Lecture Notes: Tate's thesis

September 19, 2011

1 Motivation

To prove the analytic continuation of the Riemann zeta function (1850), we start with the Gamma function:

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}$$

Substitute:

$$\Gamma(s/2) = \int_0^\infty e^{-x} x^{s/2} \frac{dx}{x}$$

And now put $\pi n^2 x$ for x:

$$\pi^{-s/2} n^{-s} \Gamma(s/2) = \int_0^\infty e^{-\pi n^2 x} x^{s/2} \frac{dx}{x}$$

Now sum over n (valid for real part s greater than 1):

$$\Lambda(x) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \omega(x) x^{s/2} \frac{dx}{x}$$

where

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

Now set

$$\theta(x) = 1 + 2\omega(x) = \sum_{n \in \mathbf{Z}} e^{-n^2 \pi x}$$

This is one of Jacobi's theta functions. Its transformation law is a consequence of the Poisson summation formula: if $f\colon \mathbf{R} \to \mathbf{C}$ is a Schwartz function (infinitely differentiable and $x^a f^{(b)}(x) \to 0$ for all $b \geq 1$, $a \in \mathbf{R}$), we have the Fourier transform

$$\hat{f}(y) = \int_{\mathbf{R}} f(x) e^{-2\pi i x y} dx,$$

and then

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \hat{f}(n).$$

The idea now is to observe that $f(x) = e^{-\pi x^2}$ is *its own Fourier transform*:

$$\begin{aligned} \hat{f}(y) &= \int_{\mathbf{R}} e^{-\pi x^2 - 2\pi i x y} dx \\ &= \int_{\mathbf{R}} e^{-\pi (x+iy)^2 + y^2} dx \\ &= f(y) \end{aligned}$$

Therefore

$$\theta(x) = \sum_{n \in \mathbf{Z}} f(n\sqrt{x}) = \frac{1}{\sqrt{x}} \sum_{n \in \mathbf{Z}} f\left(\frac{n}{\sqrt{x}}\right) = \frac{1}{\sqrt{x}} \theta(1/x)$$

or

$$\theta(1/x) = \sqrt{x}\theta(x)$$

so that

$$\begin{aligned} \omega(1/x) &= -\frac{1}{2} + \frac{1}{2}\theta(1/x) \\ &= -\frac{1}{2} + \frac{\sqrt{x}}{2}\theta(x) \\ &= -\frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x}\omega(x) \end{aligned}$$

Now apply to the Λ-function:

$$\begin{aligned}\Lambda(s) &= \int_0^\infty \omega(x)x^{s/2}\frac{dx}{x} \\ &= \int_0^1 \omega(x)x^{s/2}\frac{dx}{x} + \int_1^\infty \omega(x)x^{s/2}\frac{dx}{x} \\ &= \int_1^\infty \omega(1/x)x^{-s/2}\frac{dx}{x} + \int_1^\infty \omega(x)x^{s/2}\frac{dx}{x} \\ &= \int_1^\infty \left[-\frac{1}{2}x^{-s/2-1} + \frac{1}{2}x^{-s/2-1/2} + \omega(x)x^{\frac{1-s}{2}} + \omega(x)x^{\frac{s}{2}}\right] dx \\ &= -\frac{1}{s(1-s)} + \int_1^\infty \left(x^{s/2} + x^{(1-s)/2}\right)\omega(x)\frac{dx}{x}\end{aligned}$$

A priori this is valid only for $\Re s > 1$, but the integral converges for all s. Consequently $\Lambda(s)$ is meromorphic for all s with simple poles at 0 and 1.

What happened? We constructed a function which was its own Fourier transform, summed it over a discrete group $\mathbf{Z}$ and took its Mellin transform; Poisson summation was applied to get the FE.

2 The local story

2.1 Generalities on locally compact groups

Let G be a locally compact abelian group. The *dual group* $\hat{G}$ is another such thing. Its elements are characters $c\colon G \to S^1$. A sequence of such cs converges if it convergers uniformly on compact subsets of G. G has a Haar measure dx, and this is unique up to scaling. For the moment, if $y \in \hat{G}$, write $\langle x, y\rangle$ for $y(x)$. Given an integrable function f, we can form the Fourier transform

$$\hat{f}(y) = \int_{x\in G} f(x)\,\langle x, y\rangle\,dx$$

with respect to dx; this is a function on $\hat{G}$. There is generally a Fourier inversion formula:

$$\hat{\hat{f}}(x) = \gamma f(x^{-1})$$

for some γ which does not depend on x. One can choose dx so that $\gamma = 1$; this is called a self-dual measure. The Fourier transform puts into bijection (suitable) functions on G and on $\hat{G}$.

If $G = \mathbf{Z}$ then $\hat{G} = S^1$. If $G = \mathbf{R}$ or $\mathbf{C}$ then $\hat{G} = G$.

2.2 Harmonic analysis on local fields

In fact the additive group of a local field k is always dual to itself. We choose special isomorphisms $k \cong \hat{k}$ as follows, for $k = \mathbf{Q}_p$ or $k = \mathbf{R}$.

If $k = \mathbf{R}$, let $x \mapsto \psi_x$ be the isomorphism $G \cong \hat{G}$ with

$$\psi_x(y) = e^{-2\pi i x y}$$

If $k = \mathbf{Q}_p$, let $x \mapsto \psi_x$ be the isomorphism $G \cong \hat{G}$ with

$$\psi_x(y) = e^{2\pi i \{xy\}}$$

where $\{z\} \in \mathbf{Z}[1/p]/\mathbf{Z}$ is the fractional part of $z \in \mathbf{Q}_p$; that is, $z - \{z\} \in \mathbf{Z}_p$. In each case write $\psi = \psi_1$.

(Neat property: If $y \in \mathbf{Q}$, then the product of $\psi(y)$ over the ψ coming from each $\mathbf{Q}_v$ is 1.) If G is the additive group of a finite extension $k/\mathbf{Q}_v$, put $\psi_k(y) = \psi_{\mathbf{Q}_v}(\operatorname{Tr} y)$.

Let $\mu = dx$ be the self-dual Haar measure on k. On $\mathbf{Q}_p$, we have $\int_{\mathbf{Z}_p} dx = 1$. For $k/\mathbf{Q}_p$ we have $\int_{\mathcal{O}_k} dx = (ND)^{-1/2}$, where $D \subset k$ is the different ideal (the orthogonal complement to $\mathcal{O}_k$ under $\langle x, y \rangle = \psi_k(xy)$).

We also choose a Haar measure d^*x on $k^\times$. This is $dx/|x|$ if k is archimedean, and is normalized to satisfy $\int_{\mathcal{O}_k^\times} d^*x = (ND)^{-1/2}$ if k is non-archimedean.

2.3 Quasi-characters

These are continuous homomorphisms $c\colon k^\times \to \mathbf{C}^\times$. They are called *characters* if $c(k^\times) \subset S^1$.

Crucially, tate views the set of quasi-characters on $k^\times$ as a complex manifold (albeit generally one with infinitely many components). Say c is *unramified* if it vanishes on $\{|\alpha| = 1\}$.

If K is nonarchimedean then the unramified characters are of the form $c(\alpha) = |\alpha|^s$, where s is only well-defined up to $2\pi i/\log q$. Such a guy can be factored as $c(\pi) = \chi(\alpha)\,|\alpha|^\sigma$, where $\chi(\alpha) = |\alpha|^{s-\sigma}$ is a character.

If $K = \mathbf{R}$ or $\mathbf{C}$ the unramified characters are $c(\alpha) = |\alpha|^s$. In all cases $\sigma = \Re s$ is well-defined. We call it the real part of c.

Two quasicharacters are *equivalent* if they differ by an unramified character. We view each equivalence class of qcs as a complex manifold via the above.

2.4 The local zeta function

In Tate's theory, one attaches a zeta function to a *pair* (f, c), where c is a quasi-character and f is a test function. Tate considers a class of functions ‡ having the property that

1. f and $\hat{f}$ are continuous and L^1.
2. $f(\alpha)\,|\alpha|^\sigma$ are integrable on $k^\times$ for $\sigma > 0$. Similarly for $\hat{f}$.

Given such an f and a quasi-character c, Tate defines

$$\zeta(f, c) = \int_{k^\times} f(\alpha)c(\alpha)d^*\alpha$$

This is an analytic function of c for $\Re c > 0$.

Example: if $k = \mathbf{Q}_p$, f is the char function of $\mathbf{Z}_p$, and c is unramified of positive real part, so that $|c(\varpi)| < 1$, we get

$$\begin{aligned}\zeta(f, c) &= \int_{\mathbf{Z}_p - 0} c(\alpha)d^*\alpha \\ &= \sum_{n\geq 0} c(p)^n \int_{p^n\mathbf{Z}_p - p^{n-1}\mathbf{Z}_p} d^*\alpha \\ &= \sum_{n\geq 0} c(p)^n = (1 - c(\pi))^{-1}\end{aligned}$$

If we write $c(\alpha) = \chi(\alpha)\,|\alpha|^\sigma$ this becomes

$$\zeta(f, \chi\,|\cdot|^\sigma) = \frac{1}{1 - \frac{\chi(p)}{p^\sigma}}$$

(Note this was the result of integrating c against the self-dual function.)

On the other hand if c is ramified of conductor p^f then

$$\begin{aligned}\zeta(f, c) &= \int_{\mathbf{Z}_p - 0} c(\alpha)d^*\alpha \\ &= \sum_{n\geq 0} \int_{p^n\mathbf{Z}_p - p^{n-1}\mathbf{Z}_p} c(\alpha)d^*\alpha \\ &= \sum_{n\geq 0} c(p)^n \int_{\mathbf{Z}_p^\times} c(\alpha)d^*\alpha = 0\end{aligned}$$

If $k = \mathbf{R}$, let $f(x) = e^{-\pi x^2}$. Let $c(x) = |x|^s$ be unramified. Then

$$\begin{aligned} \zeta(f, c) &= \int_{\mathbf{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{x} \\ &= \pi^{-s/2}\Gamma(s/2). \end{aligned}$$

3 Local functional equation

For a quasi-character c, put $\hat{c}(\alpha) = |\alpha| \, c(\alpha)^{-1}$. Thus if $c(\alpha) = |\alpha|^s$, then $\hat{c}(\alpha) = |\alpha|^{1-s}$. In particular $\Re\hat{c} = 1 - \Re c$.

If $f \in \mathfrak{z}$ and $0 < \Re c < 1$, then $\zeta(f, c)$ and $\zeta(\hat{f}, \hat{c})$ both converge. Tate's local functional equation is:

$\zeta(f, c)$ extends to a meromorphic function on all c. There exists a meromorphic $\rho(c)$ with

$$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c})$$

The proof combines two computations:

One is that for $0 < \Re c < 1$, $\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \zeta(\hat{f}, \hat{c})\zeta(g, c)$. This is a formal calculation. We have

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \int\int f(\alpha)\hat{g}(\beta)c(\alpha\beta^{-1}) \, |\beta| \, d^*\alpha d^*\beta$$

Now substitute $(\alpha, \beta) \mapsto (\alpha, \alpha\beta)$:

$$\begin{aligned} \zeta(f, c)\zeta(\hat{g}, \hat{c}) &= \int\int f(\alpha)\hat{g}(\alpha\beta)c(\beta^{-1}) \, |\alpha\beta| \, d^*\alpha d^*\beta \\ &= \int\left(\int f(\alpha)\hat{g}(\alpha\beta) \, |\alpha| \, d^*\alpha\right) c(\beta^{-1}) \, |\beta| \, d^*\beta \end{aligned}$$

Remember that $d^\alpha = d\alpha / |\alpha|$. The inner integral is equal to

$$\int\int f(\alpha)g(\gamma)\psi(\alpha\beta\gamma)d\alpha d\beta,$$

and this is symmetric in f and g.

Thus $\rho(c) = \zeta(f, c)/\zeta(\hat{f}, \hat{c})$ is independent of f so long as the denominator is nonzero.

The second computation the evaluation of this quotient for a suitable f (relative to c) which makes the denominator nonzero. Tate shows that the quotient is always a meromorphic function of all c (which he makes explicit in terms of familiar functions). Therefore $\zeta(f,c)$ can be defined even for $\Re c < 0$, as the product $\rho(c)\zeta(\hat{\hat{f}},\hat{c})$ (because $\Re\hat{c} = 1 - \Re\hat{c} > 1 > 0$).

If $c(\alpha) = |\alpha|^s$, let $L(s) = \zeta(f,c)$, where f is the self-dual function. If $k = \mathbf{Q}_p$, then

$$L(s) = \int_{\mathbf{Z}_p} |\alpha|^s \, d^*\alpha = \frac{1}{1-p^{-s}}$$

(converges for $\Re s > 1$, but has obvious analytic continuation). Thus

$$\rho(c) = \frac{\zeta(f,c)}{\zeta(f,\hat{c})} = \frac{L(s)}{L(1-s)} = \frac{1-p^{s-1}}{1-p^{-s}}.$$

If $k = \mathbf{R}$ and $c(\alpha) = |\alpha|^s$, then $f(\alpha) = e^{-\pi\alpha^2}$ so that

$$L(s) = \Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

and

$$\rho(c) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(1-s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s).$$

(In the p-adic case, if c is ramified, then $\rho(c)$ is a Gauss sum.)

4 Adeles, Ideles

4.1 Restricted direct products

Let G_v be a collection of locally compact abelian groups. For almost all v, let $H_v \subset G_v$ be an open compact subgroup. Let S_∞ be the (finite) set of indices for which H_v is not defined.

For S a finite set of indices containing S_∞, let

$$G_S = \prod_{v\in S} G_v \times \prod_{v\notin S} H_v$$

This is a topological group under the product topology. (Recall that the topology of a product of spaces $\prod_v X_v$ is generated by "boxes" of the form $\prod_v U_v$ with $U_v \subset X_v$ open, subject to the restriction that $U_v = X_v$ for almost all v.) The G_S form a directed system of topological groups: When $S \subset S'$, there is an inclusion $G_S \subset G_{S'}$ as an open subgroup. The *restricted direct product* is the direct limit

$$G = \prod{}' G_v = \varinjlim_S G_S.$$

Thus an element of $\prod' G_v$ is an element of one of the G_S; *i.e.* it is a collection (x_v) of elements $x_v \in G_i$ with $x_v \in H_v$ for all but finitely many v. A subset $U \subset \prod' G_v$ is open if and only if $U \cap G_S$ is open for all S.

G is locally compact, because a compact neighborhood of 1 in any G_S serves as a compact neighborhood of 1 in G.

A sequence converges to an element in G_S if and only if almost all elements of the sequence lie in G_S, and that part of the sequence converges in G_S.

A (quasi-)character χ of G is one and the same as a collection of (quasi-)characters χ_v of G_v, subject to the restraint that $\chi_v(H_v) = 1$ for almost all v. Thus $\hat{G}$ is isomorphic as a topological group to the restricted direct product of the $\hat{G}_v$ with respect to the $H_v^\perp \subset \hat{G}_v$.

Choose Haar measures dx_v on G_v for all v such that $\int_{H_v} dx_v = 1$ for almost all v. We can then define a Haar measure dx on G which is essentially the product of the dx_v. (This is a matter of defining a compatible system of Haar measures dx_S on the G_S. The G_S are a compact group times a finite product of the G_v for $v \in S$, so this is unproblematic.) In the end, if f is an integrable function on G then

$$\int_G f(x)dx = \lim_S \int_{G_S} f(x)dx.$$

4.2 Adeles and ideles

Let K be a number field or function field. The adeles $\mathbf{A}_K$ are the restricted direct product of the K_v with respect to the $\mathcal{O}_v \subset K_v$ (for v nonarchimedean), and $K \hookrightarrow \mathbf{A}_K$ is an embedding of K as a discrete closed subring of $\mathbf{A}_K$.

The ideles $\mathbf{A}_K^\times$ are the restricted direct product of the $K_v^\times$ with respect to the $\mathcal{O}_{K_v}^\times \subset K_v^\times$. $K^\times$ embeds in $\mathbf{A}_K^\times$ as a discrete closed subgroup. It so happens that the abstract group $\mathbf{A}_K^\times$ is the same as the multiplicative group of the ring $\mathbf{A}_K$, but it topology is not the same as that induced on it as a subset of $\mathbf{A}_K$.

Lang writes J for the idele group.

There is an absolute value map

$$|\cdot| : J/K^\times \to R_{>0}^\times$$

given by $(\alpha_v) \mapsto \prod_v |\alpha_v|$. Let J^0 be the kernel. The quotient $J^0/K^\times$ is compact.

4.3 Harmonic analysis on the adele group

We have a collection of additive characters ψ_v on K_v for all v, and these have the property that $\psi_v(\mathcal{O}_v) = 1$ for almost all v, and

$$\prod_v \psi_v(a) = 1$$

for any $a \in K$. Under the self-duality $K_v \cong \hat{K}_v$, we have that $\mathcal{O}_v^\perp = D_v^{-1}$, where D_v is the different. Thus the dual of $\mathbf{A}_K$ is the restricted direct product of the K_v with respect to the D_v^{-1}, which is just $\mathbf{A}_K$ again!

If f is a rapidly decreasing function on $\mathbf{A}_K$, then as usual its Fourier transform is

$$\hat{f}(y) = \int_{\mathbf{A}_K} f(x)\psi(xy)dx$$

where dx is the self-dual Haar measure on $\mathbf{A}_K$ with respect to ψ.

4.4 Poisson summation, Riemann-Roch formula

This is a phenomenon that occurs for a pair (V, L), where V is a locally compact commutative ring together with a character ψ which induces a self-duality $V \cong \hat{V}$, and $L \subset V$ is a discrete subring which is co-compact and satisfies $L^\perp = L$. Such is the story with $(V, L) = (\mathbf{R}^n, \mathbf{Z}^n)$ or $(V, L) = (\mathbf{A}_K, K)$. The final ingredient is a fundamental domain D for V/L.

A key observation is that our ψ identifies the dual of V/L with L.

Theorem 4.1 (Poisson summation). *Let f be a continuous integrable function on V such that*

1. $\sum_{x \in L} f(x+v)$ *is uniformly convergent for all $v \in V$.*

2. $\sum_{x \in L} \left|\hat{f}(x)\right|$ *converges.*

Then

$$\sum_{x\in L}\hat{f}(x)=\sum_{x\in L}f(x)$$

Here's the proof: let $\phi(v)=\sum_{x\in L}f(x+v)$, so that $\phi(v)$ is a *periodic function* on V. Such functions are generally not integrable on V (they tend not to be rapidly decreasing!), but ϕ will be integrable on D, which is compact. Write $\hat{\phi}$ for its Fourier transform (this is to be considered as a function on L):

$$\hat{\phi}(y)=\int_D \phi(v)\psi(vy)dv$$

For $y\in L$, this is

$$\begin{aligned}\hat{\phi}(y) &= \int_D\sum_{x\in L}f(x+v)\psi(vy)dy\\ &= \sum_{x\in L}\int_{D+x}f(v)\psi(vy)dy\\ &= \int_V f(v)\psi(vy)dy\\ &= \hat{f}(y)\end{aligned}$$

Here we have used the fact that $y\in L$, so every $xy\in L$, so the change of variable we did on the second line doesn't affect the $\psi(vy)$ factor.

Our first condition on f guarantees that the Fourier inversion formula applies to ϕ. It reads:

$$\phi(0)=\hat{\hat{\phi}}(0)=\sum_{y\in L}\hat{\phi(y)}=\sum_{y\in L}\hat{f}(y)$$

which is exactly what we set out to prove.

The Riemann-Roch theorem is a consequence of the Poisson formula in the context where V is a ring.

Theorem 4.2 (Riemann-Roch theorem)**.** *Let f be a continuous integrable function on $\mathbf{A}_K$, satisfying the conditions:*

1. *For all $a\in\mathbf{A}_K^\times$, $x\in\mathbf{A}_K$, $\sum_{x\in K}f(a(x+v))$ converges uniformly for $v\in\mathbf{A}_K$.*

2. $\sum_{x\in K}\left|\hat{f}(av)\right|$ *is convergent for all* $a \in \mathbf{A}_K^\times$.

Then for all $a \in \mathbf{A}_K^\times$*:*

$$\frac{1}{|a|}\sum_{a\in K}\hat{f}\left(\frac{v}{a}\right) = \sum_{a\in K} f(av)$$

To prove the theorem, one applies Poisson summation to $g(x) = f(ax)$, noting that

$$\begin{aligned}\hat{g}(y) &= \int_{\mathbf{A}_K} f(ax)\psi(xy)dx \\ &= \frac{1}{|a|}\int_{\mathbf{A}_K} f(x)\psi(xy/a)dx \\ &= \frac{1}{|a|}\hat{f}\left(\frac{y}{a}\right)\end{aligned}$$

5 The global functional equation

5.1 Hecke characters

Recall that $J = \mathbf{A}_K^\times$ and $J^0 \subset J$ is the kernel of the absolute value map $J \to \mathbf{R}_{>0}$. A *Hecke character* χ is a quasi-character of $J/K^\times$. Since $J^0/K^\times$ is compact, the restriction of χ to $J^0/K^\times$ is unitary. Thus $|\chi(a)|$ only depends on a modulo J^0, which is to say that $|\chi(a)|$ only depends on $|a|$. There must be some $\sigma \in \mathbf{R}$ with $|\chi(a)| = |a|^\sigma$; this σ is called the *real part* of χ.

This might be a good time to classify Hecke characters in the case of $K = \mathbf{Q}$. The short story is that the finite-order Hecke characters are essentially the primitive Dirichlet characters, and that any Hecke character is of the form $\chi|\ |^s$, where $s \in \mathbf{C}$.

First, notice we have an isomorphism

$$J/\mathbf{Q}^\times \cong \hat{\mathbf{Z}}^\times \times \mathbf{R}_{>0}.$$

Given any idele $a = (a_v)$, one can find a rational number $b \in \mathbf{Q}^\times$ with $|a_v|_v = |b|_v$ for all finite places v (this is because $\mathbf{Q}$ has class number one), and furthermore b can be chosen to have the same sign as a_∞. Then $b^{-1}a \in \hat{\mathbf{Z}}^\times \times \mathbf{R}_{>0} \subset J$ shall be the image of a. The map is well defined because the intersection of $\mathbf{Q}^\times$ with $\hat{\mathbf{Z}}^\times \times \mathbf{R}_{>0}$ is $\{1\}$. The inverse is the inclusion $\hat{\mathbf{Z}}^\times \times \mathbf{R}_{>0} \to J$ followed by the quotient $J \to J/\mathbf{Q}^\times$.

Now suppose ψ is a primitive Dirichlet character on $(\mathbf{Z}/N\mathbf{Z})^\times$. We can then define a Hecke character χ of finite order on $J/\mathbf{Q}^\times$ via

$$J/\mathbf{Q}^\times \cong \hat{\mathbf{Z}}^\times \times \mathbf{R}_{>0} \to \hat{\mathbf{Z}}^\times \to (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times,$$

where the last map is χ. One should work through the details of how χ works locally. If χ_v is the restriction of χ to $\mathbf{Q}_v$, then (for instance) χ_∞ is the unique character of $\mathbf{R}^\times$ with $\chi_\infty(-1) = \psi(-1 \mod N)$. Here's another example: if p is a finite prime, let N_0 be the largest divisor of N which is relatively prime to p. Let p_0 be an integer with $p_0 \equiv p \pmod{N_0}$ and $p_0 \equiv 1 \pmod{p}$. Then $\chi_p(p) = \psi(p_0)^{-1}$.

5.2 Global zeta function

Tate defines a class $\mathfrak{z}$ of functions f on $\mathbf{A}_K$ which have the following nice properties:

1. f and $\hat{f}$ are continuous and integrable.
2. The average $\sum_{x \in K} f(a(x+v))$ converges uniformly as a function of $x \in D$ and a in any compact subset of J. Similarly for $\hat{f}$
3. $a \mapsto f(a)\,|a|^\sigma$ is integrable on I for all $\sigma > 1$.

For such f, and for a quasi-character c of $J/K^\times$, Tate defines the zeta function

$$\zeta(f,c) = \int_J f(a)c(a)da$$

This is convergent as long as the real part of c is > 1.

Just as in the local theory, one divide quasi-characters c of $J/K^\times$ into equivalence classes, each class being parameterized by a complex variable s. It then makes sense to ask whether $\zeta(f,c)$ can be analytically continued to all s.

Theorem 5.1. *The zeta function $\zeta(f,c)$ has an analytic continuation to quasi-characters. The only poles are at $c(a) = 1$ and $c(a) = |a|$; these are* *simple and the poles are* $-\kappa f(0)$ *and* $\kappa\hat{f}(0)$*, respectively. We have the functional equation*

$$\zeta(f,c)=\zeta(\hat{f},\hat{c}).$$

Here κ *is the volume of a fundamental domain for* $J/k^\times$*:*

$$\kappa=\frac{2^{r_1}(2\pi)^{r_2}hR}{\sqrt{|d_K|}w}$$

Proof. We have the absolute value map $J\to\mathbf{R}_{>0}$. It is helpful to provide a splitting for this map, $R_{>0}\to J$, for instance as $t\mapsto(t,1,1,\dots)$ (where the first slot is one of the archimedean places), so that $J=R_{>0}\times J^0$. Choose a measure d^*b for J^0 which satisfies $d^*a=(dt/t)d^*b$. (This is the measure inherent in the definition of κ above. We won't be going into the evaluation of κ; this is quite analogous to the derivation of the class number formula for the Dedekind zeta function.) Then

$$\zeta(f,c)=\int_0^\infty\left[\int_{J^0}f(tb)c(tb)d^b\right]\frac{dt}{t}=\int_0^\infty\zeta_t(f,c)\frac{dt}{t},$$

say.

Now use the fact that J^0/K^* is compact. Let E be a fundamental domain for this quotient. We may write

$$\begin{aligned}
\zeta_t(f,c) &= \sum_{x\in K^\times}\int_{xE}f(tb)c(tb)d^*b\\
&= \sum_{x\in K^\times}\int_{E}f(xtb)c(tb)d^*b\\
&= \int_E\left[\sum_{x\in K}f(xtb)\right]c(tb)d^*b-f(0)\int_E c(tb)d^*b\\
&= \int_E\frac{1}{|tb|}\left[\sum_{x\in K}\hat{f}\left(\frac{x}{tb}\right)\right]c(tb)d^*b-f(0)\int_E c(tb)d^*b\\
&= \int_E\left[\sum_{x\in K}\hat{f}\left(\frac{xb}{t}\right)\right]\hat{c}\left(\frac{b}{t}\right)d^*b-f(0)\int_E c(tb)d^*b\\
&= \zeta_{1/t}(\hat{f},\hat{c})-f(0)\int_E c(tb)d^*b+\hat{f}(0)\int_E\hat{c}\left(\frac{b}{t}\right)d^*b
\end{aligned}$$

As for the remaining integrals: they are zero unless c is trivial on J^0, which is to say that $c(a) = |a|^s$ for some s. If that is the case then

$$\int_E c(tb)d^*b = \kappa t^s.$$

Plugging this all in to our original expression for $\zeta(f,c)$ gives:

$$\begin{aligned}\zeta(f,c) &= \int_0^1 \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(f,c)\frac{dt}{t} \\ &= \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_0^1 \zeta_{1/t}(\hat{f},\hat{c})\frac{dt}{t} + \kappa\left\{\frac{\hat{f}(0)}{s-1} - \frac{f(0)}{s}\right\} \\ &= \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \kappa\left\{\frac{\hat{f}(0)}{s-1} - \frac{f(0)}{s}\right\}\end{aligned}$$

where the expression in braces means that it is only included when $c(a) = |a|^s$.

Recall that the defining integral for $\zeta(f,c)$ converged for $\Re c > 1$. So of course the first integral converges on that region as well. But the integrand in the first integral gets *smaller* as $\Re c$ decreases, so that in fact the first integral converges for all c, period. The same argument holds for the second integral: this integral converges a priori for $\Re\hat{c} > 1$, but all the better for $\Re\hat{c}$ smaller than this. Therefore the whole expression converges for all c, save for the poles at $s = 0$ and $s = 1$ when $c(J^0) = 1$. Finally, the expression is unchanged when the pair (f,c) is replaced by $(\hat{f},\hat{c})$. □

5.3 The finale

Let χ be a *character* of $J/K^\times$. Tate now applies his global functional equation to a special function f, relative to χ, which is essentially the product of his special f_v relative to the components χ_v of χ. An important point is that if χ is unramified at the finite place v, which is the case for almost all v, then f_v is simply the characteristic function of $\mathcal{O}_v$, in which case the local zeta function is (up to a factor $ND_v^{-1/2}$ which only appears finitely many times)

$$\zeta(f_v, \chi_v\,||^s) = \left(1 - \frac{\chi(\pi_v)}{N\pi_v^s}\right)^{-1}$$

The *Hecke L-function* $L(s,\chi)$ is the product of the above factors over all unramified places v.

The zeta function of Tate's f differs from $L(s,\chi)$ at finitely many factors:

$$\zeta(f,\chi|\,|^s) = \prod_{v\in S} \zeta(f_v,\chi_v|\,|_v^s) \times \prod_{v\notin S} ND_v^{-1/2} \times L(s,\chi)$$

Here S is the set of ramified places of χ, together with the infinite places.

Tate's main theorem on the global zeta function will now imply an analytic continuation and functional equation for $L(s,\chi)$; the huge advantage is that the constant appearing in the functional equation can be factored as a product of local factors, derived from the quantities $\rho(c)$ appearing in the local theory.