MA 843 Assignment 3, due Thurs. 10/13

1. Suppose $\sum_{n\geq 1} a_n n^{-s}$ converges for $\Re s$ sufficiently large to a function $L(s)$. Suppose that $L(s)$ extends to an entire function of s, which is bounded in vertical strips, and that $\Lambda(s) = (2\pi)^{-s}\Gamma(s)L(s)$ satisfies the functional equation
$$\Lambda(s) = i^k \Lambda(k-s).$$
Show that there exists a cusp form $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ with $L(f,s) = L(s)$. You may use the fact that there is such a thing as the inverse Mellin transform, and also the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

2. In the context of the previous problem, show that f is an eigenform if and only if $L(f,s)$ admits a factorization
$$L(f,s) = \prod_p \left(1 - \frac{a_p}{p^s} + \frac{p^{k-1}}{p^{2s}}\right)^{-1}$$

3. Let A be a positive definite symmetric $4k \times 4k$ matrix with integer coefficients. Let's suppose that A is *even*, meaning that the diagonal entries of A are even integers. Also assume that A^{-1} is integral and even. To A we associate the quadratic form $Q(x) = {}^t x A x$, where x is a column vector, and the theta function
$$\theta_Q(z) = \sum_x e^{\pi i z Q(x)}$$
where the sum runs over all integral vectors in $\mathbb{Z}^{4k}$. Use Poisson summation (on $\mathbb{R}^{4k}/\mathbb{Z}^{4k}$!) to show that $\theta_Q \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$

4. In a variation on the above idea, let $K/\mathbb{Q}$ be the imaginary quadratic field of discriminant D, and let χ be a nontrivial character of the ideal class group of K. Let

$$f_\chi(z) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) q^{N\mathfrak{a}}$$

where the sum runs over nonzero integral ideals of $\mathfrak{a}$.

   (a) Let $L(\chi, s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \sum_{n \geq 1} a_n n^{-s}$ be the corresponding Hecke L-series. Show that

$$L(\chi, s) = \prod_p \left(1 - \frac{a_p}{p^s} + \frac{\omega_D(p)}{p^{2s}}\right)^{-1},$$

   where ω_D is the Legendre symbol $\omega_D(p) = (D/p)$. Also show that $|a_p| \leq 2$ for all p.

   (b) (Optional.) This suggests that $f = f_\chi$ is a modular form of weight 1 and character ω_D. In fact, f is a newform in $S_1(|D|, \omega_D)$. Sketch a proof of this fact using Weil's converse theorem (p. 17 of Gelbart).

   (c) By class field theory, χ corresponds to an unramified character ψ of $\mathrm{Gal}(\overline{K}/K)$. Let ρ be the 2-dimensional Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ induced from ψ. Show that $L(\rho, s) = L(f, s)$.