MA 542 Midterm, due Friday, Mar. 24

1. (a) Let $I \subset \mathbf{R}[x]$ be the set of polynomials $f(x)$ for which $f(2) = f'(2) = f''(2) = 0$. Prove that this set forms an ideal, and find a polynomial $g(x)$ for which $I = g(x)R[x]$.
   (b) Let $J \subset \mathbf{R}[x]$ be the set of polynomials $f(x)$ for which $f(2) = f'(3) = 0$. Show that J is not an ideal.

2. Find a maximal ideal of $\mathbf{Z}[x, y]$.

3. Let R be a commutative ring, and let $a \in R$ be nilpotent. Show that a is contained in every prime ideal of R.

4. Factor $x^4 + x^3 + 2x^2 + x + 1$ into irreducibles in $\mathbf{Z}_5[x]$.

5. Consider the field $\mathbf{Z}_3(\alpha)$ and $\mathbf{Z}_3(\beta)$, where α is a root of $x^2 + 1$, and β is a root of $x^2 + x + 2$. Describe an isomorphism between $\mathbf{Z}_3(\alpha)$ and $\mathbf{Z}_3(\beta)$.

6. Let $\alpha \in \mathbf{C}$ be algebraic over $\mathbf{Q}$, and let $f(x) \in \mathbf{Q}[x]$ be its minimal polynomial. Show that if $g(x) \in \mathbf{Q}[x]$ has α as a root, then $f(x)$ divides $g(x)$ in $\mathbf{Q}[x]$.

7. Let $z = e^{2\pi i/5}$, so that $z^5 = e^{2\pi i} = 1$.
   (a) Find the minimal polynomial for z over $\mathbf{Q}$.
   (b) Show that $2^{1/3}$ does not lie in $\mathbf{Q}(z)$.