LECTURE MARCH 27: DETERMINANTS

In this chapter we learn what the determinant of a square matrix is. The notion is related to *area* and *volume*.

1. Determinants of 1×1 matrix

A 1×1 matrix is just a scalar: (a). The determinant of (a) is a. Nothing too exciting here.

2. Determinants of 2×2 matrices: definition and first properties

The determinant of a 2×2 matrix has a simple formula:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We already saw that the expression $ad - bc$ appeared in the formula for the inverse of an invertible 2×2 matrix. Let's investigate some of the properties of the 2×2 determinant.

To get your head in the right place, let's start with a question: how do you know when two fractions a/b and c/d are equal? You cross multiply: the two fractions are equal when $ad = bc$.

Theorem 2.1. *Let A be a 2×2 matrix. Then A is invertible if and only if $\det A \neq 0$.*

Proof. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If $\det A = 0$ it means $ad - bc = 0$. Then there's a linear combination of the columns of A which equals 0:

$$d \begin{pmatrix} a \\ c \end{pmatrix} - c \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ad - bc \\ cd - cd \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If c and d are not both zero, this is a nontrivial linear combination of the column vectors which equals 0; it means the columns of A are linearly DEpendent, and thus A is not invertible.

On the other hand, if c and d are both zero, then $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, which can't have linearly independent rows, so that in any case A is not invertible.

Conversely, suppose A is not invertible. This means the columns of A are linearly dependent. This means that one column is a multiple of the other, say $\begin{pmatrix} a \\ b \end{pmatrix} = x \begin{pmatrix} c \\ d \end{pmatrix}$. Then $ad - bc = (cx)d - (dx)c = x(cd - cd) = 0$. ∎

We might even offer a different proof. Start row-reducing A. Remember that for a square matrix A, row reducing produces a row of 0s at the bottom if and only if A is not invertible.

If $a \neq 0$, we would get from $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}$. This has a row of 0s at the bottom if and only if $d = bc/a$, which is the same as $ad - bc = 0$. (Try to fill in the rest of this proof in the case $a = 0$.)

3. Determinants of 2×2 matrices: area

Once again let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. The next important fact about determinants has to do with the *parallelogram* whose vertices are at $(0, 0,$ (a, b), (c, d) and at $(a + c, b + d)$. Call this parallelogram P.

Theorem 3.1. *The area of P is $|\det A|$. The sign of $\det A$ is determined by the right-hand rule.*

The "right hand rule" means that if the first vector (a, b) is found counterclockwise from (c, d), then $\det A < 0$. If (a, b) is clockwise from (c, d), then $\det A > 0$.

4. Determinants of 2×2 matrices: additivity properties

The determinant is a function

$$\det\colon M_{2\times 2}(\mathbf{R}) \to \mathbf{R}.$$

Is it a linear transformation? If so, it would satisfy $\det(A + B) = \det A + \det B$. Let's try $A = I_2$, $B = I_2$. $\det(I_2 + I_2) = \det(2I_2) = 4$. But $\det I_2 + \det I_2 = 1 + 1 = 2$. Therefore the determinant map is not a linear transformation.

It does however satisfy "linearity in each column" and "linearity in each row." Observe:

$$\begin{aligned}
\det \begin{pmatrix} a + a' & b \\ c + c' & d \end{pmatrix} &= (a + a')d - b(c + c') \\
&= (ad - bc) + (a'd - bc') \\
&= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det \begin{pmatrix} a' & b \\ c' & d \end{pmatrix}
\end{aligned}$$

Let's also observe:

$$\begin{aligned}
\det \begin{pmatrix} xa & b \\ xc & d \end{pmatrix} &= xad - xbc \\
&= x \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

This is like saying: if you take a parallelogram, and scale one of its sides by a factor x, that multiplies the area by x.

If A is again 2×2, what is $\det(xA)$? Answer is $x^2 \det A$.

How many square inches are there in a square foot? Answer: $12^2 = 144$.

There's one other important property, which is the $\det A$ is *alternating* in its rows and columns.

Observe:

$$\begin{aligned}
\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} &= bc - ad \\
&= -(ad - bc) \\
&= -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

5. What is an $n \times n$ determinant?

We hope to define a function

$$\det\colon M_{n\times n}(\mathbf{R}) \to \mathbf{R}$$

which has a few nice properties:

(1) $\det A \neq 0$ if and only if A is invertible.
(2) Given an $n \times n$ matrix A, the rows (or columns) of A make a *parallelepiped* in $\mathbf{R}^n$, whose volume is $|\det A|$.
(3) $\det A$ is linear in its rows and columns.
(4) $\det cA = c^n \det A$ for a scalar c.
(5) $\det A^t = \det A$.
(6) If you switch two rows (or columns) of A, that switches the sign of $\det A$.
(7) $\det I_n = 1$.

In fact, there is a unique function that has all of these properties!