MA746 Final, due Tuesday, May 8

1. Let X be a curve over an algebraically closed field k (meaning a non-singular projective variety of dimension 1). Assume that the genus of X is 2. You may also assume that the characteristic of k is 0.

   (a) Let K be the canonical divisor of X. Show that $\ell(K) = 2$ and that K determines a morphism $f\colon X \to \mathbf{P}^1$ of degree 2.

   (b) Use the Riemann-Hurwitz formula to show that f is branched over six points in $\mathbf{P}^1$, with ramification degree 2 at each one.

   (c) Show that X does *not* admit a closed immersion into $\mathbf{P}^2$.

   (d) Show that if D is a divisor of degree 5 on X, then D induces a closed immersion $X \to \mathbf{P}^3$.

2. For this exercise, you may use the following fact: Let $r_i \geq 2$ be a finite collection of integers, and let $R = \sum_i (1 - 1/r_i)$. Then if $R > 2$, then $R - 2 \geq 1/42$.

   Let X be a curve of genus $g \geq 2$ over an algebraically closed field of characteristic 0, and let G be a finite group of automorphisms of X. Assume the existence of a quotient curve $f\colon X \to Y$, meaning that the fibers of f are exactly the orbits of G. Use the Riemann-Hurwitz formula to show that $\#G \leq 84(g-1)$.

   (The bound is sharp for infinitely many g. For instance, if X is the *Klein quartic* $x^3y + y^3z + z^3x = 0$, then X has genus 3, and the automorphism group of X is isomorphic to $\mathrm{PSL}_2(\mathbf{F}_7)$, a simple group of order 168.)