MA 843 Assignment 2, due Sept. 24

September 13, 2013

1. Let $f: X \to Y$ be a finite morphism of degree d between smooth projective curves over a field K.

   (a) Show that if D is a divisor on Y of degree d, then $\deg f^*(D) = d \deg D$.

   (b) If D is a principal divisor on Y, show that $f^*(D)$ is also principal.

   (c) Show that if D is a divisor on X of degree d, then $\deg f_*(D) = \deg D$. (Very easy.)

   (d) If D is a principal divisor on X, show that $f_*(D)$ is also principal. (The morphism f induces a map of function fields $K(Y) \to K(X)$ which presents $K(X)$ as a finite extension of $K(Y)$. Thus there is a norm map $K(X)^* \to K(Y)^*$, which is key to this part.)

   From this we can conclude that a correspondence from X to Y induces a well-defined map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$.

2. Let A be an abelian variety over $\mathbb{F}_q$. Let $P(T) \in \mathbb{Z}[T]$ be the characteristic polynomial of the qth power Frobenius map acting on ℓ-adic Tate module of A ($\ell \nmid q$). Let $\alpha_1, \ldots, \alpha_{2g}$ be its complex roots. Show that for all $n \geq 1$, $\#A(\mathbb{F}_{q^n}) = \prod_{i=1}^{2g}(1 - \alpha_i^n)$.

3. For abelian varieties A, A', show that the dimension of the $\mathbb{Q}$-vector space $\mathrm{Hom}(A, A') \otimes \mathbb{Q}$ is at most $4 \dim A \dim A'$. (If you like, assume that the base field is $\mathbb{C}$, so that A and A' are complex tori. In general, one uses Tate modules.)

4. Let A be an abelian variety of dimension g over a field F. A is said to be of GL_2-type if the $\mathbb{Q}$-algebra $\mathrm{End}\, A \otimes \mathbb{Q}$ contains a subfield K of degree

g over $\mathbb{Q}$. In this case, for every prime ℓ unequal to the characteristic of the base field of A, we get an action of $K_\ell := K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ on the rational Tate module $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Show that $V_\ell A$ is a free K_ℓ-module of rank 2. If λ is a prime of K above ℓ, we get a surjective map $K_\ell \to K_\lambda$. Show that $V_\lambda = V_\ell A \otimes_{K_\ell} K_\lambda$ is a free K_λ-module of rank 2. We therefore get a Galois representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{K_\lambda} V_\lambda \approx \mathrm{GL}_2 K_\lambda$.

5. Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cuspidal eigenform of weight 2 for $\Gamma_0(N)$. Its coefficients lie in some totally real number field K. Let S be the set of real embeddings of K, so that $\{f^\sigma\}_{\sigma\in S}$ is the Galois orbit of f. Let A be the modular abelian variety associated to $\{f^\sigma\}_{\sigma\in S}$. Look up what the L-function of an abelian variety over $\mathbb{Q}$ is, and show that

$$L(A,s) = \prod_{\sigma\in S} L(f,s),$$

at least up to finitely many bad places. (It turns out that the L-functions are equal on the nose, but this is harder.)