

Adelic Modular Forms

October 31, 2011

1 Motivation

Hecke's theory is concerned with a family of finite-dimensional vector spaces $S_k(N, \chi)$, indexed by weights, levels, and characters. The Hecke operators on such spaces already provide a very rich theory. It will be very advantageous to pass to the adelic setting, however, for the same reasons that Hecke characters on number fields should be studied in the adelic setting (rather than as homomorphisms out of the group of fractional ideals). In Tate's thesis, we learned that once we view Hecke characters as idele class characters, we can apply the tools of harmonic analysis on locally compact groups. We're going to do something very similar with modular forms: we'll view them as special functions (called automorphic forms) on an adelic group $GL(2)$. Some advantages of this point of view are (at least):

- For a Hecke eigenform f , it will become clear what the “local components” of f are supposed to mean.
- Furthermore, the local components of f determine the Euler factors of $L(f, s)$, and as well the local factors appearing in its functional equation, just as in Tate's thesis.
- In order to generalize the notion of modular forms to other number fields (or function fields) in a uniform way, there is little alternative to the adelic theory.

As for the last point, one can survive for a time studying Hilbert modular forms, Maass waveforms, Bianchi modular forms, etc, as these all can still be realized as functions on a symmetric space. But that will never quite reveal

the unity of these constructions: they are all avatars of automorphic forms (on $GL(2)$ over a totally real field, on $GL(2)$ over \mathbf{Q} but with nonzero eigenvalue for the Laplacian, and on $GL(2)$ over an imaginary quadratic field). To say nothing of other groups! The field's most important number-theoretic results (the Langlands-Tunnell theorem, the Taniyama-Shimura conjecture, the Sato-Tate conjecture) really do rely on the theory of automorphic forms on (at least) the groups $GL(n)$ for $n > 2$.

In the end, automorphic forms on $GL(2)$ over \mathbf{Q} are supposed to be functions on the quotient $GL_2(\mathbf{Q}) \backslash GL_2(\mathbf{A}_{\mathbf{Q}})$. But at the moment it will be convenient to take a different tack, which avoids some difficulties at the infinite place. The goal is to create a large vector space M_k , admitting an action of $GL_2(\mathbf{A}_{\mathbf{Q}}^{\text{fin}})$, which also contains every $M_k(N, \chi)$ for every level N and character χ .

2 Smooth Representations

First we will need to make a detour into representation theory. Let G be a topological group, certainly not assumed compact. We will want to study the category of representations $\rho: G \rightarrow GL(V)$ which has some of the nice properties of the theory in the compact case. There are a few natural choices for how to do this. One is to give V an inner product which makes V into a Hilbert space, and to require that $\rho(g)$ be unitary for each g . Such a representation of G is called *unitary*. This is certainly an important concept.

Another (perhaps simpler) thing to do is to give V the *discrete topology*, and require that $G \times V \rightarrow V$ be a *continuous map*. This is equivalent to saying that for each $v \in V$, the stabilizer of v in G is an open subgroup. Such a representation is called *smooth*. If $\rho: G \rightarrow GL(V)$ is any representation, and $v \in V$, then v is smooth if its stabilizer in G is open. Thus V is smooth when all of its vectors are. Let V^{sm} be the set of smooth vectors in V ; it is easy to check that V^{sm} is a vector subspace of V which is preserved by G .

Smooth representations are going to be few and far between if G doesn't have many open subgroups. For instance if $G = SL_2(\mathbf{R})$, there aren't any nontrivial smooth representations! ($GL_2(\mathbf{R})$ has a smooth one-dimensional representation, $\text{sgn} \circ \det$.) But p -adic groups such as $GL_2(\mathbf{Q}_p)$ have many interesting smooth representations.

One way to construct smooth representations is through smooth induction. If $H \subset G$ is a subgroup, and $\rho: H \rightarrow GL(W)$ is a representation, let

$V = \text{sm-Ind}_H^G W$ be the *smoothly induced representation*: that is, V is the space of functions $f: G \rightarrow W$ which satisfy the properties

- $f(hg) = f(g)$ for all $h \in H, g \in G$,
- There exists an open subgroup $K \subset G$ (depending on f) such that $f(gK) = f(g)$ for all $h \in H$.

Then V has an action of G , call it η , defined by $\eta(g)f(g') = f(g'g)$. Note that $\text{sm-Ind}_H^G W = (\text{Ind}_H^G W)^{\text{sm}}$.

This is how one constructs the *principal series* representations of $G = \text{GL}_2(\mathbf{Q}_p)$. Choose two characters χ_1, χ_2 of \mathbf{Q}_p^\times , let $\chi = \chi_1 \times \chi_2$ be the corresponding character of the Borel subgroup $B \subset G$, and let $\pi(\chi_1, \chi_2) = \text{sm-Ind}_B^G \chi$. Irreducible representations of G appearing in the $\pi(\chi_1, \chi_2)$ are known as the principal series representations. (In practice, however, it is more convenient to normalize things a bit differently, so that duals work the right way, but more on this later.)

3 The space M_k of adelic modular forms

Let W_k be the space of holomorphic functions on the upper half-plane \mathcal{H} with the property that, for all $\gamma \in \text{GL}_2^+(\mathbf{Q})$, $f|_{\gamma, k}$ is bounded on the region $\Im z > y_0$ for every $y_0 > 0$. The weight k is going to be fixed for now, so we'll just write $f|_\gamma$ for $f|_{\gamma, k}$. Give this space a (left) action $r_k: \text{GL}_2^+(\mathbf{Q}) \rightarrow W_k$ by

$$r_k(\gamma)f = f|_{\gamma^{-1}}$$

Now define a smooth representation $\rho_k: \text{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\text{fin}}) \rightarrow \text{GL}(M_k)$ by

$$M_k = \text{sm-Ind}_{\text{GL}_2^+(\mathbf{Q})}^{\text{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\text{fin}})} W_k$$

As we will see, this space contains all the modular forms of weight k at once!

3.1 Strong approximation for $\text{GL}(2)$

Just as

$$(\mathbf{A}_{\mathbf{Q}}^{\text{fin}})^\times = \mathbf{Q}_{>0}^+ \hat{\mathbf{Z}}^\times,$$

there is a decomposition

$$\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}}) = \mathrm{GL}_2^+(\mathbf{Q}) \mathrm{GL}_2(\hat{\mathbf{Z}}).$$

(The analogy isn't perfect, because the first decomposition is a direct product, while the second one is far from being a direct product: $\mathrm{GL}_2^+(\mathbf{Q}) \cap \mathrm{GL}_2(\hat{\mathbf{Z}}) = \mathrm{SL}_2(\mathbf{Z})$.) Better yet, we have the more general

Theorem 3.1. *Let $K \subset \mathrm{GL}_2(\hat{\mathbf{Z}})$ be an open subgroup such that $\det : K \rightarrow \hat{\mathbf{Z}}^\times$ is surjective. Then*

$$\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}}) = \mathrm{GL}_2^+(\mathbf{Q})K$$

The restriction on K is necessary, because the image of $\mathrm{GL}_2^+(\mathbf{Q})K$ under \det is $\mathbf{Q}_{>0}^\times \det K$, and this will not equal $(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})^\times$ unless $\det K = \hat{\mathbf{Z}}^\times$.

We'll prove the result for $K = \mathrm{GL}_2(\hat{\mathbf{Z}})$. The key observation we need is that

$$\mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$$

is surjective, which isn't difficult to prove.

Suppose $g \in \mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})$. After translating g by $\mathrm{GL}_2^+(\mathbf{Q})$ it is clear we can assume that g 's entries lie in $\hat{\mathbf{Z}}$. Let $L = \mathbf{A}^{\mathrm{fin}} \oplus \mathbf{A}^{\mathrm{fin}}$. Since g_p is almost always in $\mathrm{GL}_2(\mathbf{Z}_p)$, $g(L)$ must have finite index in L . Apply the structure theorem of finite abelian groups to $L/g(L)$: there exist $r|s$ such that $L/g(L) \cong \mathbf{Z}/r\mathbf{Z} \oplus \mathbf{Z}/s\mathbf{Z}$. This means exactly that

$$g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \begin{pmatrix} r & \\ & s \end{pmatrix} \mathrm{GL}_2(\hat{\mathbf{Z}}).$$

Write $g = u \begin{pmatrix} r & \\ & s \end{pmatrix} w$ with $u, w \in \mathrm{GL}_2(\hat{\mathbf{Z}})$. Let $N = rs$. Let $d > 0$ be an integer with $d \equiv \det u \pmod{N}$. Then the reduction of $\begin{pmatrix} d & \\ & 1 \end{pmatrix}^{-1} u$ modulo N lies in $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. Lift it to an integral matrix $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, so that $\gamma \equiv \begin{pmatrix} d & \\ & 1 \end{pmatrix}^{-1} u \pmod{N}$. Write $z = \gamma^{-1} \begin{pmatrix} d & \\ & 1 \end{pmatrix}^{-1} u$, so that $z \in \hat{\Gamma}(N)$

(that is, z is congruent to 1 modulo N). Then

$$\begin{aligned}
g &= u \begin{pmatrix} r & \\ & s \end{pmatrix} w \\
&= \begin{pmatrix} d & \\ & 1 \end{pmatrix} \gamma \gamma^{-1} \begin{pmatrix} d & \\ & 1 \end{pmatrix}^{-1} u \begin{pmatrix} r & \\ & s \end{pmatrix} w \\
&\in \mathrm{GL}_2^+(\mathbf{Q}) z \begin{pmatrix} r & \\ & s \end{pmatrix} \mathrm{GL}_2(\hat{\mathbf{Z}}) \\
&\in \mathrm{GL}_2^+(\mathbf{Q}) \begin{pmatrix} r & \\ & s \end{pmatrix}^{-1} z \begin{pmatrix} r & \\ & s \end{pmatrix} \mathrm{GL}_2(\hat{\mathbf{Z}}).
\end{aligned}$$

The theorem is proved with the calculation that conjugation by $\begin{pmatrix} r & \\ & s \end{pmatrix}$ moves $\hat{\Gamma}(N)$ into a subgroup of $\mathrm{GL}_2(\hat{\mathbf{Z}})$.

The theorem applies to a few adelic subgroups of interest: the groups $K_0(N)$, $K_1(N)$, and $K(N)$, which consist of matrices in $\mathrm{GL}_2(\hat{\mathbf{Z}})$ congruent modulo N to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$, respectively. The intersections of these groups with $\mathrm{GL}_2^+(\mathbf{Q})$ are $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$, respectively.

Let K be one of $K_0(N)$, $K_1(N)$, $K(N)$, and let $\Gamma = K \cap \mathrm{GL}_2^+(\mathbf{Q})$. We will now show how to associate to $f \in M_k(\Gamma)$ an element $\phi_f \in M_k$. Given $g \in \mathrm{GL}_2(\mathbf{A}^{\mathrm{fin}})$, find $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ and $h \in K$ with $g = \gamma h$. Then define

$$\phi_f(g) = f|_{\gamma^{-1}}$$

This is well-defined because if $\gamma' h'$ is another decomposition, then $\gamma^{-1} \gamma' = h' h^{-1} \in \mathrm{GL}_2^+(\mathbf{Q}) \cap K = \Gamma$, so that

$$f|_{\gamma^{-1}} = f|_{(\gamma^{-1} \gamma')(\gamma')^{-1}} = f|_{(\gamma')^{-1}}.$$

Also, ϕ_f is smooth because it is K -invariant.

Therefore there is a map

$$\begin{aligned}
M_k(\Gamma) &\rightarrow M_k \\
f &\mapsto \phi_f
\end{aligned}$$

which is injective. The maps are compatible in an obvious sense: for instance when $N|N'$, the map $M_k(\Gamma(N)) \rightarrow M_k$ factors through $M_k(\Gamma(N)) \rightarrow$

$M_k(\Gamma(N'))$. Thus there is an injective map

$$\varinjlim_N M_k(\Gamma(N)) \rightarrow M_k.$$

Note that each $M_k(\Gamma(N))$ has an action of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$, so that *a priori* the limit $M_k = \varinjlim M_k(\Gamma(N))$ has an action of $\varinjlim \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}) = \mathrm{SL}_2(\hat{\mathbf{Z}})$. The action of the larger group $\mathrm{GL}_2(\mathbf{A}^{\mathrm{fin}})$ makes M_k somewhat easier to work with.

One benefit is that the appearance of modular forms with character can now be explained very neatly: the center $(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})^{\times}$ of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})$ acts on M_k , and the subgroup $\mathbf{Q}_{>0}^{\times} \subset (\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})^{\times}$ acts trivially on M_k , so the action of the center factors through an action of $(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})^{\times} / \mathbf{Q}_{>0}^{\times} = \hat{\mathbf{Z}}^{\times}$. Thus M_k splits up as a direct sum of subspaces $M_k(\chi)$, where χ runs over the set of characters of $\hat{\mathbf{Z}}^{\times}$; *i.e.* Dirichlet characters. One must check that if χ is a Dirichlet character modulo N , then the inverse image of $M_k(\chi)$ under $M_k(\Gamma_1(N)) \rightarrow M_k$ is $M_k(N, \chi)$.

Of course there is also a cuspidal version of M_k , call it S_k , in which the original space W_k is redefined so that every $f|_{\gamma}(x + iy)$ is not just bounded as $y \rightarrow \infty$, but approaches the limit 0 uniformly in x .

Our goal in the coming lectures is to prove the following theorem, which elegantly encapsulates Hecke's theory in the language of representation theory. By a *cuspidal newform* of weight k , we mean a normalized Hecke eigenform $f \in S_k(N, \chi)^+$ (for some N and χ). As we have seen, a newform is determined completely by its Hecke eigenvalues. For such a newform f , let $\pi_f \subset M_k$ be the smallest $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})$ -stable subspace of S_k containing ϕ_f .

Theorem 3.2. 1. *For each newform f of weight k , π_f is irreducible.*

2. *S_k is the direct sum of the π_f as f runs through all newforms of weight k .*

Recall that a Hecke character χ is necessarily the product of local characters χ_v . In a similar fashion, our representation π_f of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})$ is the *restricted tensor product* of representations $\pi_{f,p}$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. We now turn our attention to these local representations.