MA 843 Assignment 6, due Thurs. 11/17

1. Let $\mathfrak{g}$ be a finite-dimensional, real Lie algebra. Let $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ be the *Killing form*: $B(X,Y)=\operatorname{Tr}((\operatorname{ad}X)(\operatorname{ad}Y))$, where $\operatorname{ad}X\colon\mathfrak{g}\to\mathfrak{g}$ is the linear map $Y\mapsto[X,Y]$. The Killing form has the property of $\mathfrak{g}$-invariance: $B([X,Y],Z)=B(X,[Y,Z])$. Assume that B is nondegenerate (by the *Cartan criterion*, this is the case if and only if $\mathfrak{g}$ is semisimple). Let $X_1,\dots,X_n$ be a basis for $\mathfrak{g}$, and let $X_1^*,\dots,X_n^*$ be the dual basis with respect to B. Let

   $$\Delta=\sum_{i=1}^{n}X_iX_i^*\in U(\mathfrak{g})$$

   be the Casimir operator. Show that Δ lies in the center of $U(\mathfrak{g})$.

2. Now let $\mathfrak{g}=\mathfrak{sl}(2)$. Let $r=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $l=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&\\&-1\end{pmatrix}$, so that $\{r,l,h\}$ be a basis for $\mathfrak{g}$. Show that the Casimir operator is (up to a scalar) equal to $2rl+2lr+h^2$.

3. Let $\rho\colon \mathrm{SL}_2(\mathbb{R})\to\mathrm{GL}(V)$ be the representation (by right-translation) of $\mathrm{SL}_2(R)$ on the space $V=C^\infty(\mathrm{SL}_2(\mathbb{R}))$ of smooth functions on $\mathrm{SL}_2(\mathbb{R})$. Let $k\in\mathbb{Z}$, and let $V[k]$ be the space of vectors $\phi\in V$ with

   $$\rho\left(\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}\right)\phi=e^{ik\theta}\phi.$$

   One can parametrize $g\in\mathrm{SL}_2(\mathbb{R})$ by variables x,y,θ like so:

   $$g=\begin{pmatrix}y^{1/2}&xy^{-1/2}\\&y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}.$$

   (Note that $g(i)=x+iy$. Also note that the rotation matrix is the clockwise one. I don't like that, but to change it runs against all the notation in the books of Bump and Lang.) Thus any $\phi \in C^\infty(\mathrm{SL}_2(\mathbb{R}))$ can be written as a function $\phi(x, y, \theta)$ of those three variables. In terms of this parametrization, the elements R, L, H, Δ, considered as differential operators on $C^\infty(\mathrm{SL}_2(\mathbb{R}))$, are

$$\begin{aligned}
R &= e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right) \\
L &= e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta}\right) \\
H &= -i\frac{\partial}{\partial \theta} \\
\Delta &= 4y^2\left(\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 4y\frac{\partial^2}{\partial x \partial \theta}.
\end{aligned}$$

(See Lang's book $\mathrm{SL}_2(\mathbb{R})$, pp. 1141-115 and p. 198, or Bump, p. 155. You can also work this out for yourself but I don't really recommend doing so.)

(a) Show that there exists an isomorphism $f \mapsto \phi_f$ between $C^\infty(\mathcal{H})$ and $V[k]$, via
$$\phi_f(x, y, \theta) = e^{ik\theta} f(x + iy).$$

(b) Suppose $\phi = \phi_f$ belongs to $V[0]$ and happens to be an eigenvector for Δ, with $\Delta\phi = \lambda\phi$. Show that f is an eigenvector for the differential operator
$$4y^2\left(\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$
with the same eigenvalue λ. (The above operator, when multiplied by $-1/4$, is the *Laplace-Beltrami operator* for the Riemannian manifold $\mathcal{H}$.)

(c) Suppose that $k > 0$ and $\phi = \phi_f \in V[k]$ is a vector with $L\phi = 0$. Let $g \in C^\infty(\mathcal{H})$ be
$$g(x + iy) = y^{-k/2} f(x + iy).$$
Show that g is *holomorphic.*