MA 843 Assignment 3, due Oct. 17

October 11, 2013

1. (a) Show that the action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathcal{H}_g$ by fractional linear transformations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (A + BZ)(C + DZ)^{-1}$$

   is (a) well-defined (in the sense that $C+DZ$ is invertible), (b) transitive, and (c) the stabilizer of $iI_g \in \mathcal{H}_g$ is isomorphic to the unitary group $U(n)$. Thus $\mathcal{H}_g$ can be identified with $\mathrm{Sp}_{2g}(\mathbb{R})/U(n)$.

   (b) Show that $Z \mapsto (Z - iI_g)(Z + iI_g)^{-1}$ defines an isomorphism of complex manifolds between $\mathcal{H}_g$ onto the set of symmetric complex $g \times g$ matrices Z such that $I_g - Z^*Z$ is positive definite. Show that this last set is an open and bounded subset of the space of symmetric complex $g \times g$ matrices.

2. (a) Let V be a finite-dimensional real vector space of dimension $2g$. Show that any two complex structures on V are conjugate by an element of $\mathrm{GL}(V)$.

   (b) Now assume V is equipped with a symplectic structure (a non-degenerate alternating form ψ). Recall that a complex structure $J \in \mathrm{Sp}(V)$ is positive if $\psi(v, Jv) > 0$ for all $v \neq 0$. Show that any two positive complex structures are conjugate by an element of $\mathrm{Sp}(V)$.

   (c) Finally, show that the subgroup of $\mathrm{Sp}(V)$ which stabilizes J is isomorphic to $U(n)$, so that (by the previous exercise), the space of positive complex structures on V is $\mathcal{H}_g$.

3. Recall that $\mathbb{S}$ is the real algebraic group defined as follows: for all $\mathbb{R}$-algebras A, $\mathbb{S}(A)$ is the group of expressions $a + bi$ (with $a, b \in A$) for which $a^2 + b^2$ is a unit; the group operation follows the same rule that governs multiplication of complex numbers. Thus $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, and $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$. Let V be a finite-dimensional real vector space, and let $G = GL(V)$ (considered as a real algebraic group). Show rigorously that there is a bijection between homomorphisms $h\colon \mathbb{S} \to G$ and Hodge structures on V, which carries h to the Hodge structure defined by

$$V^{pq} = \left\{ v \in V_\mathbb{C} \;\middle|\; h(z)v = z^{-p}\overline{z}^{-q}v, \quad \text{all } z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \right\}$$

4. Let V be a real vector space of even dimension $2g$, equipped with a nondegenerate symplectic form $\Psi\colon V \times V \to \mathbb{R}$. This exercise is about classifying the possible polarized weight 1 Hodge structures on V.

   (a) Show that giving a weight 1 polarized Hodge structure on V (with polarization Ψ) is the same thing as a giving a complex subspace $V^{1,0} \subset V_\mathbb{C}$ which satisfies the conditions $\Psi|_{V^{1,0}} = 0$ and $\Psi(u, \overline{u}) > 0$ for all nonzero $u \in V^{1,0}$.

   (b) Show that $\mathcal{H}_g$ classifies polarized Hodge structures of weight 1 on V.

5. Let V be a real vector space of dimension n, equipped with a nondegenerate symmetric form $\Psi\colon V \times V \to \mathbb{R}$. Let $h^{2,0}$ and $h^{1,1}$ be nonnegative integers with $2h^{2,0} + h^{1,1} = n$. This exercise is about classifying the possible polarized weight 2 Hodge structures on V for which $\dim V^{pq} = h^{p,q}$.

   (a) Show that there aren't going to be any polarized weight 2 Hodge structures on V unless the signature of Ψ is $(2h^{2,0}, h^{1,1})$.

   (b) Assuming this is the case, show that giving a weight 2 polarized Hodge structure on V (with polarization Ψ) with $\dim V^{pq} = h^{p,q}$ is the same thing as giving a complex subspace $V^{2,0} \subset V_\mathbb{C}$ of dimension $h^{2,0}$ which satisfies the conditions that $\Psi|_{V^{2,0}} = 0$ and $\Psi(u, \overline{u}) > 0$ for all nonzero $u \in V^{2,0}$.

(c) Assume now that $h^{2,0} = 1$, and that the signature of Ψ is $(2, n-2)$. Show that the set of Hodge structures is parametrized by

$$D = \left\{ u \in \mathbb{P}(V_{\mathbb{C}}) \;\middle|\; \Psi(u,u) = 0,\ \Psi(u,\overline{u}) > 0 \right\},$$

and that D is a complex manifold. Here D is a period space of "K3 type". A *K3 surface* is a simply-connected 2-dimensional Kähler manifold with $h^{2,0} = 1$. The primitive H^2 of a K3 surface has $h^{2,0} = 1$ and $h^{1,1} = 19$. With those parameters, a family of K3 surfaces $X \to S$ induces a period map $\tilde{S} \to D$.