Representations of adele groups

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1 Admissible representations of a locally profinite group

1.1 Admissibility

Let G be locally profinite[1]. Recall that a representation $\pi\colon G \to \mathrm{GL}(V)$ is *admissible* if it is smooth and has the property that V^K is finite-dimensional for all open compact $K \subset G$.

If $\sigma\colon K \to \mathrm{GL}(W)$ is a representation of K, let $V[\sigma]$ be the σ*-isotypic subspace* of V. This is the sum of the images of all K-equivariant maps $W \to V$.

Proposition 1.1. *A representation $\pi\colon G \to \mathrm{GL}(V)$ is admissible if and only if the following two conditions hold:*

1. *For all irreducible representations σ of K, $V[\sigma]$ is finite-dimensional.*
2. *$V = \bigoplus_\sigma V[\sigma]$, where σ runs over all smooth irreducible representations of K.*

Proof. Left as exercise. □

[1] Much of this lecture is adapted from David Rohrlich's manuscript on automorphic representations.

1.2 Schur's lemma

As a consequence we have *Schur's lemma* for irreducible admissible representations $\pi\colon G \to \mathrm{GL}(V)$. This states that $\mathrm{Hom}_G(V, V) = \mathbf{C}$. Proof: $f\colon V \to V$ commutes with the action of G, we would like to show that f has an eigenvalue λ. (Note that the existence of an eigenvector for f is not at all guaranteed, because V may be infinite-dimensional.) Then the λ-eigenspace of f is a nonzero G-stable subspace of V, hence it is all of V, hence f acts on V as λ. To show that f has an eigenvalue, let K be an open compact subgroup. By the proposition, there is a smooth representation σ of K with $V[\sigma]$ finite-dimensional and nonzero. Now observe that f preserves $V[\sigma]$. Since $V[\sigma]$ is finite-dimensional, it has an eigenvalue.

1.3 Admissible unitary representations

Unitary representations V have the nice property that for $W \subset V$ a G-invariant subspace, the complement $W^\perp$ is also G-invariant. This suggests that V should be a direct sum of irreducible G-invariant subspaces. But this may fail to be true: V might not even be equal to the direct sum of W and $W^\perp$! It turns out that we are saved by the property of admissibility.

Theorem 1.2. *Let G be locally profinite, and let $\pi\colon G \to \mathrm{GL}(V)$ be an admissible unitary representation. Then V is the orthogonal direct sum of irreducible representations.*

This will follow from the following two lemmas:

Lemma 1.3. *If $W \subset V$ is G-invariant, then $V = W \oplus W^\perp$.*

Lemma 1.4. *If $V \neq 0$ then V contains an irreducible G-invariant subspace.*

Given the lemmas, the proof of the theorem proceeds as follows. Let $\mathcal{S}$ be the set whose members are families of mutually orthogonal irreducible subspaces of V. Put a partial order on $\mathcal{S}$ under inclusion. Then every chain in $\mathcal{S}$ has an upper bound, namely the union. Therefore by Zorn's lemma, $\mathcal{S}$ has a maximal element S. Put

$$U = \bigoplus_{W \in S} W.$$

We claim $U = V$. Otherwise, the first lemma gives $V = U \oplus U^\perp$, and the second lemma shows that $U^\perp$ contains an irreducible representation W. Then $S \cup \{W\}$ is strictly greater than S, contradiction.

Now, the proof of Lemma 1: Since V is admissible, it enough to show that

$$V[\sigma] = W[\sigma] \oplus W^\perp]\sigma]$$

for every smooth irreducible representation of K. Since $V[\sigma]$ is finite-dimensional, we have

$$V[\sigma] = W[\sigma] \oplus \big(W[\sigma]^\perp \cap V[\sigma]\big)$$

We claim that $W^\perp[\sigma] = W[\sigma]^\perp \cap V[\sigma]$. Certainly $W^\perp[\sigma] \subset V[\sigma]$ and $W^\perp[\sigma] \subset W^\perp \subset W[\sigma]^\perp$, so that we have one inclusion, $W^\perp[\sigma] \subset W[\sigma]^\perp \cap V[\sigma]$.

On the other hand, $V[\sigma]$ is orthogonal to $W[\tau]$ for all $\tau \neq \sigma$, and $W[\sigma]^\perp$ is orthogonal to $W[\sigma]$, so that the intersection $V[\sigma] \cap W[\sigma]^\perp$ is orthogonal to

$$\left(\bigoplus_{\tau \neq \sigma} W[\tau] \right) \bigoplus W[\sigma] = W,$$

so that $V[\sigma] \cap W[\sigma]^\perp \subset W^\perp$, which implies that $V[\sigma] \cap W[\sigma]^\perp$ (since it is σ-isotypic) is contained in $W^\perp[\sigma]$. This is the other required inclusion.

Proof of Lemma 2: Left as exercise.

2 Representations of restricted direct product groups

Let G_v be a collection of locally profinite groups. For almost all v, let $K_v \subset G_v$ be a compact open subgroup. Let S_0 be the finite set of v containing those for which K_v is not defined. We need to make the following important assumption: the Hecke algebra $\mathcal{H}(G_v, K_v)$ is commutative for all $v \notin S$. This mimics the situation of $G_v = \mathrm{GL}_2(K_v)$, $K_v = \mathrm{GL}_2(\mathcal{O}_v)$.

For all v, suppose $\pi_v \colon G_v \to \mathrm{GL}(V_v)$ be a smooth admissible representation. Assume that for almost all $v \notin S_0$, π_v is "spherical", in the sense that $V_v^{K_v} \neq 0$. Assume that S contains all those v for which π_v is not spherical. For $v \notin S_0$, $\dim V_v^{K_v} = 1$ is 1-dimensional (because the Hecke algebra is commutative); let $\xi_v \in V_v$ be a nonzero spherical vector.

Now define the *restricted tensor product* $\pi = \bigotimes'_v \pi_v$ as follows. For every finite $S \supset S_0$, let

$$\pi_S = \bigotimes_{v \in S} \pi_v;$$

this is a representation of $\prod_{v \in S} G_v$ on a vector space V_S. In fact it is a representation of

$$G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} K_v$$

by having each K_v ($v \notin S$) act trivially on V_S. For $S \subset S'$, define a map $V_S \to V_{S'}$ by tensoring a vector $x \in V_S$ with $\otimes_{v \in S' \setminus S} \xi_v$. This map is compatible with the actions of G_S and $G_{S'}$ and the inclusion $G_S \hookrightarrow G_{S'}$, because the K_v act trivially on the ξ_v. Then set

$$\bigotimes_v{}' \pi_v = \varinjlim_S \pi_S;$$

this is a representation of

$$G = \varinjlim_S G_S.$$

Thus π is the vector space spanned by symbols $\otimes_v x_v$, with $x_v = \xi_v$ for almost all $v \notin S_0$, modulo the usual relations regarding addition and scalar multiplication. Note that it is smooth and admissible (follows easily from the corresponding properties of the π_v. Also, if each π_v is irreducible, then so is π.

Should each π_v be a unitary representation with norm $\|x\|_v$, one can give π the structure of a unitary representation. For this, one assumes that each ξ_v, $v \notin S$, is a unit vector with respect to the inner product on V_v. Then the norm of $\otimes_v x_v$ is defined to be $\prod_v \|x_v\|_v$.

A smooth representation π of G is *decomposable* if it isomorphic to some $\bigotimes'_v \pi_v$.

2.1 The factorizability theorem

Theorem 2.1. *Suppose π is a unitary admissible irreducible representation of $G = \prod'_v G_v$ on a vector space V. Then π is decomposable, and its irreducible factors are uniquely defined.*

Proof. The first observation is that for each v, $\pi|_{G_v}$ is the sum of subrepresentations which are all isomorphic to one particular representation, call it $\pi_v\colon G_v \to \mathrm{GL}(V_v)$. Proof: $\pi|_{G_v}$ is again an admissible unitary representation, so it contains an irreducible G_v-invariant subspace V_v. Let π_v be the representation of G_v on V_v. Then (key observation coming up) $V[\pi_v]$ is preserved by the action of G. Therefore $V = V[\pi_v]$. By the previous theorem, $\pi|_{G_v}$ is the direct sum of representations all isomorphic to π_v.

We claim that π_v is spherical for almost every v. Since π is smooth, an arbitrary nonzero vector in V is fixed by a compact open $K \subset G$. By definition of the topology on G, K contains K_v for almost all v. Since $\pi|_{K_v}$ is the direct sum of representations all isomorphic to π_v, π_v has an K_v-fixed vector for almost all v. Enlarge S_0 to include all the v for which π_v is not spherical. Choose spherical unit vectors $\xi_v \in V_v$ for all $v \notin S_0$.

For each v we may define V^v to be the space of G_v-equivariant linear maps $V_v \to V$ which are smooth with respect to the evident action π^v of $G^v = \prod'_{w\neq v} G_w$. Since π_v embeds into π by its very definition, there do exist nonzero G_v-equivariant linear maps $f\colon V_v \to V$. Furthermore these are automatically smooth. Indeed, let $x \in V_v$ is a nonzero vector, let $K^v \subset G^v$ fix $f(x)$. Then K^v fixes $f(\pi_v(g)x)$ for all $g \in G_v$ (since the actions of G_v and G^v commute). Since π_v is irreducible, the $\pi_v(g)x$ must span V_v. Thus K^v fixes $f(y)$ for all $y \in V_v$, which is to say that K^v fixes f. Therefore π^v is nonzero.

We have a nonzero G-equivariant map $V_v \otimes V^v \to V$ given by "evaluation": $x \otimes f \mapsto f(x)$, and since both sides are irreducible, we have an isomorphism

$$V \cong V_v \otimes V^v.$$

This can be generalized: if S is a finite set of indices, let π_S be the representation of $\prod_{v\in S} G_v$ on $V_S = \bigotimes_{v\in S} V_v$; we have a factorization

$$V \cong V_S \otimes V^S.$$

For S containing S_0, we claim that V^S has a unique spherical vector up to scaling. Existence is because V itself has an S-spherical vector, hence so does V^S. Uniqueness is because $\mathcal{H}(\prod'_v G_v, \prod_v K_v)$ is commutative (this follows from the local statement). Let ξ^S be a nonzero spherical vector in V^S.

Let π_S be the representation of G_S on V_S, defined by having K_v act trivially for $v \notin S$. Whenever $S \subset S'$, we have a G_S-equivariant map $V_S \to$

$V_{S'}$ given by tensoring with $\otimes_{v\in S'\setminus S}\xi_v$. We have $\bigotimes' \pi_v = \varinjlim \pi_S$ by definition. We must produce a nonzero G-equivariant map $\bigotimes' \pi_v \to \pi$. This is done by producing compatible maps $\pi_S \to \pi$ for each $S \supset S_0$ by means of the embedding

$$V_S \subset V_S \otimes V^S \cong V$$

given by tensoring with ξ_S. (One must perhaps rescale the ξ_S to ensure compatibility.) $\square$

3 Representations of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$ arising from cusp forms

Let $k \geq 1$. Recall we have a space S_k of adelic modular forms. This was the smooth induction from $\mathrm{GL}_2^+(\mathbf{Q})$ up to $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$ of the space of holomorphic functions on $\mathcal{H}$ which vanish at the cusps, under the action $f \mapsto f|_{\gamma^{-1},k}$.

Proposition 3.1. *S_k is a unitary admissible representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$.*

Proof. (Sketch) First, admissibility. Let $K \subset \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$ be a compact open subgroup. We want to show that S_k^K is finite-dimensional. First observe that the double coset space

$$\mathrm{GL}_2^+(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})/K$$

is finite. (By conjugating K one may assume that it is a finite-index subgroup of $\mathrm{GL}_2(\hat{\mathbf{Z}})$. Then use strong approximation: $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}}) = \mathrm{GL}_2^+(\mathbf{Q})\,\mathrm{GL}_2(\hat{\mathbf{Z}})$.) Let C be a set of coset representatives for this double coset space. For each $c \in C$, let $\Gamma_c = \mathrm{GL}_2^+(\mathbf{Q}) \cap cKc^{-1}$. Then there is a map

$$\begin{array}{rcl} S_k^K & \to & \bigoplus_{c\in C} S_k(\Gamma_c) \\ \phi & \mapsto & (\phi(c))_{c\in C}, \end{array}$$

which is in fact an isomorphism (Exercise). Therefore S_k^K is finite-dimensional and S_k is admissible. (In fact M_k is admissible as well, by the same argument.)

To give S_k the structure of an inner-product space, it will be useful to interpret elements of S_k as functions not on $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$ but on all of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$. First note that

$$\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}}) = \mathrm{GL}_2(\mathbf{Q})(\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}}) \times \mathrm{GL}_2^+(\mathbf{R})),$$

where $\mathrm{GL}_2(\mathbf{Q})$ is embedded diagonally in $\mathrm{GL}_2(\mathbf{A_Q})$.

Given $\phi \in S_k$, define a function Φ on elements

$$g = \gamma(g_{\mathrm{fin}}, g_\infty) \in \mathrm{GL}_2(\mathbf{Q})(\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}}) \times \mathrm{GL}_2^+(\mathbf{R}))$$

by

$$\Phi(g) = \left(\phi(g_{\mathrm{fin}})|_{g_\infty,k}\right)(i)$$

(Do check this is well-defined!) Then Φ is invariant under left multiplication by $\mathrm{GL}_2(\mathbf{Q})$. It is also invariant under right multiplication by an open subgroup $K \subset \mathrm{GL}_2(\hat{\mathbf{Z}})$ (depending on Φ). Finally, Φ is invariant under multiplication by the group $Z(\mathbf{R})^+$ of diagonal matrices $\begin{pmatrix} a & \\ & a \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R})$ with $a > 0$, whereas $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ transfoms Φ into $(-1)^k\Phi$. Let $\phi_1, \phi_2 \in S_k$ have images Φ_1, Φ_2 under this correspondence; we define

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathrm{GL}_2(\mathbf{Q})Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{A_Q})} \Phi_1(g)\overline{\Phi_2(g)}d\mu(g)$$

Here $d\mu$ is a Haar measure on the locally compact group $Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{A_Q})$; the integral is understood to mean the integral over a fundamental domain for $\mathrm{GL}_2(\mathbf{Q})Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{A_Q})$ inside of $Z(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{A_Q})$. The integral is well-defined because the Φ_i are left $\mathrm{GL}_2(\mathbf{Q})$-invariant, and the integrand is $Z(\mathbf{R})$-invariant. It is convergent because the ϕ_i were cusp forms, although we do not check the details here. Finally, $\langle \phi_1, \phi_2 \rangle$ is $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$-invariant because $d\mu$ is a Haar measure. If the ϕ_i arose from cusp forms f_i lying in a common space $S_k(\Gamma)$, then $\langle \phi_1, \phi_2 \rangle$ equals the Peterssen inner product $\langle f_1, f_2 \rangle$ up to a scalar which only depends on Γ (and on $d\mu$). □

As a corollary, we find that S_k decomposes as a direct sum of irreducible representations, each of which is an admissible representation π of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$. By the factorizability theorem, each π is the restricted direct product of representations π_p of $\mathrm{GL}_2(\mathbf{Q}_p)$, almost all of which are spherical. By Casselman's theorem on the new vector, there exists for each p an integer c_p (almost always 0) and a unit vector ϕ_p in the space of π_p for which

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\phi_p = \chi_p(a)\phi_p,$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)$ with $c \equiv 0 \pmod{p^{c_p}}$. Here χ_p is the central character of π_p.

Note that ϕ_p is a spherical vector for almost all p, so that $\phi = \otimes_p \phi_p$ is a well-defined unit vector in the space of π. Let $N = \prod_p p^{c_p}$. Then ϕ is invariant under $K_1(N)$. Since $K_1(N)$ surjects via the determinant map onto $\hat{\mathbf{Z}}^\times$, we have $S_k^{K_1(N)} \cong S_k(\Gamma_1(N))$, so that $\phi = \phi_f$ for a classical cusp form $f \in S_k(\Gamma_1(N))$. Since the center of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})$ acts on ϕ through χ, we have $f \in S_k(N, \chi)$.