MA542 Final, due Wednesday May 10

1. Find an isomorphism between $\mathbf{Q}[x]/(x^2 - 10x + 26)$ and $\mathbf{Q}(i)$.

2. Let $\zeta = e^{2\pi i/12}$. Then $G(\mathbf{Q}(\zeta)/\mathbf{Q}) \cong U_{12} = \{1, 5, 7, 11\}$. Find the subfield of $\mathbf{Q}(\zeta)$ fixed by the subgroup $\{1, 11\}$.

3. Prove that there exists a normal extension $F/\mathbf{Q}$ with $G(F/\mathbf{Q}) \cong \mathbf{Z}_5$.

4. Let E/F be a normal extension such that $G(E/F)$ is a simple group. Show that if $F \subsetneq K \subsetneq E$ is an intermediate extension, then K/F is not normal.

5. Let $F = \mathbf{Q}(\sqrt{2 + \sqrt{2}})$. Show that $F/\mathbf{Q}$ is normal, and identify the group $G(F/\mathbf{Q})$.

6. Let F be a field, and let $f(x) = x^3 + bx^2 + cx + d \in F[x]$ be a cubic polynomial. Let $\alpha_1, \alpha_2, \alpha_3 \in \overline{F}$ be the roots of $f(x)$. The *discriminant* of $f(x)$ is defined as

   $$\Delta = (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_1 - \alpha_3)^2.$$

   Show that $\Delta \in F$.

7. Continuing from the previous problem, assume that $f(x)$ is irreducible and that its splitting field E/F is normal with $G(E/F) \cong S_3$. Let K be the fixed field of $A_3 \subset S_3$ (this is the subgroup of cyclic permutations of $\alpha_1, \alpha_2, \alpha_3$). Show that $K = F(\sqrt{\Delta})$.