MA711 Final Problem Set

Due Dec. 17

Please read Chapter 22.4 of Royden and answer the following questions.

1. (Poincaré recurrence.) Let T be a measure-preserving transformation on a finite measure space $(X, \mathcal{M}, \mu)$. Let A be a measurable set. Show that for almost all $x \in A$, there are infinitely many natural numbers n for which $T^n(x) \in A$. (Hint: let $A_n = \bigcup_{k=n}^{\infty} T^{-k}(A)$; show that all the A_n have the same measure. In the end you want to show that $\bigcup_{n \geq 1} A \backslash A_n$ has measure zero.) Thus for almost all $x \in A$, the sequence $x, T(x), T^2(x), \ldots$ returns to A infinitely many times.

2. Give a counter-example to Poincaré recurrence in the case that the measure space is not finite.

3. Let $X = [0,1] \backslash \mathbb{Q}$ be the set of irrationals in the unit interval, under Lebesgue measure m. Let $T \colon X \to X$ be the transformation which takes a decimal expansion $.a_1 a_2 \ldots$ and returns $.a_2 a_3 \ldots$. Show that T is measure-preserving, and then show it is ergodic.

4. Notation as in the previous problem. An irrational number $x \in X$ is *normal* if every string of decimal digits $b_1 \ldots b_i$ is contained in the decimal expansion of x with the expected frequency. That is, if $N(k)$ is the number of times $b_1 \ldots b_i$ appears as a substring of the first k decimal digits of x, then
$$\lim_{k \to \infty} \frac{N(k)}{k} = \frac{1}{10^i}.$$
Use the ergodic theorem (Thm. 15 in the Chapter) to show that almost every $x \in X$ is normal. (On the other hand, it is extremely difficult to show that particular irrational numbers are normal.)

5. Let ν be the *Gauss measure* on $X = [0,1] \backslash \mathbb{Q}$, defined by
$$\nu(E) = \frac{1}{\log 2} \int_E \frac{1}{1+t} dm(t).$$
Show that ν and m (the Lebesgue measure) are absolutely continuous with respect to one another. Also show that the transformation $U(x) = \{1/x\}$ is measure-preserving with respect to ν, where $\{z\}$ means the fractional part of z.

6. For $x \in X$, the continued fraction expansion of x is

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}, a_i \in \mathbb{N}.$$

We abbreviate this as $x = [a_1, a_2, \ldots]$. Note that $U([a_1, a_2, \ldots])$ equals $[a_2, a_3, \ldots]$, and that a_i is the greatest integer below $1/U^{i-1}(x)$. It turns out that U is ergodic with respect to the Gauss measure ν (but don't prove this). Show that for almost all x, each natural number n occurs in the string $a_1, a_2, \ldots$ with frequency $\log_2[(n+1)^2/n(n+2)]$.