MA 843 Assignment 2, due Tues. 9/27

1. Show that $K \subset \mathbf{A}_K$ is a discrete subgroup.

2. Show that the topology on the ideles $\mathbf{A}_K^\times$ is *not* the same as the subset topology induced from the adeles $\mathbf{A}_K$.

3. Let χ be a nontrivial Dirichlet character on $\mathbb{Z}/p\mathbb{Z}$. Then χ can be considered as a character of $\mathbf{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$, and it makes sense to talk about the restriction χ_v of χ to $\mathbb{Q}_v^\times$ for each place v of $\mathbb{Q}$. Follow Tate's thesis to compute $\rho(\chi_v \,|\,|^s)$ as a function of s, for each v. Then write down the functional equation for $L(\chi, s)$.

4. Let K be the function field of a complete nonsingular curve $C/\mathbf{F}_q$. This exercise shows how the classical Riemann-Roch formula drops out of the adelic version we talked about in lecture. Let $\operatorname{Div} C$ be the free abelian group on the set of places of K, and let $\operatorname{Pic} C$ be the quotient of $\operatorname{Div} C$ by the group of those divisors of the form $\operatorname{div} g = \sum_v \operatorname{ord}_v(g)$, where $g \in K^\times$. Also let $\psi_v \colon K_v \to \mathbb{C}^\times$ be a collection of local additive characters, such that $\psi_v(\mathcal{O}_v) = 1$ for almost all v, and such that $\psi = \prod_v \psi_v$ is trivial on $K^\times$.

   (a) For each place v, let $c_v \in \mathbb{Z}$ be the least integer (please see note beneath) for which ψ_v is trivial on $\mathfrak{p}_v^{c_v}$. Let $\kappa = \sum_v c_v v \in \operatorname{Div} C$. Show that the image of κ in $\operatorname{Pic} C$ is independent of the choice of ψ. This image is called the "canonical bundle" of C.

   (b) For any divisor $D = \sum_v a_v$, let $U(D)$ be the (compact) set of adeles (x_v) with $\operatorname{ord}_v x_v \geq a_v$. Let $\mathcal{L}(D)$ be the union of $U(D) \cap K^\times$ with 0. This is a finite-dimensional $\mathbf{F}_q$-vector space (being the intersection of compact with discrete, it is set-theoretically finite); we write $\#\mathcal{L}(D) = q^{\ell(D)}$. Then $\ell(D)$ only depends on the image of D in $\operatorname{Pic} C$. Let $g = \ell(\kappa)$; this number is the genus of C.

With respect to the self-duality on $\mathbf{A}_K$ induced by ψ, show that $U(D)^\perp = U(\kappa - D)$ for $D \in \operatorname{Div} K$.

(c) Let dx be the self-dual Haar measure on $\mathbf{A}_K$ with respect to ψ. Show that $\int_{U(0)} dx = q^{-\frac{1}{2}\deg\kappa}$.

(d) Let $D \in \operatorname{Div} K$. Apply the adelic Riemann-Roch formula to the characteristic function of $U(D)$ and deduce that $\ell(D) = \ell(\kappa - D) + \deg D - \frac{1}{2}\deg\kappa$. Plug in $D = 0$ to the above to derive the formula $\deg\kappa = 2g-2$. Therefore $\ell(D) = \ell(\kappa - D) + \deg D + 1 - g$.

In the context of the above exercise, it isn't obvious that ψ_v couldn't simply be *trivial* on K_v for some v, which case the integer c_v wouldn't be well-defined. In fact, ψ_v must be a nontrivial character of K_v for every v. I will explain why in two steps.

First, I want to explain why there exists some character ψ of $\mathbf{A}_K/K$ which has the desired property. In the number field case, Tate does this "by hand" by constructing a special ψ in the case of $K = \mathbb{Q}$ and extending it to all number fields through the trace map. In the function field case, there is a nice conceptual construction. Choose any nonzero differential form ω on C (this will be allowed to have poles at some places). Let ψ_0 be a nontrivial additive character of the (finite) field of constants of C. Then define a character of $\mathbf{A}_K$ by

$$\psi_\omega((f_v)) = \prod_v \psi_0\left(\operatorname{Res}_v f_v \omega\right)$$

Then $\psi_{\omega,v} = \psi_\omega|_{K_v}$ is certainly nontrivial for every v, because you can always find an f_v so that $f_v\omega$ has whatever desired residue. The local character $\psi_{\omega,v}$ is trivial on $\mathcal{O}_v$ for almost all v, including those places at which ω is holomorphic. Furthermore, ψ_ω is trivial on K, because if $f \in K$ then $\sum_v \operatorname{Res}_v f_v \omega = 0$ by the residue theorem!

(The divisor κ constructed from this ψ_ω has associated line bundle equal to the canonical bundle in the usual sense. Sections of this bundle over U are those functions f on U for which $f\omega$ is holomorphic on U. But this bundle is isomorphic via $f \mapsto f\omega$ to the bundle whose sections over U are holomorphic differential forms on U.)

Now I must explain why *any* character ψ of $\mathbf{A}_K/K$ has the desired property. Our character $\psi = \psi_\omega$ sets up a self-duality on $\mathbf{A}_K$, with adeles $a \in \mathbf{A}_K$ being in one-to-one correspondence with characters $\psi_a(x) = \psi(ax)$. Of these characters, which are trivial on K? This is the same as asking what $K^\perp$ is. But now we apply Tate's argument (Thm. 4.1.4 on p. 331 of Cassels-Frohlich): the quotient $K^\perp/K$ is a discrete subset of the compact set $\mathbf{A}_K/K$, hence it is finite. On the other hand $K^\perp$ is a K-vector space, which is only possible if $K^\perp = K$.

Thus every nontrivial character of $\mathbf{A}_K$ trivial on K is of the form ψ_a with $a \in K^\times$. Since a has nontrivial v-component, $\psi_v = \psi \circ a$ cannot be trivial on K_v.