Hodge Structures

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1 A few examples of symmetric spaces

The upper half-plane $\mathcal{H}$ is the quotient of $\mathrm{SL}_2(\mathbb{R})$ by its maximal compact subgroup $SO(2)$. More generally, Siegel upper-half space $\mathcal{H}_g$ is the quotient of $\mathrm{Sp}_{2g}(\mathbb{R})$ by its maximal compact subgroup $U(g)$. $\mathcal{H}_g$ is a complex manifold, with an action of $\mathrm{Sp}_{2g}(\mathbb{R})$ by holomorphic automorphisms. Thus if $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{R})$ is a discrete subgroup, then $\Gamma\backslash\mathcal{H}_g$ has a chance at being an algebraic variety. In fact, if $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ is a congruence subgroup, then this is in fact the case; the theorem of Baily-Borel shows that $\Gamma\backslash\mathcal{H}_g$ is an open subset of a projective variety, known as a Siegel modular variety.

Just how general is this process? If we start with a group such as $\mathrm{SL}_n(\mathbb{R})$, and take its quotient by (say) the compact subgroup $\mathrm{SO}(n)$, the result can be characterized as the set of positive definite quadratic forms of rank n and determinant 1. Its dimension is $n(n+1)/2-1$. For $n = 3$, this is an odd number, so there can't be a complex structure on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$. For arithmetic subgroups $\Gamma \subset \mathrm{SL}_3(\mathbb{Z})$, one can still form the quotient $\Gamma\backslash \mathrm{SL}_3(\mathbb{R})/SO(3)$ to get an interesting manifold, but it won't be an algebraic variety.

What about $\mathrm{SL}_2(\mathbb{C})$? That group has a complex structure at least. $\mathrm{SL}_2(\mathbb{C})$ acts on the set of 2×2 hermitian matrices M, by the action $g \cdot M = gMg^*$, and preserves the determinant. The determinant, considered as a quadratic form on the space of hermitian matrices, has signature $(1, 3)$. Thus we get a homomorphism $\mathrm{SL}_2(\mathbb{C}) \to O(1, 3)$, whose kernel is just $\{\pm I\}$. The Lorentz group $O(1, 3)$ acts transitively on the set $\mathcal{H}^3$ of 4-tuples (t, x, y, z) such that $t^2-x^2-y^2-z^2 = 1$, $t > 0$; the stabilizer of $(1, 0, 0, 0)$ is $O(1)\times O(3)$ which is the maximal compact subgroup. Hence the symmetric space for the complex Lie group $\mathrm{SL}_2(\mathbb{C})$ is actually $\mathcal{H}^3$, a hyperbolic 3-manifold!

In the following, we'll get to the bottom of which spaces G/K have complex structures, where G is a real Lie group and $K \subset G$ is a maximal compact subgroup. In fact, there's a special class of these, the hermitian symmetric domains, which are the most interesting and which are required to define Shimura varieties in general.

2 Classifaction of PPAVs

Consider the problem of classifying principally polarized abelian varieties (PPAVs) over $\mathbb{C}$. As we know, If $A/\mathbb{C}$ is an abelian variety of dimension g, then as topological spaces we have $A = V/\Lambda$, where $\Lambda = H_1(A, \mathbb{Z})$ and $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. To give a polarization $\lambda\colon A \to A^\vee$ is to give a Riemann form

$$E\colon \Lambda \times \Lambda \to \mathbb{Z};$$

that is, an alternating bilinear form satisfying the properties

1. The extension $E_{\mathbb{R}}\colon V \times V \to \mathbb{R}$ satisfies $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$, and
2. If $H(v, w) = E_{\mathbb{R}}(iv, w) + iE_{\mathbb{R}}(v, w)$, then $H\colon V \times V \to \mathbb{C}$ is a positive definite hermitian form on V.

The condition that λ be principal corresponds to the condition that E is a perfect pairing, so that it identifies Λ with its $\mathbb{Z}$-linear dual.

Thus if A is a principally polarized abelian variety of dimension g, then $\Lambda = H_1(A, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$ equipped with a perfect symplectic form E. We can find a basis $x_1, \ldots, x_{2g}$ for Λ with respect to which E has some convenient form, such as

$$J_0 = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Let Ψ be the corresponding symplectic form on $\mathbb{Z}^{2g}$, so that

$$\Psi(v, w) = v^t J_0 w.$$

Now we can ask a slightly different problem. Begin with $\mathbb{Z}^{2g}$ together with its symplectic form Ψ. We will classify pairs (A, λ, α) where (A, λ) is a PPAV and $\alpha\colon \mathbb{Z}^{2g} \to H_1(A, \mathbb{Z})$ is an isomorphism *which is compatible with the symplectic forms on either space*:

$$\Psi(v, w) = E(\alpha(v), \alpha(w)),\ v, w, \in \mathbb{Z}^{2g}.$$

Proposition 2.0.1. *Triples (A, λ, α) are in bijection with points of Siegel upper half space*

$$\mathcal{H}_g = \left\{ X + iY \in M_{g\times g}(\mathbb{C}) \;\middle|\; X^t = X,\ Y > 0. \right\}$$

Pairs (A, λ) are classified by the quotient $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$.

(In the above definition of $\mathcal{H}_g$, X and Y are real matrices, and $Y > 0$ means that Y is positive definite. Thus $\mathcal{H}_1$ is the usual upper half plane.)

The second claim follows from the first: if (A, λ) is a PPAV, then there always exists an α of the required form, and any two such differ by an automorphism of $\mathbb{Z}^{2g}$ with preserves Ψ; that is, by an element of Sp_{2g}.

To prove the first claim, we need to think of $\mathcal{H}_g$ not as a subset of $M_{g\times g}(\mathbb{C})$ but rather as the quotient $\mathrm{Sp}_{2g}(\mathbb{R})/U(g)$. The symplectic group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on $\mathcal{H}_g$ via fractional linear transformations, and the stabilizer of iI_g is $U(g)$ (exercise).

First, we'll start with a triple (A, λ, α) and construct the appropriate coset in $\mathrm{Sp}_{2g}(\mathbb{R})/U(g)$. Write $A = V/\Lambda$ We can use α to identify the symplectic lattice $(\mathbb{Z}^{2g}, \Psi)$ with (Λ, E). Now when we tensor with $\mathbb{R}$, we have an isomorphism between $\mathbb{R}^{2g}$ and $\Lambda \otimes \mathbb{R} = V$, which is a complex vector space. The action of i on the V corresponds to a map $J\colon \mathbb{R}^{2g} \to \mathbb{R}^{2g}$ with $J^2 = -I$; that is, J is a *complex structure* on $\mathbb{R}^{2g}$. Furthermore, we have the properties that $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$ and $E_{\mathbb{R}}(iv, v) > 0$ for nonzero $v \in V$, and these correspond to the properties

1. J is *symplectic*: $\Psi_{\mathbb{R}}(Jv, Jw) = \Psi_{\mathbb{R}}(v, w)$
2. J is *positive*: $\Psi_{\mathbb{R}}(v, Jv) > 0$ for $v \in \mathbb{R}^{2g}$, $v \neq 0$.

Going the other way, if J is a complex structure on $\mathbb{R}^{2g}$ which is symplectic and positive, then $\mathbb{R}^{2g}$ becomes a complex vector space V (where the action of i is J), and E becomes a principal Riemann form on $\mathbb{Z}^{2g}$, so that $A = V/\mathbb{Z}^{2g}$ becomes a PPAV together with an isomorphism of its H_1 with $\mathbb{Z}^{2g}$.

How to classify symplectic and positive complex structures J on $\mathbb{R}^{2g}$? Let J_0 be any such structure (such as the matrix J_0 noted above). Then any other such structure J is conjugate to J_0 by a symplectic matrix S: $J = SJ_0S^{-1}$ (exercise). Here $S \in Sp_{2g}(\mathbb{R})$ is a matrix which is well-defined up to the group of matrices U which commute with J_0. But commuting with

J_0 means that U is $\mathbb{C}$-linear on $V = \mathbb{R}^{2g}$ (considered as a complex vector space). Furthermore, our assumption on J_0 means that the form

$$H(v, w) = \Psi(v, iw) + i\Psi(v, w)$$

turns $\mathbb{R}^{2g}$ into a hermitian inner product space; the fact that S is symplectic means that S is unitary with respect to H. Thus S is well-defined as an element of $\mathrm{Sp}_{2g}(\mathbb{R})/U(g)$.

3 The formalism of Hodge structures

In the foregoing discussion we considered the moduli space of complex structures J on a vector space $V \cong \mathbb{R}^{2g}$, which preserved some extra structure (a symplectic pairing), and which were subject to some positivity condition. The moduli space ended up being the quotient of a real algebraic group $\mathrm{Sp}_{2g}(\mathbb{R})$ by a maximal compact subgroup. We're now going to introduce Hodge structures on a real vector space, which are a generalization of complex structures.

If V is a real vector space of dimension d, then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector space of dimension d. Let τ be complex conjugation; then $1 \otimes \tau$ is a $\mathbb{C}$-semilinear endomorphism of $V_{\mathbb{C}}$, which we denote as $v \mapsto \overline{v}$.

Let V be a real vector space. A *Hodge structure* on V is a bigrading $V_{\mathbb{C}} = \bigoplus_{p,q} V^{pq}$, such that $\overline{V}^{pq} = V^{qp}$. A Hodge structure is of type $S \subset \mathbb{Z} \times \mathbb{Z}$ if $V^{pq} = 0$ for all $(p, q) \notin S$.

The idea behind this definition comes from cohomology. Let $X/\mathbb{C}$ is a smooth projective variety of dimension d. Then X has the structure of a real manifold of dimension $2d$. We have the singular cohomology groups $H^i(X, \mathbb{Z}) = H^i_{\mathrm{sing}}(X, \mathbb{Z})$, which are zero outside the range $i = 1, 2, \ldots, 2d$. For each i, the pairing of i-forms and i-cochains gives an isomorphism

$$H^i(X, \mathbb{R}) \cong H^i_{\mathrm{dR}}(X, \mathbb{R}),$$

where $H^i_{\mathrm{dR}}(X, \mathbb{R})$ is closed i-forms modulo exact i-forms. What's more, the complex space $H^i_{\mathrm{dR}}(X, \mathbb{C})$ breaks up according to the Hodge decomposition

$$H^i_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the subspace of $H^i_{\mathrm{dR}}(X, \mathbb{C})$ consisting of classes having a representative which is of type (p, q). Complex conjugation maps (p, q)-forms onto (q,p)-forms, so that we get a Hodge structure on each $H^i(X,\mathbb{R})$. (By the Dolbeault theorem, $H^{p,q}(X)$ is isomorphic to $H^q(X,\Omega^p)$, where Ω^p is the sheaf of holomorphic p-forms.)

We let $\operatorname{Fil}^p V$ be the direct summand

$$\operatorname{Fil}^p V = \bigoplus_{p'\geq p} V^{p'q},$$

so that $\operatorname{Fil}^p V \subset V_{\mathbb{C}}$ is a complex subspace, and $\operatorname{Fil}^p V$ decreases with p. Say that V has weight n if $p+q=n$ for every pair (p,q) with $V^{pq}\neq 0$. If V has weight n, note that the filtration $\{\operatorname{Fil}^p V\}$ determines the Hodge structure completely, since

$$V^{pq} = \operatorname{Fil}^p V \cap \overline{\operatorname{Fil}^q V}.$$

An *integral Hodge structure* is a $\mathbb{Z}$-module Λ together with a (real) Hodge structure on $\Lambda\otimes\mathbb{R}$. Thus in the above situation, $H^i(X,\mathbb{Z})$ is an integral Hodge structure.

Let's see how complex structures are examples of Hodge structures. If V is a real vector space, and $J\in GL(V)$ is a complex structure, then $V_{\mathbb{C}}$ breaks up into two subspaces $V^{1,0}$ and $V^{0,1}$, these being the subspaces where $J=i$ and $J=-i$, respectively. Thus a complex structure is the same as a Hodge structure of type $\{(1,0),(0,1)\}$.

In the situation of abelian varieties, we have to be careful about duals. If A is an abelian variety, then the Hodge decomposition applied to A makes $H^1(A,\mathbb{Z})$ into an integral Hodge structure of type $\{(1,0),(0,1)\}$. But $H_1(A,\mathbb{Z})$, its $\mathbb{Z}$-linear dual, is a Hodge structure of type $\{(-1,0),(0,-1)\}$.

If V is a Hodge structure, then define an action $h\colon \mathbb{C}^\times \to \mathrm{GL}(V)$ as follows. First define an action h of $\mathbb{C}^\times$ on V_C by

$$h(z)(v^{pq}) = z^{-p}\overline{z}^{-q}v^{pq},\quad \text{all } v^{pq}\in V^{pq}.$$

Then since $\overline{V}^{pq}=V^{qp}$, we have $h(z)(\overline{v})=\overline{h(z)(v)}$ for all $v\in V_{\mathbb{C}}$. Thus $h(z)$ preserves the real subspace $V\subset V_{\mathbb{C}}$, this being the space of vectors invariant under $v\mapsto\overline{v}$. We get an action h of $\mathbb{C}^\times$ on V.

It will be important to view this action in terms of algebraic groups. Let $\mathbb{S}$ be the restriction of scalars of $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$. Thus, for an $\mathbb{R}$-algebra A, $\mathbb{S}(A)$ is the group of pairs $(a,b)\in A\times A$ which satisfy $a^2+b^2\in A^\times$, under the multiplication law which tells you how to multiply complex numbers $a+bi$ together. We have $\mathbb{S}(\mathbb{R})=\mathbb{C}^\times$. The following is an exercise:

Proposition 3.0.2. *Morphisms of real algebraic groups* $\mathbb{S} \to \mathrm{GL}(V)$ *correspond to Hodge structures on* V*.*

This suggests defining Hodge structures relative to an algebraic group other than $\mathrm{GL}(n)$. So let G be a real algebraic group, and say that a Hodge structure on G is a homomorphism $h\colon \S \to G$ of real algebraic groups. If V is a faithful representation of G, then h induces a Hodge structure on V as above.

A *polarization* of a Hodge structure h on V is an alternating bilinear map of Hodge structures $\Psi\colon V \times V \to \mathbb{R}$ for which $\Psi(v, Jw)$ is positive definite, where $J = h(i)$. The statement that Ψ be a morphism of Hodge structures is the statement that $\Psi(Jv, Jw) = \Psi(v, w)$.

Example 3.0.3. The set of polarized Hodge structures on $\mathbb{R}^{2g}$ of type $\{(-1, 0), (0, -1)\}$ is the Siegel upper half-space $\mathcal{H}_g$.