MA 843 Assignment 1, due Thurs. 9/15

For these exercises, $G = \mathrm{GL}_2(\mathbb{F}_q)$ and $B \subset G$ is the subgroup of upper-triangular matrices. V is the space of complex-valued functions on $\mathbb{F}_{q^2}$, and $r\colon \mathrm{SL}_2(\mathbb{F}_q) \to \mathrm{GL}(V)$ is the representation defined in lecture on Thursday 9/8. Note that r depends on a choice of nontrivial additive character $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$. It is defined as the unique representation of $\mathrm{SL}_2(\mathbb{F}_q)$ on V satisfying

$$\begin{aligned} r\left(\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}\right)\Phi(x) &= \psi(zN(x))\Phi(x) \\ r\left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\right)\Phi(x) &= \Phi(ax) \\ r\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\Phi(x) &= \hat{\Phi}(x^q) \end{aligned}$$

We write $N^1(\mathbb{F}_{q^2})$ for the kernel of the norm map $N\colon \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$. The following is an adaptation to the finite field case of material from Weil and Jacquet-Langlands on representations of GL_2 over a nonarchimedean local field.

1. Let χ_1, χ_2 be two characters of $\mathbb{F}_q^\times$. Recall that $\pi(\chi_1, \chi_2)$ is the representation induced from the character $\chi = (\chi_1, \chi_2)$ of B. Construct an isomorphism between $\pi(\chi_1, \chi_2)$ and $\pi(\chi_2, \chi_1)$.

2. Let k be a field. Show that $\mathrm{SL}_2(k)$ really is the group generated by elements $t_a = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $u_z = \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$ and $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ ($a \in k^\times$, $z \in k$) modulo the obvious relations among the t_as and those among the u_zs, along with

$$\begin{aligned} wt_a &= t_{a^{-1}}w \\ w^2 &= t_{-1} \\ wu_aw &= t_{-a^{-1}}u_{-a}wu_{-a^{-1}}. \end{aligned}$$

That is, show that these relations suffice to define $\mathrm{SL}_2(k)$.

Once one shows that the operators $r(u_z)$, $r(t_a)$ and $r(w)$ defined above satisfy the appropriate relations (I'm not asking you to do this), the existence of r is established.

3. Let θ be a character of $N^1(\mathbb{F}_{q^2})$. Let Θ be any character of $\mathbb{F}_{q^2}^\times$ which extends θ. Show that $r|_{V[\theta]}$ has a unique extension to a representation $\pi_\Theta \colon G \to \mathrm{GL}(V[\theta])$ which satisfies

$$\pi_\Theta\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) f(y) = \Theta(x) f(xy)$$

whenever $N(x) = a$.

4. Suppose that Θ is a character of $\mathbb{F}_{q^2}^\times$, with $\Theta \neq \Theta^q$. Show that the restriction of π_Θ to the subgroup $\begin{pmatrix} 1 & \mathbb{F}_q \\ & 1 \end{pmatrix} \subset G$ decomposes into the sum over the nontrivial characters of that group, each appearing with multiplicity one. All of those characters are G-conjugate; deduce from this that π_Θ is irreducible. Such a $\pi = \pi_\Theta$ is called a *cuspidal* representation of G.

5. Let $W(\psi)$ be the space of complex-valued functions on G satisying

$$W\left(\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} g\right) = \psi(a) W(g)$$

for all $a \in \mathbb{F}_q$. Note that $W(\psi)$ has a natural action of G by right-translation. Let $\pi = \pi_\Theta$ be a cuspidal representation of G as in the previous exercise. For $f \in V[\theta]$, define an element W_f in $W(\psi)$ by

$$W_f(g) = (\pi(g) f)(1)$$

Let $W(\pi, \psi) \subset W(\psi)$ be the span of the W_f. Show that $f \mapsto W_f$ is an isomorphism of $V[\theta]$ onto $W(\pi, \psi)$. Also show that $W(\pi, \psi)$ is exactly the π-isotypic part of $W(\psi)$. $W(\pi, \psi)$ is called a *Whittaker model* of π. (In the theory of automorphic representations of GL_n, Whittaker models play a role in the definition of the L-function.) Does every irreducible representation of G have a Whittaker model?