* We just observed that we can find invertible matrices $E$ and $G$ such that

$$D = EAG,$$

where $D$ has 1s going down the diagonal, and 0s everywhere else. Apply transpose to both sides of the equation:

$$D^t = (EAG)^t = G^t A^t E^t.$$

It's easy to observe that the rank of $D$ is the same as the rank of $D^t$, since in both cases it's the number of 1s that appear. Now we use the fact that multiplying on either side by an invertible matrix doesn't change the rank. Thus

$$\mathrm{rank(A^t) = rank(D^t) = rank(D) = rank(A)}.$$

$\square$

## 4. Finding the inverse of a matrix

There's another application of elementary operations, having to do with invertible matrices. Remember from the chapter on invertible matrices, that all of the following mean the same thing for an $n \times n$ matrix:

- $A$ is invertible.
- The columns of $A$ are linearly independent.
- The columns of $A$ span $\mathbf{R}^n$.
- The rank of $A$ is $n$.
- The rows of $A$ are linearly independent.
- The rows of $A$ span $\mathbf{R}^n$.
- The determinant of $A$ is nonzero (that's for later).

So far we don't really have a method for finding the inverse of a matrix, unless it's a $2 \times 2$ matrix. Let's review that case first: if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $A$ is invertible if and only if $ad - bc \neq 0$. If that's the case, then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

To figure out what to do in general, let's think about what happens when you row-reduce an invertible $n \times n$ matrix $A$ by only row operations. Since the rank of $A$ is $n$, each column must be a pivot column, and what's more, the "staircase" of the form is just the diagonal:

$$\begin{pmatrix} a_1 & * & * \\ 0 & a_2 & * \\ 0 & 0 & a_3 \end{pmatrix}$$

Here the $a_i$ are all nonzero, and the $*$s are arbitrary. We can multiply row $i$ by $1/a_i$ to get the diagonal entries to be 1, and then we can use those 1s to turn all the $*$s into 0.

As a result, an invertible matrix can be turned into the identity matrix by row operations.

This is the same as saying that there are elementary matrices $E_1, \dots, E_p$ such that

$$E_p \cdots E_1 A = I_n.$$

Multiply on the right by $A^{-1}$ to obtain:

$$E_p \cdots E_1 = A^{-1}.$$

Said another way:

$$A^{-1} = E_p \cdots E_1 I_n.$$

This formula is the theoretical basis for the following method to find the inverse of a matrix.

**Example 4.1.** *Find the inverse of the matrix*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Take the matrix, and place it to the left of the identity matrix in a big augmented matrix:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Then apply row reduction to reduce the left-side matrix into the identity:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Switch the first two rows:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Use the last row to kill of the entries above the 1:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Thus, the inverse of the matrix is

$$\begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

This technique also tells you when the $n \times n$ matrix is not invertible. If it were not invertible, row reducing would produce a full row of 0s at the bottom of the matrix.