MA 843 Assignment 5, due Tues. 11/8

1. Let π be a representation of a locally profinite group G on a vector space V. Let $K \subset G$ be a compact open subgroup. Show that π is admissible if and only if both of the following conditions hold:

   (a) For all irreducible representations σ of K, $V[\sigma]$ is finite-dimensional.

   (b) $V = \bigoplus_\sigma V[\sigma]$, where σ runs over all smooth irreducible representations of K.

   (In particular, you must show that these two conditions imply that π is smooth.)

2. Let (π, V) be a nonzero admissible unitary representation of G. Let $K \subset G$ be a compact open subgroup, and let σ be a smooth irreducible representation of K for which $V[\sigma] \neq 0$ (this must exist, by the previous exercise). Let M be a subspace of minimal positive dimension such that $M = W[\sigma]$ for some G-invariant subspace W of V. (This M actually exists because $V[\sigma]$ itself is such a space.) Let

$$U = \bigcap_{W[\sigma]=M} W,$$

   where the intersection runs over G-invariant subspaces W of V such that $W[\sigma] = M$. Show that U is irreducible. Therefore V contains an irreducible representation.

3. Let $K \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ be a compact open subgroup. Let C be a set of coset representatives for

$$\mathrm{GL}_2^+(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbf{A}_{\mathbb{Q}}^{\mathrm{fin}})/K.$$

For each $c \in C$, let $\Gamma_c = \mathrm{GL}_2^+(\mathbb{Q}) \cap cKc^{-1}$. Show that the map

$$\begin{array}{rcl} M_k^K & \to & \bigoplus_{c \in C} M_k(\Gamma_c) \\ \phi & \mapsto & (\phi(c))_{c \in C}, \end{array}$$

makes sense (*i.e.* each $\phi(c)$ lies in $M_k(\Gamma_c)$) and is an isomorphism.