Modular abelian varieties and the Eichler-Shimura relation

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The goal of this lecture is to construct the Galois representation arising from a cuspidal eigenform of weight 2.

1 Modular abelian varieties

The main players here are the modular curve $X_0(N)$, and its Jacobian, which we write as $J_0(N)$. Both objects are defined over $\mathbb{Q}$. For each prime p not dividing N we have we Hecke correpondence T_p between $X_0(N)$ and itself:

$$X_0(N) \xleftarrow{\ f\ } X_0(Np) \xrightarrow{\ g\ } X_0(N)$$

The curve $X_0(Np)$ (or at least an affine part of it) classifies triples (E, C_N, C_p), where E is an elliptic curve, C_N is a cyclic subgroup of E of order N, and C_p is a cyclic subgroup of order p. There is an obvious map $X_0(Np) \to X_0(N)$ defined by forgetting C_p, but there is also another one where (E, C_N, C_p) gets sent to $(E/C_p, (C_N + C_p)/C_p)$. These two maps give the correspondence T_p. (There is also a definition of T_n for general n.)

Recall that correspondences between curves induce maps between Jacobians. Thus we get endomorphisms $T_n \colon J_0(N) \to J_0(N)$. Let $\mathbb{T}$ be the subalgebra of $\operatorname{End} J_0(N)$ generated by the T_n. We write $\mathbb{T}_{\mathbb{Q}}$ and $\mathbb{T}_{\mathbb{C}}$ for $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{C}$, respectively. The tangent space at the origin of $J_0(N)$

is the space $H^0(X_0(N), \Omega^1_{X_0(N)/\mathbb{Q}})$, so $\mathbb{T}$ acts on this space as well. Meanwhile, we know that $H^0(X_0(N), \Omega^1_{X_0(N)/\mathbb{Q}})$ is a rational model for the space $S_2(\Gamma_0(N))$ of modular forms of weight 2 and level N, so we get an action of $\mathbb{T}_C$ on $S_2(\Gamma_0(N))$. It isn't too hard to see that this action is the same as the usual action of Hecke operators on modular forms. Further, the map $\operatorname{End} J_0(N) \to \operatorname{End} H^0(X_0(N), \Omega^1_{X_0(N)/\mathbb{Q}})$ (given by sending a morphism to its derivative at the origin) is injective (this is a general fact about abelian varieties). Thus $\mathbb{T}_\mathbb{C} \to \operatorname{End} S_2(\Gamma_0(N))$ is injective. It is therefore valid to say that $\mathbb{T}$ is the subring of $\operatorname{End} S_2(\Gamma_0(N))$ generated by the T_n.

Lemma 1.0.1. *The pairing*

$$\begin{aligned} S_2(\Gamma) \times \mathbb{T}_\mathbb{C} &\to \mathbb{C} \\ (f, T) &\mapsto a_1(T(f)) \end{aligned}$$

is perfect (it identifies $\mathbb{T}_\mathbb{C}$ with the dual of $S_2(\Gamma)$).

Proof. Suppose $f \in S_2(\Gamma_0(N))$ satisfies $a_1(T_n(f)) = 0$ for all n. But $a_1(T_n(f)) = a_n(f)$, so that $f = 0$.

Now suppose $T \in \mathbb{T}_\mathbb{C}$ satisfies $a_1(T(f)) = 0$ for all f. Then for all n and f we have $0 = a_1(TT_n(f)) = a_1(T_nT(f)) = a_n(T(f))$, so that $T(f) = 0$. Since T kills all the fs, we have $T = 0$. □

Lemma 1.0.2. *There is a bijection between the set of normalized eigenforms in $S_2(N)$ and the set of $\mathbb{C}$-algebra homomorphisms $\mathbb{T}_\mathbb{C} \to \mathbb{C}$.*

Proof. (Easy.) □

Lemma 1.0.3. *Let $f = \sum_n a_n q^n$ be an eigenform in $S_2(\Gamma_0(N))$ which is normalized (meaning that $a_1 = 1$). Then the a_n are algebraic numbers (in fact they are algebraic integers) which are totally real. If $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $f^\sigma = \sum_n a_n^\sigma q^n$ is also an eigenform.*

Proof. $\mathbb{C}$-algebra homomorphisms $\mathbb{T}_\mathbb{C} \to \mathbb{C}$ are the same as $\mathbb{Q}$-algebra homomorphisms $\mathbb{T}_\mathbb{Q} \to \mathbb{C}$. Since $\mathbb{T}_\mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-algebra, the image of any such homomorphism has to lie in a number field $K \subset \mathbb{C}$, which we can assume is Galois over $\mathbb{Q}$. Then if σ is an automorphism of K, then $T \mapsto \lambda(T)^\sigma$ is a $\mathbb{Q}$-algebra homomorphism $\mathbb{T}_\mathbb{Q} \to \mathbb{C}$, which corresponds to an eigenform having q-expansion $\sum_n a_n^\sigma q^n$.

The coefficient a_n is real because it is the eigenvalue of a self-adjoint operator on a Hilbert space. The same is true for each a_n^σ, so in fact a_n is totally real. □

Lemma 1.0.4. *Prime ideals of $\mathbb{T}_{\mathbb{Q}}$ correspond to Galois orbits of eigenforms.*

Proof. An eigenform f corresponds to a homomorphism $h_f\colon \mathbb{T}_{\mathbb{Q}} \to \mathbb{C}$, whose kernel must be a prime ideal and whose image is a number field K. If f^σ is a conjugate eigenform, then h_{f^σ} is the composite $\mathbb{T}_{\mathbb{Q}} \to K \to K^\sigma \to \mathbb{C}$, which has the same kernel.

Conversely, if $P \subset \mathbb{T}_{\mathbb{Q}}$ is prime, then $\mathbb{T}_{\mathbb{Q}}/P$ is an integral domain which is finite over $\mathbb{Q}$, hence a number field K, and then the embeddings of K into $\mathbb{C}$ give the required eigenforms. □

Let f be a normalized eigenform of weight 2. The coefficients of f generate a number field K. Let $P \subset \mathbb{T}_{\mathbb{Q}}$ be the corresponding prime ideal, so that $\mathbb{T}_{\mathbb{Q}}/P = K$. Note that P only depends on the Galois orbit of f. We want to associate to P an abelian variety quotient of $J_0(N)$. Let $P_{\mathbb{Z}} = P \cap \mathbb{T}$, and put

$$A_f = J_0(N)/PJ_0(N),$$

so that A_f is an abelian variety over $\mathbb{Q}$ admitting endomorphisms by $\mathbb{T}/P_{\mathbb{Z}}$, an order in $\mathbb{T}_{\mathbb{Q}}/P = K$.

We can explain what A_f is on the level of complex tori. We have $J_0(N)(\mathbb{C}) = S_2(\Gamma_0(N))/H^1(X_0(N),\mathbb{Z})$. Let $V_f = S_2(\Gamma_0(N))/PS_2(\Gamma_0(N))$, so that V_f is the $\mathbb{C}$-span of the Galois conjugates of f. Also let Λ_f be the quotient of $H^1(X_0(N),\mathbb{Z})$ by $T_{\mathbb{Q}}H^1(X_0(N),\mathbb{Q}) \cap H^1(X_0(N),\mathbb{Z})$. Then $A_f(\mathbb{C}) = V_f/\Lambda_f$, so that A_f has dimension equal to $[K:\mathbb{Q}]$.

The ℓ-adic Tate module $T_\ell A_f$ has rank $2g$ over $\mathbb{Z}_\ell$, where $g = \dim A_f = [K:\mathbb{Q}]$. Let λ be a prime of K above λ. Then $V_\lambda := T_\ell A_f \otimes_{\mathbb{Z}_\ell} K_\lambda$ is a K_λ-vector space of dimension 2 (exercise). We get a Galois representation

$$\rho_{f,\lambda}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{K_\lambda} V_\lambda \approx \mathrm{GL}_2(K_\lambda).$$

2 The Eichler-Shimura relation

The next step is to see why $\rho_{f,\lambda}$ has the properties that it does, namely:

1. $\rho_{f,\lambda}$ is unramified outside of N and ℓ.

2. If p is a prime not dividing $N\ell$, then $\mathrm{tr}\,\rho_{f,\lambda}(\mathrm{Frob}_p) = a_p(f)$ (the eigenvalue of T_p on f).

The first point follows as soon as you know that $X_0(N)$ has a model over $\mathbb{Z}[1/N]$ which is smooth at all the primes not dividing N. This is a result of Deligne-Rapoport, who describe a model of $X_0(N)$ even over $\mathbb{Z}$. Let's just call this model $X_0(N)$. Deligne and Rapoport even give a description of $X_0(N)$ modulo those primes dividing N. For now, all we need is a result about $X_0(Np)_{\mathbb{F}_p}$.

Theorem 2.0.5. *$X_0(Np)_{\mathbb{F}_p}$ is a union of two copies of $X_0(N)_{\mathbb{F}_p}$ which intersect transversely at the supersingular points. The morphism $X_0(Np) \to X_0(N)$ which forgets the level p structure restricts to one copy as the identity map and the other as Frobenius. The morphism $X_0(Np) \to X_0(N)$ which quotients by the level structure does the reverse.*

The morphism $\mathrm{Frob}_p \in \mathrm{End}\, J_0(N)_{\mathbb{F}_p}$ has degree p. This implies that $p\,\mathrm{Frob}_p^{-1}$ makes sense as an endomorphism of $J_0(N)_{\mathbb{F}_p}$.

Theorem 2.0.6. *In $\mathrm{End}\, J_0(N)_{\mathbb{F}_p}$ we have*

$$T_p = \mathrm{Frob}_p + p\,\mathrm{Frob}_p^{-1}$$

Theorem 2.0.7. *The matrix $\rho_{f,\lambda}(\mathrm{Frob}_p)$ has trace $a_p(f)$ and determinant p.*

For a discussion of the proof, see William Stein's lecture notes: `http://modular.math.washington.edu/edu/Fall2003/252/lectures/11-19-03/11-19-03.pdf`