MA542 Midterm, due Friday, Mar. 25

1. The complex number $z = e^{2\pi i/5}$ is a 5th root of unity: $z^5 = 1$. Find the minimal polynomial of z over $\mathbf{Q}$.

2. Suppose a nonzero complex number α is a root of a polynomial of degree n with rational coefficients. Show that $1/\alpha$ is also a root of a polynomial of degree n with rational coefficients.

3. (a) Let $I \subset \mathbf{R}[x]$ be the set of polynomials $f(x)$ for which $f(2) = f'(2) = f''(2) = 0$. Prove that this set forms an ideal, and find a polynomial $g(x)$ for which $I = g(x)R[x]$.

   (b) Let $J \subset R[x]$ be the set of polynomials $f(x)$ for which $f(2) = f'(3) = 0$. Show that J is not an ideal.

4. Let F be a finite field of characteristic p. Show that the function $f\colon F \to F$ defined by $f(a) = a^p$ is a) a ring homomorphism, b) injective, and c) surjective.

5. Factor $x^9 - x$ into irreducibles in $\mathbf{Z}_3[x]$.

6. Factor 65 into irreducibles in $\mathbf{Z}[i]$.

7. Let p be a nonzero prime element of an integral domain D. This means that whenever p divides a product ab with $a, b \in D$, it must divide a or b. Show that p is irreducible.