Representations of locally profinite groups

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The reference here is the book of C. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), §1.

1 The Haar measure

1.1 Locally profinite groups

Let G be a *locally profinite group*. This means that G is a Hausdorff topological group such that there exists a system of neighborhoods of $1 \in G$ consisting of compact open subgroups. If $K' \subset K$ are two compact open subgroups of G, then the coset space K/K' is compact and discrete, hence finite. If K is a fixed compact open in G, consider the map

$$K \to \varprojlim_{K'} K/K'$$

where K' runs over all open normal subgroups of K. This map is injective because K is Hausdorff and surjective because K is compact. Thus K is actually *profinite*.

1.2 Construction of the Haar measure

It is a simple matter to prove the existence of a Haar measures on G, at least compared to other sorts of locally compact groups. First, some definitions.

Let $C_c^\infty(G)$ be the space of compactly supported complex-valued smooth functions on G, where "smooth" here means right K-invariant for some compact open K. An equivalent definition is that $C_c^\infty(G)$ is the space of compactly supported complex-valued functions on G which are locally constant.

(Indeed, if f is a smooth function it is certainly locally constant. Conversely, if f is compactly supported and locally constant, it means that for each point g in the support of f, there is a compact open subgroup K_g for which $f|_{gK_g}$ is constant. The gK_g cover the support of f, so there is a finite subcover, say $\{gK_g\}_{g\in S}$; then f is right K-invariant for $K = \bigcap_{g\in S} K_g$.)

There are two obvious ways to define a representation of G on $C_c^\infty(G)$, corresponding to left and right translation. These are

$$\begin{aligned} \lambda, \rho\colon G &\to \mathrm{GL}(C_c^\infty(G)) \\ \lambda(g)f(x) &= f(g^{-1}x) \\ \rho(g)f(x) &= f(xg) \end{aligned}$$

A *left Haar integral* is a non-zero linear functional $I\colon C_c^\infty(G) \to \mathbf{C}$ which is invariant under $\lambda(G)$ (that is, $I(\rho(g)f) = I(f)$ and which satisfies $I(f) \geq 0$ whenever $f \geq 0$ is nonnegative. A right Haar integral is the same, except it is required to be invariant under $\rho(G)$.

We'll now show that a right Haar measure exists. For every $K \subset G$ compact open, let $C_c^\infty(G)^K$ be the subspace of $C_c^\infty(G)$ fixed by $\lambda(K)$. This space has a basis $\{f_{Kg}\}$ consisting of the characteristic functions f_{Kg} of the cosets Kg in $K\backslash G$. Define $I_K\colon C_c^\infty(G)^K \to \mathbf{C}$ by $I_K(f_{gK}) = 1$. Then I_K has the property that $I_K(f) \geq 0$ whenever $f \geq 0$ is in $C_c^\infty(G)^K$. Since $\rho(h)f_{Kg} = f_{Kgh^{-1}}$, I_K is $\rho(G)$-invariant. It is easy to see that I_K is up to scaling the only functional on $C_c^\infty(G)^K$ which is $\rho(G)$-invariant.

We can now build a right Haar integral I out of the I_K. Let K be an open compact subgroup of G, and let $K_j \subset K$ be a family of open compact subgroups of K with $\cap K_j = 1$. By definition of smoothness,

$$C_c^\infty(G) = \bigcup_{j\geq 1} C_c^\infty(G)^{K_j}.$$

Now define I as follows. For $f \in C_c^\infty(G)^{K_j}$, let

$$I(f) = \frac{1}{[K : K_j]} I_{K_j}(f);$$

then $I(f)$ does not depend on the index j. Then I is $\rho(G)$-invariant. The uniqueness of I follows from the uniqueness of the individual I_K.

The Haar integral is usually written like this:

$$I(f) = \int_G f(x)\,d\mu_G(x),\ f \in C_c^\infty(G)$$

where the symbol μ_G is the *Haar measure.* μ_G is a left or right Haar measure as I is a left or right Haar integral. We can extend the domain of the Haar integral to other locally constant functions f as well: for example if H_i is a covering of G by compact opens such that $f|_{H_i}$ is constant, let $x_i^* \in H_i$ be any element. Then define

$$\int_G f(x) d\mu_G(x) = \sum_i f(x_i^*) \int_{G_i} d\mu_G,$$

provided the sum converges absolutely.

1.3 The modular character

Generally the left and right Haar measures do not agree. Let μ_G be a left Haar measure. For $g \in G$, consider the functional on $C_c^\infty(G)$:

$$f \mapsto I(\rho(g)f) = \int_G f(xg) d\mu_G(x).$$

This is a left Haar integral, so it must equal $\delta_G(g)^{-1} I(f)$ for some positive real scalar $\delta_G(g)$. Then for $g, h \in G$ we have

$$\delta_G(gh) I(\rho(gh)f) = I(f) = \delta_G(h) I(\rho(h)f) = \delta_G(g)\delta_G(h) I(\rho(g)\rho(h)f),$$

so that $\delta_G \colon G \to R_{>0}^\times$ is a homomorphism. In brief, we have $d\mu_G(gx) = d\mu_G(x)$ because $d\mu_G$ is a left Haar measure, but

$$\frac{d\mu_G(xg)}{d\mu_G(x)} = \delta_G(g).$$

The function δ_G is called the *module* (or modular quasi-character) of G.

The module measures disagreement between the left and right Haar measures of G. The functional

$$f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x)$$

is a *right* Haar integral on G, which equals μ_G if and only if $\delta_G = 1$. In this case we say G is *unimodular*.

If G is compact, there is a unique left Haar measure which assigns to an open compact $K \subset G$ the measure $1/[G : K]$. This is also a right Haar measure. Thus G is unimodular.

Generally, if $K \subset G$ is a compact open subgroup then $\delta_G|_K = 1$. This implies that δ_G is continuous.

1.4 The example of $G = \mathrm{GL}_2(F)$ and its Borel subgroup

Let F be a p-adic field. For the group $G = \mathrm{GL}_2(F)$, a Haar measure can be defined as follows. Let $A = M_2(F)$, so that $G = A^\times$. Let μ_A be a Haar measure on (the additive group of) A. It is not difficult to see that $d\mu_A(gx) = d\mu_A(xg) = |\det g|^2\, d\mu_A(x)$ for $g \in G$.

If $f \in C_c^\infty(G)$, then the function on $A = M_2(F)$ which is $f(x)\,|\det x|^{-2}$ on G and zero elsewhere lies in $C_c^\infty(A)$. We define

$$\int f d\mu_G = \int_G f(x)\,|\det x|^{-2}\, d\mu_A(x);$$

this is both a left and right Haar measure on G. Thus G is *unimodular.*

Now let $B \subset G$ be the subgroup of upper-triangular matrices. It turns out that B is not unimodular.

Proposition 1.1. *The module of B is* $\begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto |d/a|$.

Proof. We have $B = T \rtimes N$, where $T = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \right\}$ and $N = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right\}$. Let $d\mu_T$ and $d\mu_N$ be Haar measures on T and N, respectively. We define an integral on B as follows: if $f \in C_c^\infty(B)$, set

$$I(f) = \int_T \int_N f(tn) d\mu_T(t) d\mu_N(n).$$

We claim that $I(f)$ is $\lambda(B)$-invariant. If $t_0 \in T$, then $I(\lambda(t_0)f) = I(f)$ (use the change of variables $t \mapsto t_0 t$). If $n_0 \in N$, then

$$\begin{aligned} I(\lambda(n_0)f) &= \int_T \int_N f(n_0^{-1} tn) d\mu_T(t) d\mu_N(n) \\ &= \int_T \int_N f(t(t^{-1}n_0^{-1}t)n) d\mu_T(t) d\mu_N(n), \end{aligned}$$

and now observe that $t^{-1}n_0^{-1}t =: (n_0^{-1})^t \in N$, so we can turn this into $I(f)$ by the change of variable $n \mapsto n_0^t n$. Thus $I(f) = \int_B f d\mu_B$ is a left Haar integral.

But now consider $I(\rho(t_0)f)$ for an element $t_0 = \begin{pmatrix} a & \\ & d \end{pmatrix} \in T$:

$$\begin{aligned} I(\rho(t_0)f) &= \int_T \int_N f(tnt_0) d\mu_T(t) d\mu_N(n) \\ &= \int_T \int_N f(tt_0 n^{t_0}) d\mu_T(t) d\mu_N(n) \\ &= \int_T \int_N f(tn^{t_0}) d\mu_T(t) d\mu_N(n) \\ &= |a/d| \int_T \int_N f(tn) d\mu_T(t) d\mu_N(n). \end{aligned}$$

In the final step, notice that if $n = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ then $n^{t_0} = \begin{pmatrix} 1 & a^{-1}db \\ & 1 \end{pmatrix}$, so that $d\mu_N(n^{t_0}) = |d/a| \, d\mu_N(n)$. On the other hand $I(\rho(t_0)f) = \delta_B(t_0)^{-1} I(f)$, so this shows that $\delta_B(t_0) = |d/a|$. A simple change of variables shows that $I(\rho(n_0)f) = I(f)$ for $n_0 \in N$, so that $\delta_B|_N = 1$. The proposition is proved. □

2 Contragredients, admissibility, and induced representations

When G is a finite group, and $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of G, the contragredient of ρ is the representation $\check{\rho}\colon G \to \mathrm{GL}(\check{V})$ defined on the linear dual $\check{V}$ of V by $\check{\rho}(g)f(v) = f(\rho(g^{-1})v)$ for $v \in V$, $f \in \check{V}$, $g \in G$. When $H \subset G$ is a subgroup and $\sigma\colon H \to \mathrm{GL}(V)$ is a finite-dimensional representation of H, we have the induced representation $\operatorname{Ind} \sigma$ of G; this satisfies the rules

- $\mathrm{Hom}_G(\operatorname{Ind} \sigma, \rho) = \mathrm{Hom}_H(\sigma, \rho|_H)$
- $(\operatorname{Ind} \sigma)^\vee \cong \operatorname{Ind} \check{\sigma}$.

When G is a locally profinite group, these rules must be adjusted by the modular characters coming from G and H.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a smooth representation. The *contragredient* $\check{\rho}\colon G \to \mathrm{GL}(\check{V})$ is the space of smooth vectors in the linear dual $\mathrm{Hom}(V, \mathbf{C})$. (Note that the vector space $\check{V}$ might itself depend on G!)

Proposition 2.1. *The following are equivalent.*

- *The natural map $V \to \check{\check{V}}$ is an isomorphism.*
- *For every compact open subgroup $K \subset G$, V^K is finite-dimensional.*

(Exercise!) Such a V satisfying the conditions is called *admissible.*

Let $H \subset G$ be a closed subgroup, and let $\theta \colon H \to \mathbf{C}^\times$ be a character. Let $C_c^\infty(H\backslash G, \theta)$ be the space of functions $f \colon G \to H$ which satisfy the properties

1. $f(gK) = f(g)$ for a compact open $K \subset G$ (*i.e.* f is K-smooth)
2. $f(hg) = \theta(h)f(g)$, for $h \in H$, $g \in G$
3. The image of the support of f in $H\backslash G$ is compact.

Then $C_c^\infty(H\backslash G, \theta)$ admits a representation of G under right translation; we call this action ρ, as usual. Note that $C_c^\infty(H\backslash G, \theta)$ isn't quite the same as $\text{sm-}\operatorname{Ind}\theta$, because the functions f are required to be compactly supported modulo H.

Let $\delta_{H\backslash G} \colon H \to \mathbf{C}^\times$ the character defined by $\delta_{H\backslash G}(h) = \delta_H(h)^{-1}\delta_G(g)$.

Proposition 2.2. $\operatorname{Hom}_G(C_c^\infty(H\backslash G, \theta), \mathbf{C}) \neq 0$ *if and only if* $\theta = \delta_{H\backslash G}$*, in which case this* Hom *has dimension 1.*

(Contrast the case of finite or compact groups, in which $\operatorname{Hom}_G(\operatorname{Ind}\theta, \mathbf{C})$ would be nonzero if and only if $\theta = 1$. Also note that when $H = 1$, this corresponds to the existence and uniqueness of the right Haar integral on G.)

For a full proof, see the Bushnell-Henniart book, §1.3.3 Proposition. We sketch a proof of the "if" direction. Assume $\theta = \delta_{H\backslash G}$. Consider the map $C_c^\infty(G) \to C_c^\infty(H\backslash G, \delta_{H\backslash G})$, $f \mapsto \tilde{f}$, where

$$\tilde{f}(g) = \int_H \delta_G(h)^{-1} f(hg) d\mu_H(h).$$

(Do check that this actually lies in $C_c^\infty(H\backslash G, \delta_{H\backslash G})$.) We claim that $f \mapsto \tilde{f}$ is surjective. It is enough to check that $C_c^\infty(G)^K$ lies in the image of $f \mapsto \tilde{f}$ for each compact open K. If HgK is a double coset in $H\backslash G/K$, define a function $F_{g,K}$ on G as follows: let the support of $F_{g,K}$ be HgK, and define

$$F_{g,K}(hgk) = \delta_{H\backslash G}(h).$$

(This is well-defined because if $h'gk' = hgk$, then $(h')^{-1}h \in gKg^{-1} \in H$ lies in a compact subgroup of H, hence in the kernel of both δ_G and δ_H.) It is not hard to see that the functions $F_{g,K}$ span $C_c^\infty(H\backslash G, \delta_{H\backslash G})^K$, and that up to scaling $F_{g,K}$ is the only function in $C_c^\infty(H\backslash G, \delta_{H\backslash G})^K$ supported on HgK.

Now consider the function $f_{g,K} \in C_c^\infty(G)$ which is the characteristic function of gK. We have

$$\tilde{f}_{g,K}(x) = \int_H \delta_G(h)^{-1} f_{g,K}(hx) d\mu_H(h),$$

and this is supported on HgK. Therefore it is a multiple of $F_{g,K}$; in fact

$$\tilde{f}_{g,K} = \mu_H(H \cap gKg^{-1}) F_{g,K}.$$

Thus $f \mapsto \tilde{f}$ is surjective.

Let $I\colon C_c^\infty(G) \to \mathbf{C}$ be the *right* Haar integral:

$$I(f) = \int_G \delta_G(g)^{-1} f(g) d\mu_G(g).$$

We claim that I factors through the required map $C_c^\infty(H\backslash G, \theta) \to \mathbf{C}$. We need to show that if $\tilde{f} = 0$ then $I(f) = 0$.

It suffices to show this when the support of f is in HgK for some $g \in G$. (This is because if f is supported on HgK then so is $\tilde{f}$.) Write $HgK = \coprod_i h_i gK$. Then

$$\begin{aligned} \tilde{f}(g) &= \int_H \delta_G(h)^{-1} f(hg) d\mu_H(x) \\ &= \mu_H(H \cap gKg^{-1}) \sum_i \delta_G(h_i)^{-1} f(h_i g) \end{aligned}$$

is zero by hypothesis, so that

$$\begin{aligned} I(f) &= \int_G \delta_G(x)^{-1} f(x) d\mu_G(x) \\ &= \sum_i f(h_i g) \int_{h_i gK} \delta_G(x)^{-1} d\mu_G(x) \\ &= \mu_G(K) \delta_G(g)^{-1} \sum_i \delta_G(h_i)^{-1} f(h_i g) \end{aligned}$$

is zero as well.

Let $I_{H\backslash G}$ be the G-invariant linear form on $C_c^\infty(H\backslash G, \delta_{H\backslash G})$. One may use the symbol $\mu_{H\backslash G}$ to write

$$I_{H\backslash G}(f) = \int_{H\backslash G} f d\mu_{H\backslash G}$$

This $\mu_{H\backslash G}$ is a *semi-invariant* measure on $H\backslash G$. It is positive in the sense that it takes nonnegative values on nonnegative functions.

2.1 The duality theorem

Theorem 2.3. *Suppose $H \subset G$ is a closed subgroup with G/H compact. Let $\sigma\colon H \to \mathrm{GL}(W)$ be a smooth representation of H. There is an isomorphism*

$$\left(\text{sm-Ind}_H^G \sigma\right)^\vee \cong \text{sm-Ind}_H^G \left(\delta_{H\backslash G} \otimes \check{\sigma}\right).$$

Let $C_c^\infty(H\backslash G, W)$ be the space of smooth functions $f\colon G \to W$ for which $f(hg) = \sigma(h)f(g)$ and which have compact support on $H\backslash G$. Then $C_c^\infty(H\backslash G, W)$ is the same thing as $\text{sm-Ind}_H^G \sigma$: any function $f\colon G \to W$ is automatically compactly supported modulo H, because $H\backslash G$ is compact. The natural pairing $W \otimes \check{W} \to \mathbf{C}$ induces a bilinear map

$$C_c^\infty(H\backslash G, W) \otimes C_c^\infty(H\backslash G, \check{W} \otimes \delta_{H\backslash G}) \to C_c^\infty(H\backslash G, \delta_{H\backslash G})$$

which is then composed with the semi-invariant measure $\mu_{H\backslash G}$ to arrive at the desired pairing. To actually show the pairing is perfect, we refer to Bushnell-Henniart, §1.3.5 Theorem.

2.2 Unitary induction and unitary representations

We call a smooth representation $\rho\colon G \to \mathrm{GL}(V)$ *unitary* if there exists a Hermitian inner product on V with respect to which $\rho(g)$ is a unitary operator for each $g \in G$. (Some authors use the term pre-unitary for this, reserving the term unitary for those V which are Hilbert spaces.)

For instance, a 1-dimensional representation $\chi\colon G \to \mathbf{C}^\times$ is unitary if and only if $|\chi(g)| = 1$ for all $g \in G$.

Let $H \subset G$ be a closed subgroup with $H\backslash G$ compact. If σ is a representation of H, write

$$\iota_H^G \sigma = \text{sm-Ind}_H^G \left(\delta_{H\backslash G}^{1/2} \otimes \sigma\right).$$

Then

$$\left(\iota_H^G \sigma\right)^\vee = \iota_H^G \check{\sigma}$$

Proposition 2.4. *If σ is unitary then so is $\iota_H^G \sigma$.*

Let's sketch out the proof in the case that σ is a (1-dimensional) character. Define a Hermitian inner product on $\iota_H^G \sigma = \operatorname{Ind} \sigma \otimes \delta_{H\backslash G}^{1/2}$ by

$$\langle f_1, f_2 \rangle = \int_{H\backslash G} f_1(g)\overline{f_2(g)} d\mu_{H\backslash G}.$$

The integrand $F(g) = f_1(g)\overline{f_2(g)}$ has the property that for $h \in K$,

$$F(hg) = \sigma(h)\overline{\sigma(h)}\delta_{H\backslash G}(h)F(g) = \delta_{H\backslash G}(h)F(g),$$

so that F belongs to $C_c^\infty(H\backslash G, \delta_{H\backslash G})$, ensuring that the integral makes sense. The right G-invariance of $\delta_{H\backslash G}$ now ensures that the action of G is unitary with respect to $\langle f_1, f_2 \rangle$.