MA746 Midterm, due Wednesday, 3/21

1. (Ex 2.4 in Hartshorne) Let A be a ring and X a scheme. Show that there is a bijection between the set of morphisms of schemes $X \to \operatorname{Spec} A$ and the set of ring homomorphisms $A \to \Gamma(X, \mathcal{O}_X)$.

2. Read the proof of Thm. 7.1 in Hartshorne, which describes when a tuple of global sections $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ of an invertible sheaf on a scheme X over a ring A can be used to give a morphism $\phi\colon X \to \mathbf{P}^n_A$. At the end of the proof, Hartshorne says "It is clear from the construction...that $\mathscr{L} \cong \phi^*(\mathcal{O}(1))$, and that the sections s_i correspond to $\phi^*(x_i)$ under this isomorphism." This is an important point. Write up a proof of these claims.

3. Let k be an algebraically closed field, and let $i\colon C \hookrightarrow \mathbf{P}^2_k$ be a cubic curve. That is, C is the closed subscheme of the projective plane cut out by a homogeneous polynomial $f \in k[x, y, z]$ of degree 3. Assume that C is nonsingular. Let ∞ be a *flex point* of C, meaning a point whose tangent line meets C to order 3. (If the characteristic of k is not 2 or 3, one can do a change of variables so that C is the curve $y^2z = x^3 + axz^2 + bz^3$, and that $\infty = [0 : 1 : 0]$. You may assume this if it helps.)

   (a) Let $\mathscr{L} = i^*(\mathcal{O}_{\mathbf{P}^2_k}(1))$. Show that $\mathscr{L} \cong \mathscr{L}(3\infty)$.

   (b) Show that $P \sim Q$ in $\operatorname{Cl}(C)$ if and only if $P = Q$.

   (c) Let P, Q, R be distinct closed points in $C \setminus \{\infty\}$. Show that in $\operatorname{Cl}(C)$, $P + Q + R \sim 3\infty$ if and only if P, Q, R are collinear. (Of course there is a natural generalization to the case that P,Q and R are not all distinct.)

   (d) Let $\operatorname{Cl}^\circ(C)$ be the kernel of the degree homomorphism $\operatorname{Cl}(C) \to \mathbf{Z}$. Show that $P \mapsto P - \infty$ is a bijection from the set of closed points of C onto $\mathrm{Cl}^\circ(C)$. This is an explanation for the classical group law on the points of an elliptic curve.

This also shows that $\mathrm{Pic}^\circ(C) = \mathrm{Cl}^\circ(C)$, a priori only a group, carries the structure of a projective variety. This is true for all nonsingular projective curves C; $\mathrm{Pic}^\circ C$ is called the Jacobian variety of C, a so-called "abelian variety".