$(\mathfrak{g}, K)$-modules

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We now initiate the study of automorphic forms on GL_2 as spaces of functions on the quotient $\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_Q)$. It is this formulation that will allow us to generalize the notion of automorphic form to global fields other than $\mathbf{Q}$ and groups other than GL_2. The presence of archimedean places poses a significant hurdle, though. It will become necessary to devise a notion of admissibility for representations of $\mathrm{GL}_2(\mathbf{R})$ parallel to the notion of admissibility for representations of locally profinite groups G such as $\mathrm{GL}_2(\mathbf{Q}_p)$. One very nice feature of such representations is that their restriction to a maximal compact subgroup K decomposes as a direct sum of irreducible representations of K, each of which appears only finitely many times. One can then classify representations of G according to which irreducible representations of K they contain.

One problem with extending this definition to representations of (say) $\mathrm{SL}_2(\mathbf{R})$ is that such representations, when infinite-dimensional, tend to be really huge (at least uncountable), whereas there are only countably many irreducible representations of the maximal compact subgroup $K = \mathrm{SO}_2(\mathbf{R})$. (Note that $\mathrm{SL}_2(\mathbf{Q}_p)/\mathrm{SL}_2(\mathbf{Z}_p)$ is countable, while $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$ is the upper half-plane.) The workaround, developed by Harish-Chandra, is the notion of a $(\mathfrak{g}, K)$-module for a Lie group G. It isn't actually a representation of G at all!

Here is our motivation: suppose $\pi\colon G \to \mathrm{GL}(V)$ is a continuous representation of G on a Hilbert space V, but not necessarily a unitary one. Recall that $\operatorname{Lie} G$ is the tangent space to the identity of G, and that each $X \in \operatorname{Lie} G$ defines a one-parameter subgroup $t \mapsto \exp(tX)$ of G. A vector $v \in V$ is C^1 if, for all $X \in \mathfrak{g} = \operatorname{Lie} G$, the derivative

$$\pi(X)v := \frac{d}{dt}\pi(\exp(tX))v|_{t=0} = \lim_{t\to 0}\frac{\pi(\exp(tX))v - v}{t}$$

is defined. The vector $v \in V$ is C^∞, or smooth, if $\pi(X_1)\cdots\pi(X_n)v$ is defined for every sequence $X_1, \ldots, X_n$ of elements of $\mathfrak{g}$. Let V^∞ be the subspace of smooth vectors. It is a representation of G. It will be unlikely that $V = V^\infty$, but for $G = \mathrm{GL}_n(\mathbf{R})$ it is the case that V is dense in V^∞ (see Bump, p. 190). There is a representation $\pi\colon \mathfrak{g} \to \operatorname{End} V^\infty$, called the *infinitesimal action*. This places us in the realm of representations of Lie algebras, but there is still a problem in that V^∞ is very large, yet generally not a Hilbert space.

Let $K \subset G$ be a maximal compact subgroup of G. By the Peter-Weyl theorem, V decomposes as a Hilbert direct sum of irreducible unitary representations of K. We say V is *admissible* if each isomorphism class of irreducible representation of K appears only finitely often in such a decomposition.

Let $\pi\colon G \to \mathrm{GL}(V)$ be an admissible representation; assume that the restriction of π to K is unitary. (This can always be assumed, by averaging the inner product on V over K.) We have

$$V = \hat{\bigoplus} V[\sigma],$$

where σ runs over the set of isomorphism classes of unitary irreducible representations of K. Then each $V[\sigma]$ is finite-dimensional. Let V^{fin} be the algebraic direct sum

$$V^{\mathrm{fin}} = \bigoplus V[\sigma].$$

Vectors lying in V^{fin} are called *K-finite*. A vector $v \in V$ belongs to V^{fin} if and only if the space spanned by $\pi(k)v$, $k \in K$, is finite-dimensional. V^{fin} has the following virtues (see Bump, p. 197):

1. V^{fin} is dense in V,
2. $V^{\mathrm{fin}} \subset V^\infty$,
3. V^{fin} is invariant under the action of $\pi(\mathfrak{g})$,

but it also has the vice of not being G-invariant. Nevertheless, V^{fin} melds together two structures which are both quite algebraic (read: tractable) in nature, namely that it has an action of $\mathfrak{g}$, and also an action of K with respect to which it is admissible.

A $(\mathfrak{g}, K)$-module is a vector space V together with representations $\pi\colon \mathfrak{g} \to \operatorname{End}(V)$ and $\pi\colon K \to \mathrm{GL}(V)$ such that:

1. V is the (algebraic) direct sum of finite-dimensional irreducible representations of K,

2. The infinitesimal action of K on G agrees with the restriction of $\pi\colon \mathfrak{g} \to \mathrm{End}(V)$ to $\mathfrak{k}$.

3. For $X \in \mathfrak{g}$, $k \in K$, we have $\pi(k)\pi(X)\pi(k^{-1}) = \pi((\mathrm{Ad}\, k)X)$ as operators on V.

Furthermore, V is *admissible* if each irreducible representation of K appears only finitely many times in V.

1 Classification of admissible $(\mathfrak{g}, K)$-modules for $\mathrm{GL}_2(\mathbf{R})$

Let $G^+ = \mathrm{GL}_2^+(\mathbf{R})$, $K = \mathrm{SO}(2)$. Then $\mathfrak{g}$ is spanned by $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, and $z = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. Then z lies in the center of $\mathfrak{g}$, whereas

$$\begin{aligned} [h, r] &= 2r \\ [h, l] &= -2l \\ [r, l] &= h. \end{aligned}$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let

$$-4\Delta = h^2 + 2rl + 2lr \in U(\mathfrak{g}).$$

Then Δ lies in the center of $U(\mathfrak{g})$. Indeed this Δ is the *Casimir operator* (normalized to agree with convention).

Let (π, V) be an admissible $(\mathfrak{g}, K)$-module. Since $K \cong \mathbf{R}/2\pi\mathbf{Z}$, V is the direct sum of finite-dimensional spaces $V[k]$ on which $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ acts as the scalar e^{ikt}.

Notice that if $W = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, and $x \in V[k]$, then

$$\pi(W)x = \frac{d}{dt}\pi(e^{tW})x|_{t=0} = \frac{d}{dt}\pi\left(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\right) x|_{t=0} = \frac{d}{dt} e^{ikt} x|_{t=0} = ikx$$

Thus if

$$H = -iW = \begin{pmatrix} & i \\ -i & \end{pmatrix} \in \mathfrak{g}_{\mathbf{C}},$$

then $\pi(W)$ acts on $V[k]$ as the scalar k.

The element H has eigenvalues $1, -1$, just like h. The two are conjugate: $H = C^{-1}hC$, where $C = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. After conjugating by C, one arrives at a new basis for $\mathfrak{g}_{\mathbf{C}}$:

$$\begin{aligned} R &= C^{-1}rC = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\ L &= C^{-1}lC = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\ H &= C^{-1}hC = \begin{pmatrix} & i \\ -i & \end{pmatrix} \\ Z &= C^{-1}zC = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{aligned}$$

(Some of our notation differs from Bump's, because Bump parametrizes SO(2) by the rotation matrix $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$.) This basis has the same commutation relations as $\{r, l, h\}$.

Simply write Xx for $\pi(X)x$. If $x \in V[k]$ we have

$$HRx = (HR - RH)x + RHx = [H, R]x + RHx = 2Rx + kRx = (k+2)Rx,$$

so that $RV[k] \subset V[k+2]$. Similarly $LV[k] \subset V[k-2]$.

Now assume that V is irreducible. Then Δ must act on V as a constant, say λ. Let $x \in V[k]$ be nonzero. Let U be the span of $x, R^n x, L^n x$ $(n > 0)$. It is easy to show that U is invariant under both $\mathfrak{g}$ and $K = \mathrm{SO}(2)$. (Invariance under K is self-evident, since K acts as a scalar on each $V[\ell]$. Invariance under Z and Δ follows from Schur's lemma, since these guys act as scalars. Say Δ acts as λ on V. We have $-4\Delta = H^2 + 2RL + 2LR = H^2 + 2H + 4LR$, so if $w \in V[\ell]$ we have

$$-4\lambda w = (\ell^2 + 2\ell)w + 4LRw,$$

implying that if $w \in \mathbf{C}R^n x$, then $LRw \in \mathbf{C}Rw$, etc.) Since V is irreducible, this sum must be all of V. Therefore each $V[n]$ is at most one-dimensional (and is zero unless $n \equiv k \pmod 2$). Some more tinkering shows that if $x \in V[k]$ is nonzero, and $Rx = 0$, then $\lambda = -\frac{k}{2}(1+\frac{k}{2})$. Likewise if $x \in V[k]$ is nonzero, and $Lx = 0$, then $\lambda = \frac{k}{2}(1-\frac{k}{2})$. Indeed, if $Rx = 0$, then

$$-4\lambda x = \Delta x = (H^2 + 2H + 4LR)x = (k^2 + 2k)x,$$

etc.

Already this shows that if V is an irreducible $(\mathfrak{g}, K)$-module, then there are four possibilities:

1. One is that $V = \bigoplus_{k \equiv \varepsilon \pmod 2} V[k]$, with each $V[k] \neq 0$ one-dimensional, for some $\varepsilon \in \{0, 1\}$. Here there is no *a priori* restriction on the eigenvalue λ.

2. Another is that there is an integer k with

$$V = V[k] \oplus V[k+2] \oplus V[k+4] \oplus \dots,$$

(all spaces nonzero), such that $LV[k] = 0$, in which case $\lambda = \frac{k}{2}(1-\frac{k}{2})$.

3. Similarly there could be an integer k with

$$V = V[k] \oplus V[k-2] \oplus V[k-4] \oplus \dots,$$

such that $RV[k] = 0$, in which case $\lambda = -\frac{k}{2}(1+\frac{k}{2})$.

4. Finally, we could have

$$V = V[2-k] \oplus V[4-k] \oplus \dots \oplus V[k-4] \oplus V[k-2],$$

in which case $\lambda = \frac{k}{2}(1-\frac{k}{2})$ once again.

Now suppose $G = \mathrm{GL}_2(\mathbf{R})$; the maximal compact subgroup of G is Ø(2). Ø(2) is a semidirect product of §0(2) by an element of order 2, namely $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. If V is an irreducible admissible $(\mathfrak{g}, Ø(2))$-module, then there are two possibilities: either the restriction of V to $(\mathfrak{g}, \mathrm{SO}(2))$ is still irreducible, or else the restriction of V to $(\mathfrak{g}, \mathrm{SO}(2))$ is the sum of two irreducibles which are swapped by w. Note that w swaps $V[k]$ with $V[-k]$. There are the following possibilities:

1. $V = \bigoplus_{k \equiv \varepsilon \pmod 2} V[k]$, with each $V[k] \neq 0$ one-dimensional, for some $\varepsilon \in \{0, 1\}$.

2. There exists an integer $k \geq 1$ with

$$V = V[\pm k] \oplus V[\pm(k+2)] \oplus \dots,$$

and $\lambda = \frac{k}{2}(1 - \frac{k}{2})$.

3. There exists $k \geq 2$ with

$$V = V[2-k] \oplus V[4-k] \oplus \cdots \oplus V[k-4] \oplus V[k-2].$$

In fact all three possibilities occur. In the first case, V can be modeled on the space of K-finite vectors in an induced representation $\pi(\chi_1, \chi_2)$. (It is possible to read off the eigenvalues of Δ and z on V from the χ_i, but not necessary for us now.) This is the *principal series*.

In the second case, V can be modeled (up to twisting by a 1-dimensional character) on the space of K-finite vectors in a certain representation $\mathcal{D}_k$ of $\mathrm{GL}_2^+(\mathbf{R})$ defined as follows: $\mathcal{D}_k$ is the space of holomorphic functions f on the upper half plane which satisfy

$$\int_{\mathcal{H}} |f(z)|^2 \, y^k \frac{dx\, dy}{2} < \infty,$$

with the action of $g \in \mathrm{GL}_2^+(\mathbf{R})$ defined by $f \mapsto f|_{g^{-1},k}$. (If $l \geq k$ has the same parity as k, one can write down a nonzero vector $f \in \mathcal{D}_k[l]$ in terms of the coordinate w on the open unit disk, by $f(w) = w^{(l-k)/2}$.) This is the *discrete series* (if $k = 1$ it is the *limit of discrete series*).

In the third case, V can be modeled (up to twisting by a 1-dimensional character) on the representation of G on the space of homogeneous polynomials of degree $k - 2$ in 2 variables.