MA 843 Assignment 1, due Sept. 12

September 2, 2013

1. Show that there is a homeomorphism $\mathrm{SL}_2(\mathbb{R})/SO(2) \to \mathcal{H}$.

2. If $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a finite-index subgroup, then $\Gamma\backslash\mathcal{H}$ is a compact Riemann surface minus finitely many points. Show that the same cannot be true for $\mathcal{H}$ itself. (Hint: There is a bounded nonconstant holomorphic function on $\mathcal{H}$.)

3. Belyi's theorem states that an algebraic curve $X/\mathbb{C}$ has a model over a number field if and only if there exists a morphism $X \to \mathbb{P}^1$ whose branch locus has no more than 3 points. Show that if $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is any finite-index subgroup (not necessarily a congruence subgroup), then $\Gamma\backslash\mathcal{H}^*$ has a model over a number field.

4. (You may assume N is prime for the following.) Find the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Find all the branch points of the map of Riemann surfaces $X_0(N) \to X(1) = \mathbb{P}^1$, and compute the ramification index for each one. Use the Riemann-Hurwitz formula to find a formula for the genus of $X_0(N)$. For which N does $X_0(N)$ have genus 0? Genus 1?

5. Fill in the proof that $X_0(N)$ admits a model over $\mathbb{Q}$ as follows.

   (a) Show that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $j(N\gamma z)$ is a meromorphic function on $\Gamma_0(N)\backslash\mathcal{H}^*$ which is holomorphic on $\mathcal{H}$.

   (b) Let S be a set of coset representatives for $\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(N)$, and define a polynomial $g(Y)$ by

   $$g(Y) = \prod_{\gamma \in S} (Y - j(N\gamma z)).$$

Show that the coefficients of $g(Y)$ are polynomials in j, so that $g(Y) = F(j(z), Y)$ for a polynomial $F(X, Y) \in \mathbb{C}[X, Y]$. Also show that $g(Y)$ is the minimal polynomial for $j(Nz)$ over the field $\mathbb{C}(j(z))$.

(c) Start with the equation $F(j(z), j(Nz)) = 0$, and consider Fourier coefficients to show that the coefficients of $F(X, Y)$ lie in $\mathbb{Q}$.

(d) As a bonus, show that the coefficients of F are in $\mathbb{Z}$, by using the fact that the Fourier expansion of each $j(N\gamma z)$ has coefficients which are algebraic integers.