Introduction to Automorphic Representations

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1 Definition of Automorphic Representation

1.1 The Hilbert space $L^2_{\text{cusp}}(\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A_Q}), \chi)$.

Let χ be a Dirichlet character. We identify χ with a character of $\mathbf{A}_Q^\times/\mathbf{Q}^\times\mathbf{R}_{>0}^\times = \hat{\mathbf{Z}}$. Let G be the algebraic group $\mathrm{GL}(2)$, and let $Z \subset G$ be the center, so that Z is isomorphic to the multiplicative group. Let $L^2(\chi) = L^2(G(\mathbf{Q})\backslash G(\mathbf{A_Q}), \chi)$ be the space of measurable functions $\phi\colon G(\mathbf{A_Q}) \to \mathbf{C}$ which satisfy these properties:

1. $\phi\left(\begin{pmatrix} z & \\ & z \end{pmatrix} g\right) = \chi(z)\phi(g)$ for $z \in Z(\mathbf{A}_Q)$, $g \in G(\mathbf{A}_Q)$
2. $\phi(\gamma g) = \phi(g)$, $\gamma \in G(\mathbf{Q})$, $g \in G(\mathbf{A}_Q)$.
3. $\int_{Z(\mathbf{A_Q})G(\mathbf{Q})\backslash G(\mathbf{A_Q})} |\phi(g)|^2 \, dg < \infty$

The last integral is something like the Peterssen norm on modular forms. If the integral in (3) is zero, then ϕ is taken to be zero inside of $L^2(\chi)$. Then $L^2(\chi)$ is a Hilbert space under the pairing

$$(\phi_1, \phi_2) \mapsto \int_{Z(\mathbf{A_Q})G(\mathbf{Q})\backslash G(\mathbf{A_Q})} \phi_1(g)\overline{\phi_2(g)}\, dg$$

$L^2(\chi)$ admits a unitary action of $G(\mathbf{A_Q})$ via right translation.

Suppose χ is a Dirichlet character whose conductor divides $N \geq 1$. If $f \in S_k(\Gamma_0(N), \chi)$, define $\phi_f \in L^2(\chi)$ as follows. By strong approximation we have $G(\mathbf{A}_Q) = G(\mathbf{Q})K_0(N)G^+(\mathbf{R})$. If $g = \gamma\kappa g_\infty$ we set

$$\phi_f(g) = \chi(a) f|_{g_\infty,k}(i),$$

where $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then in fact $\phi_f \in L^2(\chi)$. The cuspidality of f at ∞ means exactly that $\int_{z=0}^1 f(x+z)dx = 0$ for all $x \in \mathcal{H}$, which translates into the condition that

$$\int_{\mathbf{Q}\backslash\mathbf{A}_\mathbf{Q}} \phi_f \left(\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} g \right) dz = 0$$

for all $g \in \mathrm{SL}_2(\mathbf{R})$.

Based on this observation, we define a subspace $L^2_{\mathrm{cusp}}(\chi)$ of $L^2(\chi)$ as the space of functions ϕ satisfying the additional condition

$$\int_{\mathbf{Q}\backslash\mathbf{A}_\mathbf{Q}} \phi \left(\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} g \right) dz = 0$$

for almost all $g \in G(\mathbf{A}_\mathbf{Q})$. It turns out that the representation theory of $L^2_{\mathrm{cusp}}(\chi)$ is much simpler than that of $L^2(\chi)$, in the sense that the irreducibles appearing in the former appear *discretely* in $L^2_{\mathrm{cusp}}(\chi)$. On the other hand, the complement of $L^2_{\mathrm{cusp}}(\chi)$ in $L^2(\chi)$ decomposes as a *direct integral* of irreducibles which correspond to the Eisenstein series. In what follows we will focus exclusively on $L^2_{\mathrm{cusp}}(\chi)$ rather than $L^2(\chi)$.

One can pass from the Hilbert space $L^2_{\mathrm{cusp}}(\chi)$ to the subspace $L^2_{\mathrm{cusp}}(\chi)^\infty$ of vectors which are smooth with respect to the action of $G(\mathbf{R})$. This subspace is preserved by $G(\mathbf{A}^{\mathrm{fin}})$.

1.2 Admissible representations of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$

We have developed a convenient theory of representations for both the locally profinite group $\mathrm{GL}_2(\mathbf{Q}_p)$ (p prime) and the Lie group $\mathrm{GL}_2(\mathbf{R})$. For $\mathrm{GL}_2(\mathbf{Q}_p)$ we have the notion of an admissible representation, which is a smooth representation satisfying a certain finiteness condition. For $\mathrm{GL}_2(\mathbf{R})$ we have the notion of a $(\mathfrak{g}, K_\infty)$-module, which isn't a representation of $G(\mathbf{R})$ at all but rather a vector space admitting simultaneous actions of $\mathfrak{g}$ and the maximal compact open $K_\infty = O(2)$.

Suppose V is a complex vector space together with a smooth action $G(\mathbf{A}^{\mathrm{fin}}) \to \mathrm{GL}(V)$ of the finite adele group, an action $\mathfrak{g} \to \mathrm{End}(V)$ of the Lie algebra of $G(\mathbf{R})$, and an action $K_\infty \to \mathrm{GL}(V)$ of $K_\infty = O(2)$. All of these actions will be denoted π. Assume that (π, V) (forgetting the $G(\mathbf{A}^{\mathrm{fin}})$-action) is a $(\mathfrak{g}, K_\infty)$-module in the sense we have defined previously. Call (π, V) *admissible* if the restriction of π to the maximal compact subgroup

$K = G(\hat{\mathbf{Z}})K_\infty$ of $G(\mathbf{A})$ contains each isomorphism class at most finitely many times. Such a (π, V) will be called an admissible representation of $G(\mathbf{A_Q})$, with the caveat that it isn't literally a representation of $G(\mathbf{A_Q})$ at all!

Let (π, V) be an irreducible admissible representation of $G(\mathbf{A_Q})$. By the factorizability theorem there exists for each prime p an irreducible admissible representation (π_p, V_p), almost all of which are spherical, together with an irreducible admissible $(\mathfrak{g}, K_\infty)$-module (π_∞, V_∞) such that $\pi = \bigotimes'_{p \leq \infty} \pi_p$.

Definition 1.1. A *cuspidal automorphic representation* (π, V) of $G(\mathbf{A_Q})$ with central character χ is an irreducible admissible representation of $G(\mathbf{A_Q})$ which is contained in $L^2_{\text{cusp}}(\chi)$.

Or rather we should say "contained in $L^2_{\text{cusp}}(\chi)^\infty$", because the vectors in V need to be smooth in order for the action of $\mathfrak{g}$ to make any sense.

We ought to explain to what extent the huge space $L^2_{\text{cusp}}(\chi)$ is built out of the individual cuspidal automorphic representations (π, V). It turns out that $L^2_{\text{cusp}}(\chi)$ is the Hilbert space direct sum of irreducible invariant subspaces (π, H). (See Bump's Thm. 3.3.2. This is not true of $L^2(\chi)$.) For each such (π, H), let $V \subset H$ be the subspace of K-finite vectors (and remember $K = G(\hat{\mathbf{Z}})O(2)$ here). Then V is dense in H, and π induces an irreducible admissible representation of $G(\mathbf{A})$ on V (see Thm. 3.3.4 in Bump). So in a sense the cuspidal automorphic representations with central character χ really do account for the entirety of $L^2_{\text{cusp}}(\chi)$, in the sense that the closure of their direct sum is all of $L^2_{\text{cusp}}(\chi)$.

2 Classification of cuspidal automorphic representations for $\mathrm{GL}(2)$

3 Holomorphic cusp forms

The cuspidal automorphic representations (π, V) of $\mathrm{GL}_2(\mathbf{A_Q})$ can be classified by the isomorphism class of the component π_∞ at the archimedean place.

Let us first consider the case that π_∞ is a discrete series representation of weight $k \geq 2$. This means that V_∞ is the direct sum of the spaces $V_\infty[n]$ for $n = \ldots, -k-2, -k, k, k+2, k+4, \ldots$. Let $\phi_\infty \in V_\infty[k]$ be a nonzero vector, so that $L\phi_\infty = 0$. Meanwhile for p finite, let ϕ_p be a nonzero new vector.

Recall what this means: there is a unique integer $c_p \geq 0$ for which the space of vectors $v \in V_p$ satisfying

$$\pi_p\left(\begin{pmatrix} a & b \\ cp^{c_p} & d \end{pmatrix}\right) v = \chi_p(a) v$$

is one-dimensional; let ϕ_p be any nonzero vector in that space. (Here χ_p is the central character of π_p.) When π_p is spherical, $c_p = 0$. Let $N = \prod_p p^{c_p}$ and let $\phi = \otimes_{p \leq \infty} \phi_p \in V$. By the definition of automorphic representation, ϕ actually lies in $L^2_{\text{cusp}}(\chi)$, where χ is a Dirichlet character (whose local component at p is necessarily χ_p). Note that ϕ must be invariant under the open compact subgroup $K_1(N) \subset G(\mathbf{Z})$. Define a function f on the upper half plane by

$$f(z) = y^{-k/2} \phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right), \; z = x + iy$$

Here the matrix $\begin{pmatrix} y & x \\ & 1 \end{pmatrix}$ is to be interpreted as the element of $G(\mathbf{A_Q})$ which is that matrix at the infinite place and 1 everywhere else. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$f|_{\gamma,k}(z) = (cz+d)^{-k} f(\gamma z) = (cz+d)^{-k} (y')^{-k/2} \phi\left(\begin{pmatrix} y' & x' \\ & 1 \end{pmatrix}\right),$$

where $\gamma z = x' + iy'$. Now note that

$$\begin{pmatrix} y' & x' \\ & 1 \end{pmatrix}^{-1} \gamma \begin{pmatrix} y & x \\ & 1 \end{pmatrix} = \kappa_\theta \begin{pmatrix} \sqrt{y/y'} & \\ & \sqrt{y/y'} \end{pmatrix},$$

where $\kappa_t heta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a rotation matrix satisfying $e^{-i\theta} = (cz + d)/|cz+d|$. (Hint: the product on the left side takes i to i, so it must be a rotation matrix times a scalar. Now compare determinants.) We get

$$\begin{aligned} \phi\left(\begin{pmatrix} y' & x' \\ & 1 \end{pmatrix}\right) &= \phi\left(\gamma \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \kappa_\theta^{-1} \begin{pmatrix} \sqrt{y/y'} & \\ & \sqrt{y/y'} \end{pmatrix}\right) \\ &= e^{-ik\theta} \phi\left(\gamma \begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \\ &= \frac{(cz+d)^k}{|cz+d|^k} \chi(a) \phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right). \end{aligned}$$

Putting this together with the fact that $y' = y/\left|cz+d\right|^2$, we get

$$\begin{aligned} f|_{\gamma,k}(z) &= (cz+d)^{-k}(y')^{-k/2}\phi\left(\begin{pmatrix} y' & x' \\ & 1 \end{pmatrix}\right) \\ &= (cz+d)^{-k}\frac{\left|cz+d\right|^k}{y^{k/2}}\frac{(cz+d)^k}{\left|cz+d\right|^k}\chi(a)\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \\ &= \chi(a)y^{-k/2}\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \\ &= \chi(a)f(z) \end{aligned}$$

Furthermore, by Ex. 3c of HW #6, the relation $L\phi_\infty = 0$ actually implies that f is holomorphic! The condition that ϕ is cuspidal implies that f actually belongs to $S_k(N,\chi)$. A little more work shows that f is a *newform*.

The rest is familiar territory: if $f = \sum_{n\geq 1} a_n q^n$, then for all $p \nmid N$, the local component π_p is the unramified principal series representation $\pi(\chi_1,\chi_2)$, where the χ_i are unramified characters of $\mathbf{Q}_p^\times$ satisfying $\chi_1(p)+\chi_2(p) = p^{(1-k)/2}a_p$ and $\chi_1(p)\chi_2(p) = \chi(p)$.

4 Maass cusp forms

Now suppose (π, V) is a cuspidal automorphic representation with π_∞ belonging to the principal series. For simplicity, let's assume that $V_\infty = \bigoplus_{k\in 2\mathbf{Z}} V_\infty[k]$, so that only the even weights appear in V_∞. Let λ be the eigenvalue of Δ on V_∞. Let $\phi_\infty \in V[0]$ be a nonzero vector, and let ϕ_p be a new vector for each finite p as above. Let $\phi = \otimes_{p\leq\infty}\phi_p \in V$. Define a function f on the upper half plane by

$$f(z) = \phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right),\ z = x+iy.$$

By a similar calcuation as in the previous section (in fact simpler because $k=0$), we have that $f(\gamma z) = \chi(a)f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then (see Ex. 3b of HW #6, and perhaps rescale things by 1/4) f satisfies the differential equation

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f = \lambda f$$

The cuspidal condition implies that f decays rapidly at the cusps. Such functions f are called a *Maass forms*, after Hans Maass, who discovered them in 1949 (well before this representation-theoretic interpretation).

Since f is not holomorphic, we cannot expect a Fourier expansion of the form $f(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$ as in the case of classical cusp forms. But f is still periodic, so it must admit an expression

$$f(z) = \sum_{n\in\mathbf{Z}} a_n(y) e^{2\pi i n x},\ z = x + iy$$

with a sequence of unspecified functions $a_n(y)$. In fact the condition $\Delta f = \lambda f$ determines each $a_n(y)$ up to a scalar. Explicitly: write $\lambda = 1/4 + r^2$. For any complex s, let $K_s(y)$ be the *Bessel function*

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{dt}{t}.$$

This is (up to a scalar) the unique solution to the differential equation

$$\left(y^2\frac{d^2}{dy^2} + y\frac{d}{dy} - (y^2 + s^2)\right) K_s = 0$$

which decays exponentially as $y \to \infty$. Then there are scalars a_n, $n \geq 0$, with

$$f(z) = \sum_{n\neq 0} a_n \sqrt{y} K_{ir}(2\pi\,|n|\,y) e^{2\pi i n x}$$

The a_p bear the same relationship to the local components π_p as in the previous section.

4.1 A generalized Ramanujan-Peterssen conjecture

Recall that the RP conjecture for a cuspidal weight k newform $f = \sum_{n\geq 1} a_n q^n$ was the inequality $|a_p| \leq 2p^{(k-1)/2}$. This is equivalent to saying that if π is the corresponding automorphic representation π_p is *tempered* for all p. For a spherical representation $\pi_p = \pi(\chi_1, \chi_2)$, this amounts to saying that the χ_i are unitary. (Reminder: a priori π_p is unitarizable, but this does not imply that the χ_i are unitary. There is a class of $\pi(\chi_1, \chi_2)$, the *complementary series*, which are unitary but not tempered.)

Conjecture 4.1 (Generalized RP)**.** *Let π be an automorphic representation of $G = \mathrm{GL}(2)$. Then π_p is tempered for all $p \leq \infty$.*

When π arises from a holomorphic cusp form, it means exactly the same as the usual RP conjecture for p finite, and is automatic for $p = \infty$, because unitary discrete series representations are always tempered. Furthermore, in the holomorphic case the conjecture is a theorem of Deligne. But when π arises from a Maass form f, the generalized RP conjecture has nontrivial consequences at all places. At the infinite place, it means that if π_∞ corresponds to the principal series representation $\pi(\chi_1, \chi_2)$, then (once again) the χ_i are unitary. Suppose $\chi_i(x) = |x|^{s_i} \operatorname{sgn}(x)^{\varepsilon_i}$. Let λ be the eigenvalue of Δ on π_∞. Then

$$\lambda = s(1-s), \text{ where } s = \frac{1}{2}(s_1 - s_2 + 1).$$

Therefore the RP for π_∞ amounts to the assertion that the s_i are pure imaginary and that $\lambda \geq 1/4$. (See Bump Thm. 2.6.2 and the discussion that follows. The representations with $0 < \lambda < 1/4$ are unitarizable but not tempered; this is a bit of a subtle point.) This is called *Selberg's eigenvalue conjecture.*

The RP conjecture is true for Maass forms of level 1 (ie invariant under $\mathrm{SL}_2(\mathbf{Z})$), for which the corresponding values of r are 9.533, 12.17, 13.77.... The RP for Maass forms in general is *wide open*, for both finite and infinite places. As for Selberg's eigenvalue conjecture, the best bound is currently $\lambda \geq 171/784 = .2181...$, proved by Luo, Rudnick and Sarnak in 1994. Note that the eigenvalue conjecture is really just a question about solutions to certain differential equations for functions on Riemann surfaces of the form $\Gamma\backslash\mathcal{H}$, with Γ a congruence subgroup. There doesn't seem to be any relation to number theory. However, for Γ an arbitrary discrete subgroup, the conjecture is most certainly false: there are cocompact subgroups Γ for which the smallest eigenvalue of Δ on $\Gamma\backslash\mathcal{H}$ can be made arbitrarily small[1].

5 Automorphic representations for other groups

One can now generalize the notion of automorphic representation to algebraic groups other than $\mathrm{GL}(2)_{\mathbf{Q}}$. Certainly there is no problem replacing $\mathbf{Q}$ by any number field or function field. (Note that function fields lack archimedean places, which simplifies the theory enormously!) The group GL(2) is typically

[1]See Sarnak's expository article at `http://www.ams.org/notices/199511/sarnak.pdf`.

replaced by an arbitrary *reductive group*. We will now overview a few such generalizations.

5.1 Hilbert modular forms

These are the classical cusp forms' closest cousins. If K is a totally real field of degree d, and χ a character on $\mathbf{A}_K^\times/K^\times$ which vanishes on the connected component of the identity (this would be $(R_{>0}^\times)^d$), one may consider the space $L^2(\chi) = L^2(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbf{A}_K), \chi)$. When π is an irreducible admissible subspace of $L^2_{\mathrm{cusp}}(\chi)$ such that π_v is discrete series of weight k_v at each infinite place, then a recipe similar to the above produces a family of functions $f\colon \mathbf{H}^d \to \mathbf{C}$, holomorphic in each variable and possessing an automorphicity property with respect to a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$, considered as a subgroup of $\mathrm{SL}_2(K\otimes\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})^d$. Everything can be developed from this point of view. Things get messy, however, when the narrow class number of K is greater than one.

5.2 Automorphic forms on quaternion groups

Let $D/\mathbf{Q}$ be a nonsplit quaternion algebra. For instance, D could be the noncommutative $\mathbf{Q}$-algebra generated by elements i, j satisfying $i^2 = j^2 = -1$ and $ij = -ji$. Let $\mathcal{O}$ be a maximal order in D. We have the algebraic group $G/\mathbf{Z}$ defined by $G(R) = (\mathcal{O}\otimes R)^\times$. Then $G(\mathbf{A}_\mathbf{Q})$ is the restricted direct product of the $(D\otimes\mathbf{Q}_v)^\times$ with respect to the $(\mathcal{O}_D\otimes\mathbf{Q}_v)^\times$. (Note that $G(\mathbf{A}_\mathbf{Q})$ does not depend on the choice of $\mathcal{O}$!) One has the space $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}_\mathbf{Q}))$. There is no notion of cuspidal subspace here, since G doesn't have any parabolic subgroups.

As usual we may classify the automorphic representations π by their component at ∞. If D is indefinite, then $G(\mathbf{R}) \cong \mathrm{GL}_2(\mathbf{R})$. The theory is then quite similar to the theory for GL_2, and can be formulated in terms of functions on the upper half-plane modulo certain discrete subgroups $\Gamma \subset \mathrm{SL}_2(\mathbf{R})$ (which however are not generally contained in $\mathrm{SL}_2(\mathbf{Q})$).

If D is definite, then $G(\mathbf{R}) = GU(2)$ is a group which is compact modulo its center. Therefore irreducible representations of $G(\mathbf{R})$ are finite-dimensional; there is no need to pass to $(\mathfrak{g}, K)$-modules. Suppose we wish to classify automorphic representations of G such that π_∞ equal to the trivial representation of $G(\mathbf{R})$ and such that π_p is spherical for all finite p. These are exactly the irreducible summands of the representation of $G(\mathbf{A}_\mathbf{Q}^{\mathrm{fin}})$ in

$L^2(G(\mathbf{Q})\backslash G(\mathbf{A}_Q^{\mathrm{fin}})/G(\hat{\mathbf{Z}}))$. However the double coset space

$$G(\mathbf{Q})\backslash G(\mathbf{A}_{\mathbf{Q}}^{\mathrm{fin}})/G(\mathbf{Z}) = D^\times\backslash(D\otimes\mathbf{A}_Q^{\mathrm{fin}})^\times/(\mathcal{O}\otimes\hat{\mathbf{Z}})^\times$$

is actually finite! For instance if D is the unique quaternion algebra ramified at 11 and ∞, then this set has exactly two elements. The Hecke operators $T_2, T_3, T_5, \dots$ act on $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}_Q^{\mathrm{fin}})/G(\hat{\mathbf{Z}}))$. When one diagonalizes these operators, one finds a vector on which T_p acts by $p+1$ for all $p\neq 11$ (an Eisenstein series), and another vector ϕ on which $T_2, T_3, T_5, T_7, \dots$ acts by $-2, -1, 1, -2, \dots$. Miraculously, there is a *classical* cusp form $f\in S_2(\Gamma_0(11))$ with the same Hecke eigenvalues!

Let π be the cuspidal automorphic representation corresponding to $f = q-2q^2-q^3-2q^5+\dots$, and let π' be the automorphic representation containing the vector v above. For $p\neq 11,\infty$, there is an isomorphism $G(\mathbf{Q}_p)\cong \mathrm{GL}_2(\mathbf{Q}_p)$, and the equality of Hecke eigenvalues shows that $\pi_p\cong\pi'_p$ for those primes. The relationship between π and π' is known as the *Jacquet-Langlands correspondence.* It relates the automorphic representations attached to GL(2) to those attached to its twists.

There is a local version of the correspondence. Note that π_∞ is the discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ of weight 2 and trivial central character, while π'_∞ is the trivial representation. Discrete series representations D_k of $\mathrm{GL}_2(\mathbf{R})$ are parametrized by integers $k\geq 2$ (ignoring twists by 1-dimensional characters). On the other hand, all the irreducible representations of $G(\mathbf{R}) = GU(2)$ are (again ignoring twists) symmetric powers of the tautological representation $\rho\colon GU(2)\hookrightarrow \mathrm{GL}_2(\mathbf{C})$. The local JL correspondence ties together D_k to $\mathrm{Sym}^{k-2}\rho$.

At the finite primes, there is also a local JL correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ and its inner twist, but this is much more opaque because it involves the supercuspidal representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. According to the above example, the Steinberg representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ corresponds to the trivial representation of $G(\mathbf{Q}_p)$.

5.3 GL(3)

The landscape is truly different for a higher-rank group such as GL(3). One major difference is that $\mathrm{GL}_3(\mathbf{R})$ admits no discrete series. (Harish-Chandra showed that a semisimple Lie group G admits a discrete series if and only if the rank of G equals the rank of a maximal compact subgroup of G. This shows that $\mathrm{SL}_n(\mathbf{R})$ has no discrete series for $n \geq 3$.) A related problem is that the analogue of the upper half plane, $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}(3)$, is a 5-dimensional real manifold and has no chance of being a complex manifold.

Therefore there is no direct analogue of Hecke's theory of holomorphic cusp forms for GL(3). Automorphic representations of GL(3) have more in common with Maass forms in that they are necessarily principal series representations at the infinite places.