The principal series of GL_2 over a local field

October 21, 2011

1 Principal series: definition

Let F be a nonarchimedean local field, let $G = \mathrm{GL}_2(F)$, and let B be its subgroup of upper-triangular matrices. If χ_1, χ_2 are two characters of $F^\times$, write $\chi_1 \times \chi_2$ for the character $\begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d)$ of B. Then set

$$\pi(\chi_1, \chi_2) = \iota_B^G(\chi_1 \times \chi_2) = \text{sm-Ind}\left(\chi_1 \,|\;|^{1/2} \times \chi_2 \,|\;|^{-1/2}\right)$$

Then

$$\pi(\chi_1, \chi_2)^\vee = \pi(\chi_1^{-1}, \chi_2^{-1}).$$

Another important property is that for χ a third character:

$$\pi(\chi_1, \chi_2) \otimes (\chi \circ \det) = \pi(\chi_1\chi, \chi_2\chi).$$

One can construct a large swath of irreducible representations of G by taking all irreducible subquotients of the $\pi(\chi_1, \chi_2)$. These are known as the *principal series* representations of G. To classify them, it is important to know the decomposition series of each $\pi(\chi_1, \chi_2)$. It is not necessarily the case that $\pi(\chi_1, \chi_2)$ is irreducible. For instance, $\pi(|\;|^{-1/2}, |\;|^{1/2}) = \text{sm-Ind}_B^G 1$ sits in an exact sequence

$$0 \to 1_G \to \text{sm-Ind}_B^G 1 \to \mathrm{St} \to 0.$$

Taking duals, $\pi(|\;|^{1/2}, |\;|^{-1/2})$ also fails to be irreducible; it admits 1_G as a quotient and (it turns out) St as a submodule.

1.1 Classification Theorem for principal series representations of G.

Theorem 1.1. *The principal series representations of G are:*

- *The representations $\pi(\chi_1, \chi_2)$, where $\chi_1\chi_2^{-1} \neq |\ |^{\pm 1}$. We have $\pi(\chi_1, \chi_2) \cong \pi(\chi_2, \chi_1)$,*
- *The one-dimensional representations $\chi \circ \det$.*
- *The* special representations $\mathrm{St} \otimes (\chi \circ \det)$.

We sketch a proof that $\pi(\chi_1, \chi_2) \cong \pi(\chi_2, \chi_1)$. (Reference: Bushnell-Henniart, p. 63–64.) Let $N \subset B$ be the unipotent radical, $N = \left\{ \begin{pmatrix} 1 & F \\ & 1 \end{pmatrix} \right\}$. For any representation π of G on a vector space V, let $V(N)$ be the subspace of V spanned by $v - \pi(n)v$, for $v \in V$, $n \in N$. Also let $V_N = V/V(N)$. This is the *Jacquet module* of π. It is a module for the action of the torus $T = B/N$.

In the case that $\pi = \pi(\chi_1, \chi_2)$, it is easy to see that $f \mapsto f(1)$ factors through a surjective map of T-modules $V_N \to \chi_1 |\ |^{1/2} \otimes \chi_2 |\ |^{-1/2} = \chi\delta$.

Let $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, so that $\chi^w = \chi_2 \otimes \chi_1$. We can also construct a surjective map $V_N \to \chi^w\delta$. For $f \in V$, let

$$\phi(f)(x) = \int_N f(wn)dn.$$

Then for $f \in V(N)$, $f = g - \pi(n)g$ for some $g \in V$, $n \in N$, we have $\phi(f) = 0$, so that ϕ factors through a map $V(N) \to \mathbf{C}$. For $t = \begin{pmatrix} a & \\ & d \end{pmatrix} \in T$ we have

$$\begin{aligned}
\phi(\pi(t)f) &= \int_N f(wnt)dn \\
&= \int_N f(wtt^{-1}nt)dn \\
&= \delta_B(t)^{-1} \int_N f(wtn)dn \\
&= \delta_B(t)^{-1} \int_N f(wtw^{-1}wn)dn \\
&= \delta_B(t)^{-1} \chi_1(d)\,|d|^{1/2}\,\chi_2(a)\,|a|^{-1/2}\,\phi(f) \\
&= \chi_2(a)\,|a|^{1/2}\,\chi_1(d)\,|d|^{-1/2}
\end{aligned}$$

In fact V_N is the sum of two characters of T, namely $\chi_1 |\,|^{1/2} \otimes \chi_2 |\,|^{-1/2}$, and $\chi_2 |\,|^{1/2} \otimes \chi_1 |\,|^{-1/2}$. (We have shown that V_N surjects onto the direct sum of those two characters. To show injectivity, one uses the decomposition $G = B \cup BwN$.) Therefore if $\delta = \delta_{B\backslash G}$, then

$$\pi(\chi_1, \chi_2)_N = \chi\delta^{1/2} \oplus \chi^w \delta^{1/2}.$$

Now if χ and ξ are two characters of $B/N = T$, then

$$\begin{aligned}
\mathrm{Hom}_G(\mathrm{Ind}\,\chi, \mathrm{Ind}\,\xi) &= \mathrm{Hom}_B(\mathrm{Res}\,\mathrm{Ind}\,\chi, \xi) \\
&= \mathrm{Hom}_T(\mathrm{Res}\,\mathrm{Ind}\,\chi, \xi) \\
&= \mathrm{Hom}_T((\mathrm{Ind}\,\chi)_N, \xi)
\end{aligned}$$

Writing

$$\begin{aligned}
\chi &= \chi_1 \otimes \chi_2 \\
\xi &= \xi_1 \otimes \xi_2,
\end{aligned}$$

we find

$$\begin{aligned}
\mathrm{Hom}_G(\pi(\chi_1, \chi_2), \pi(\xi_1, \xi_2)) &= \mathrm{Hom}_G(\mathrm{Ind}\,\chi\delta^{1/2}, \mathrm{Ind}\,\xi\delta^{1/2}) \\
&= \mathrm{Hom}_T\left(\chi\delta^{1/2} \oplus \chi^w\delta^{1/2}, \xi\delta^{1/2}\right) \\
&= \mathrm{Hom}_T\left(\chi \oplus \chi^w, \xi\right).
\end{aligned}$$

From this it follows that $\pi(\chi_1, \chi_2) \cong \pi(\chi_2, \chi_1)$.

2 Unitary Representations

We have already seen that if $H \subset G$ and σ is a unitary smooth representation of H then $\iota_H^G \sigma$ is a unitary smooth representation of G. It follows that $\pi(\chi_1, \chi_2)$ is unitary whenever χ_1, χ_2 are. Together with $\mathrm{St} \otimes \chi$ for χ unitary, and $\chi \circ \det$ for χ unitary, these representations form the *tempered principal series* of $G = \mathrm{GL}_2(F)$.

There are other principal series representations of G which are unitary but not tempered. These are exactly the $\pi(\chi_1, \chi_2)$, where $\chi_2 = \overline{\chi_1}^{-1}$ and $\chi_1 \chi_2^{-1}(x) = |x|^\sigma$, for $0 < \sigma < 1$. Such representations form the *complementary series*. It is nontrivial to construct a G-invariant Hermitian form on these $\pi(\chi_1, \chi_2)$ (see Godement's *Notes on Jacquet-Langlands Theory*, 1.58, Theorem 12). Suffice it to say that the tempered representations have a natural inner product, while the complementary series representations have an inner product which is somewhat contrived.

3 Spherical Representations

We need a notion of "unramified" for representations of G which is analogous to the definition of an unramified character of $F^\times$. An unramified character is trivial on $\mathcal{O}_F^\times$. We call a representation π *spherical* (or *class 1*) if it contains a vector fixed by a maximal compact subgroup of $\mathrm{GL}_2(F)$. All such subgroups are conjugate to $\mathrm{GL}_2(\mathcal{O}_F)$, so that π is spherical if and only if it has a K-fixed vector for $K = \mathrm{GL}_2(\mathcal{O}_F)$.

3.1 The Hecke algebra

Let $K \subset G$ be a compact open subgroup, and let $d\mu_G$ be the Haar measure on G which assigns 1 to K.

Let $\mathcal{H}(G)$ be the space $C_c^\infty(G)$ of compactly supported smooth functions on G, endowed with a ring structure under the operation of convolution:

$$(f_1 \star f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu_G(x)$$

This ring does not have a unit element.

If we consider the subalgebra $\mathcal{H}(G, K) \subset \mathcal{H}(G)$ of functions which are bi-invariant by K (that means $f(k_1 g k_2) = f(g)$) is a unital ring, with the characteristic function e_K of K as its unit element. (Be careful: e_K does not serve as the identity in $\mathcal{H}(G, K')$ for smaller K', let along in $\mathcal{H}(G)$.)

Let $\pi\colon G \to \mathrm{GL}(V)$ be a smooth admissible representation. We define an action of $\mathcal{H}(G)$ on V, also written π, by

$$\pi(f)v = \int_G f(g)\pi(g)v d\mu_G(g)$$

(the smoothness of f and π ensure that the integral reduces to a finite sum). Notice that

$$\pi(f_1 \star f_2) = \pi(f_1) \circ \pi(f_2),$$

so this really defines an action $\pi\colon \mathcal{H}(G) \to \mathrm{End}(V)$, analogous to the situation of a finite group, where $\mathbf{C}[G]$ acts on every representation of G. This representation will be smooth in the sense that $\mathcal{H}(G) \star V = V$. In fact there is an equivalence of categories between smooth G-representations and smooth $\mathcal{H}(G)$-modules.

Another important observation is that the action of $\mathcal{H}(G, K)$ sends every vector in V into the finite-dimensional space V^K:

$$\pi(k)\pi(f)v = \int_G f(g)\pi(kg)v d\mu_G(g) = \int_G f(k^{-1}g)\pi(g)v d\mu_G(g) = \pi(f)v$$

In particular $\mathcal{H}(G, K)$ preserves the finite-dimensional space V^K.

Theorem 3.1. *If $\pi\colon G \to \mathrm{GL}(V)$ is a smooth irreducible representation, then either $V^K = 0$ or else π^K is an irreducible representation of $\mathcal{H}(G, K)$. The map $\pi \mapsto \pi^K$ places into bijection the set of (isomorphism classes of) smooth irreducible representations of G with a K-fixed vector, and smooth irreducible representations of $\mathcal{H}(G, K)$.*

See Bushnell-Henniart, 4.3 Theorem.

Thus the theory of smooth representations of G with a K-fixed vector can be reduced to the theory of (finite-dimensional!) representations of the (unital) ring $\mathcal{H}(K)$.

3.2 The spherical Hecke algebra

When $K = \mathrm{GL}_2(\mathcal{O}_F)$, the Hecke algebra takes a simple structure. Let ϖ be a uniformizer of F. Let T_1 be the characteristic function of $K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K$, and let T_2 be the characteristic function of $\begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} K$.

Theorem 3.2. *$\mathcal{H}(G,K)$ is the (commutative!) ring $\mathbf{C}[T_1, T_2^{\pm}]$.*

This is the *Satake isomorphism* for GL_2. The corresponding result for GL_n is covered in, *e.g.*, Bump's article at `sporadic.stanford.edu/bump/orbital.ps`.

The key here is the following *Cartan decomposition* of G:

$$G = \coprod_{i,j} K \begin{pmatrix} \varpi^i & \\ & \varpi^j \end{pmatrix} K$$

where the union runs over pairs (i,j) of integers with $i \geq j$. (The decomposition essentially follows from the structure theorem for finitely generated torsion $\mathcal{O}_F$-modules.) Thus $\mathcal{H}(G,K)$ is spanned by the characteristic functions $f_{i,j}$ of the cosets $K \begin{pmatrix} \varpi^i & \\ & \varpi^j \end{pmatrix} K$.

We can now offer a quick explanation of why $\mathcal{H}(G,K)$ is commutative. Let $\iota \colon \mathcal{H}(G,K) \to \mathcal{H}(G,K)$ be the map induced from the anti-involution $g \mapsto g^t$ of G. Then ι is an anti-involution of $\mathcal{H}(G,K)$: $\iota(f \star g) = \iota(g) \star \iota(f)$. On the other hand, ι has no effect on the $f_{i,j}$, so it must be the identity map. Therefore $\mathcal{H}(G,K)$ is commutative!

For an irreducible spherical representation π of G, the space of K-fixed vectors must be 1-dimensional, and gives a character of $\mathcal{H}(G,K)$.

Theorem 3.3. *Let χ_1, χ_2 be two unramified characters of G, such that $\pi = \pi(\chi_1,\chi_2)$ is irreducible. Then π is a spherical representation, corresponding to the following character of $\mathcal{H}(G,K)$:*

$$\begin{aligned} T_1 &\mapsto \sqrt{q}\,(\chi_1(\varpi) + \chi_2(\varpi)) \\ T_2 &\mapsto \chi_1\chi_2(\varpi) \end{aligned}$$

Conversely, if π is a spherical representation then π is principal series (but not special).

Proof. Put $\pi = \pi(\chi_1,\chi_2)$. Then π is the space of smooth functions $f \colon G \to \mathbf{C}$ which satisfy $f(bg) = \chi(b)\delta(b)^{1/2} f(g)$, $b \in B$, $g \in G$. Here $\chi = \chi_1 \otimes \chi_2$ and $\delta = \delta_{B\backslash G}$ is the character $\begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto |a/d|$ of B.

We have the Iwasawa decomposition $G = BK$, with χ trivial on BK. Therefore the function

$$\begin{aligned} f \colon G &\to \mathbf{C} \\ bk &\mapsto \chi(b)\delta(b)^{1/2} \end{aligned}$$

is well-defined and belongs to π. The function f is spherical, since it is visibly K-invariant, and has $f(1) = 1$. To see where T_1 and T_2 send f, it is enough to compute $\pi(T_1)f(1)$ and $\pi(T_2)f(1)$.

We have the following decomposition of the double coset $K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K$:

$$K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K = \begin{pmatrix}1 & \\ & \varpi\end{pmatrix}K \cup \bigcup_{\alpha}\begin{pmatrix}\varpi & \alpha \\ & 1\end{pmatrix}K.$$

(An explanation: in our discussion of modular forms, we used the $\mathbf{Q}_p$-version of the following decomposition into left K-cosets:

$$K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K = K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix} \cup \bigcup_{\alpha} K\begin{pmatrix}1 & \alpha \\ & \varpi\end{pmatrix}.$$

The claimed decomposition into right cosets follows from this one by inverting and multiplying through by the scalar ϖ.) We now calculate

$$\begin{aligned}
\pi(T_1)f(1) &= \int_G T_1(g)f(g)d\mu_G \\
&= \int_{K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K} f(g)d\mu_G \\
&= f\left(\begin{pmatrix}1 & \\ & \varpi\end{pmatrix}\right) + \sum_{\alpha} f\left(\begin{pmatrix}\varpi & \alpha \\ & 1\end{pmatrix}\right) \\
&= |\varpi|^{-1/2}\chi_2(\varpi) + q\,|\varpi|^{1/2}\chi_1(\varpi) \\
&= \sqrt{q}(\chi_1(\varpi) + \chi_2(\varpi))
\end{aligned}$$

Also,

$$\pi(T_2)f(1) = \int_{\mathrm{diag}(\varpi,\varpi)K} f(g)d\mu_G = f\left(\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}\right) = \chi_1\chi_2(\varpi)$$

as required.

Conversely, if π is spherical, let the corresponding character of $\mathcal{H}(G, K)$ send T_i to β_i. Now let χ_1, χ_2 be unramified characters of $F^\times$ with $\chi_i(\varpi)$ the roots of $X^2 - q^{-1/2}\beta_1 T + \beta_2$. (Since T_2 is invertible, $\beta_2 \neq 0$, so the roots are nonzero.) Then $\pi(\chi_1, \chi_2)$, if irreducible, has the same Hecke module of K-fixed vectors as π, so that $\pi = \pi(\chi_1, \chi_2)$. (If $\pi(\chi_1, \chi_2)$ is not irreducible, then there is a 1-dimensional representation of G whose Hecke eigenvalues match those of π.) $\square$

4 The new vector

Let π be an irreducible admissible representation of G. Even if π fails to be spherical, there is still a compact open subgroup of G for which π has a 1-dimensional space of invariants (in a sense).

For $c \geq 0$, let $K_0(c)$ be the compact open consisting of matrices whose lower left-hand entry is divisible by ϖ^c.

Theorem 4.1 (Casselman)**.** *Assume π is infinite-dimensional, with central character χ. There exists $c \geq 0$ such that the space of vectors v with $\pi(g)v = \chi(a)v$, all $g \in K_0(c)$, is non-empty. If c is the minimal such integer, then the space of such vectors is 1-dimensional.*

This minimal c is the *conductor* of π, and a nonzero vector v as in the above theorem is called a *new vector* for π.

Thus $c = 0$ if and only if π is spherical.