LECTURE MARCH 25: SYSTEMS OF LINEAR EQUATIONS

1. The system $Ax = b$

You already learned how to solve systems of linear equations. The purpose of this section is to connect what we have learned so far about vector spaces and ranks of matrices, to solutions of systems of linear equations.

The most general system of linear equations takes this form:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This is a system of m equations in n unknowns. We would like to know if it has solutions, and if so, how many, and how to describe them.

Definition 1.1. *The* solution set *of the above system is the set of vectors* $(x_1, \ldots, x_n)$ *which satisfy the equation. It is a subset of* $\mathbf{R}^n$.

Two systems are equivalent *if they have the same solution set.*

If the solution set is empty, we call the system inconsistent. *Otherwise, it's consistent.*

Examples: $2x = 8$ is consistent, with solution set $\{4\}$. This system is equivalent to $3x = 12$. The system

$$\begin{aligned} x + y &= 2 \\ y &= 4 \end{aligned}$$

has a unique solution, $(-2, 4)$. The sole equation $x + y = 2$ has infinitely many solutions, $\{(x, 2 - x) | x \in \mathbf{R}\}$. Finally,

$$\begin{aligned} 2x &= 8 \\ x &= 3 \end{aligned}$$

is inconsistent, as is $0x = 5$. Note that all inconsistent systems are equivalent!

The first order of business is to recognize that any system of linear equations can be written using matrices and vectors. More precisely, the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

or just

$$Ax = b,$$

where A is the $m \times n$ matrix (a_{ij}), and b is the column vector consisting of the b_i. The solution set is then a subset of $\mathbf{R}^n$.

Theorem 1.2. *Let $Ax = b$ be a system of linear equations, where A is an $m \times n$ matrix. If E is an invertible $m \times m$ matrix, then $Ax = b$ is equivalent to $EAx = Eb$.*

Proof. We need to show that the solution set of $Ax = b$ is the same as the solution set of $EAx = Eb$.

First, assume that x_0 is a solution to the first system, so that $Ax_0 = b$. Then $E(Ax_0) = Eb$, or $EAx_0 = Eb$, so that x_0 is also a solution to the second system.

In the other direction, suppose x_0 is a solution to $EAx_0 = Eb$. Because E is invertible, E^{-1} exists, and I can multiply both sides on the left by it: $E^{-1}EAx_0 = E^{-1}Eb$, or $Ax_0 = b$. □

(It's kind of similar to saying that the solution set to $x + y = 4$ is the same as the solution set to $x + y + 3 = 7$, because adding 3 to both sides is invertible. However, the solution set to $x + y = 4$ is not the same as the solution set to $(x + y)^2 = 16$, because the latter leaves open the possibility that $x + y = -4$.

2. Homogeneous systems

Definition 2.1. *A system $Ax = b$ is* homogeneous *if $b = 0$. Otherwise it is* inhomogeneous.

A homogeneous system $Ax = 0$ always has at least one solution, namely $x = 0$. This is the *trivial solution.*

We have already studied how to solve $Ax = 0$. The solution set has a name, the *null space* or *kernel* of A. We write $N(A)$ for this. This is a subspace of $\mathbf{R}^n$.

Theorem 2.2 (The rank-nullity theorem)**.** *Let A be an $m \times n$ matrix of rank r. The dimension of $N(A)$ is $n - r$.*

So r is the rank. If k is the dimension of the null space (the nullity), then $r + k = n$, where n is the dimension of x.

As a corollary: if $m < n$, then we must have $r \leqslant m < n$. Thus $n - r$ is positive, and so $Ax = 0$ has nontrivial solutions.

Let's review the process for describing $N(A)$ using row-reduction. The idea is that for an elementary matrix E, the systems $Ax = 0$ and $EAx = 0$ are equivalent. So we can just keep applying row operations to A to bring it into echelon form; the solution set will remain the same.

Example 2.3. *Describe the solution set of $Ax = 0$, where*

$$A = \begin{pmatrix} 2 & 1 & 7 \\ 4 & 0 & 8 \end{pmatrix}.$$

Solution: row reducing gives

$$\begin{pmatrix} 2 & 1 & 7 \\ 0 & -2 & -6 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

This last matrix is is in *reduced* row echelon form. (What is the rank of this matrix? Answer: 2. Therefore the nullity is $3 - 2 = 1$, and the solution set should be one-dimensional.) So x_3 is the independent variable, let's rename it $x_3 = t$. We have $x_1 + 2t = 0$, and $x_2 + 3t = 0$. Thus $x_1 = -2t$, $x_2 = -3t$, $x_3 = t$.

The solution set is $\{(-2t, -3t, t) | t \in \mathbf{R}\}$. This is a one-dimensional figure in three-dimensional space, it is a line.

3. Inhomogeneous systems

If $b \neq 0$, then the system $Ax = b$ is inhomogeneous. The solution set of $Ax = b$ is then not a vector space, because it doesn't contain the zero vector.

The study of $Ax = b$ is related to $Ax = 0$, the corresponding homogeneous system.

Theorem 3.1. *Let $Ax = b$ be a system of linear equations. Suppose there exists a solution x_0. Then the solution set of $Ax = b$ is*

$$x_0 + N(A) = \left\{ x_0 + y \;\middle|\; y \in N(A) \right\}.$$

Here $N(A)$ is the null space of A.

It may seem strange to write $x_0 + N(A)$, since x_0 is a vector and $N(A)$ is a whole subspace. You can think of it as the subspace $N(A)$, but translated (or displaced) by the vector x_0.

Proof. Suppose x is a solution to $Ax = b$. Then consider $y = x - x_0$. I compute

$$Ay = A(x - x_0) = Ax - Ax_0 = b - b = 0.$$

Therefore $y \in N(A)$, and so $x = x_0 + y \in x_0 + N(A)$.

Conversely, suppose $y \in N(A)$. Then let $x = x_0 + y$, and we get

$$Ax = A(x_0 + y) = Ax_0 + Ay = b + 0 = b.$$

Therefore x lies in the solution set of $Ax = b$. □

Theorem 3.2. *Any system of equations $Ax = b$ has either zero, one, or infinitely many solutions.*

Proof. If there are zero solutions, the system is inconsistent. If there is at least one solution x_0, then the solution set is $x_0 + N(A)$. If $N(A) = \{0\}$, then our solution set is just $\{x_0\}$. Otherwise, $N(A)$ has positive dimension, in which case the solution set must be infinite. □

We still need to know how to find one solution to $Ax = b$. The idea now is that $Ax = b$ is equivalent to $EAx = Eb$ for any elementary matrix E, so that $(A|b)$ has the same solution set as $(EA|Eb)$. Now keep applying row operations until A is in row echelon form.

Key fact: $Ax = b$ is inconsistent exactly when the following happens: you row reduce $(A|b)$, and you see a row of the form

$$(0\,0\,\cdots\,0|*)$$

where $* \neq 0$. Seeing such a row means that $0x_1 + \cdots + 0x_n = * \neq 0$, which is impossible.

Example 3.3. *Solve the system*

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & 0 \end{array}\right)$$

We row reduce the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Since we see a row of 0s followed by a 1, the system is inconsistent. There are no solutions. By the way, the rank of the matrix is 2.

Theorem 3.4. *Suppose $Ax = b$ is a system of linear equations, where A is an $n \times n$ invertible matrix. Then the system has a unique solution, namely $x = A^{-1}b$.*

Proof. The system $Ax = b$ is equivalent to $A^{-1}Ax = A^{-1}b$. This is the same as $x = A^{-1}b$. $\square$