Kähler manifolds and variations of Hodge structures

October 21, 2013

1 Some amazing facts about Kähler manifolds

The best source for this is Claire Voisin's wonderful book *Hodge Theory and Complex Algebraic Geometry, I.*

There are Riemannian manifolds, symplectic manifolds, and hermitian manifolds, all of which are manifolds X together with some kind of tensor (a map from $T_pX \otimes T_pX$ to $\mathbb{R}$ or $\mathbb{C}$) at every point, which is (respectively) symmetric and positive definite, alternating, or hermitian. (To be a hermitian manifold, you need to first be a complex manifold, so that T_pX is a complex vector space.)

A *Kähler manifold* is a manifold X where all the above adjectives apply: Riemannian, symplectic, complex, hermitian. More precisely, a Kähler form is a complex manifold (so that each T_pX has a complex structure J) which has a symplectic form $\omega\colon \wedge^2 T_pX \to \mathbb{R}$, which is compatible with the complex structure in the sense that $\omega(Ju, Jv) = \omega(u, v)$. (Note that a symplectic form is just a differential 2-form; the definition of a symplectic manifold requires that ω be closed.) We see that $\omega(u, Jv)$ is symmetric; it is required to be positive definite (and thus it is a Riemann form). With this in place, we get a hermitian form $h\colon T_pX \times T_pX \to \mathbb{C}$ by

$$h(u, v) = \omega(u, Jv) + i\omega(u, v).$$

Sound familiar? The same structures ω and J appear on the Lie algebra of an abelian variety over $\mathbb{C}$. You can use the group operation to spread those structures around to every tangent space, so that an abelian variety is a Kähler manifold. In fact, every smooth projective variety is Kähler: projective space itself has a Kähler structure (the Fubini-Study metric), which passes to every closed complex subvariety. However, not every compact Kähler manifold is projective: let V/Λ be a complex torus, where V comes equipped with a symplectic form ω as above; then V/Λ will only be an abelian variety when the restriction of ω to $\Lambda \times \Lambda$ takes integer values.

A rich source of Kähler manifolds come from projective varieties. Projective space $\mathbb{P}^n$ has a Kähler form ω, called the Fubini-Study metric. The class $[\omega] \in H^2(\mathbb{P}^n, \mathbb{R})$ is the same as the class of a hyperplane section, so it lies in $H^2(\mathbb{P}^n, \mathbb{Z})$. If $X \subset \mathbb{P}^n$ is a closed smooth subvariety, then X inherits the Kähler form from $\mathbb{P}^n$, and the class of this form lies in $H^2(X, \mathbb{Z})$. Thus every projective variety is Kähler.

The Kodaira embedding theorem says that this is an "if and only if". That is, if X is a compact complex manifold, then X is projective if and only if it admits a Kähler form ω whose class lies in $H^2(X, \mathbb{Z})$ (a priori such a class only lies in $H^2(X, \mathbb{R})$). In the case of a complex torus $A = V/\Lambda$, we have $H^2(A, \mathbb{Z}) \cong \mathrm{Hom}(\wedge^2\Lambda, \mathbb{Z})$, and so $H^2(A, \mathbb{R}) \cong \mathrm{Hom}(\wedge^2 V, \mathbb{R})$. In this isomorphism, the class of the Kähler form ω on V corresponds to $\omega\colon \wedge^2 V \to \mathbb{R}$, and thus the Kodaira embedding theorem says that A is projective if and only if ω takes integer values on $\Lambda \times \Lambda$, which we already knew.

The cohomology of Kähler manifolds is dizzyingly rich. We'll highlight three basic properties: Hodge decomposition, Lefshetz decomposition, and polarization.

1.1 Hodge decomposition

For a compact Kähler manifold X of real dimension $2n$, the cohomology $H^k(X, \mathbb{Z})$ is an integral Hodge structure:

$$H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X).$$

Here $H^{p,q}(X)$ is the space of classes in $H^k_{\mathrm{dR}}(X, \mathbb{C})$ represented by closed forms which are everywhere of type (p, q). We have $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

In particular, the Betti numbers $\beta_k = \dim H^k(X, \mathbb{R})$ have to be even whenever k is odd. This allows us to find complex manifolds which are not Kähler. One example is the Hopf surface X obtained by quotienting

$\mathbb{C}^2\setminus\{(0,0\}$ by the contraction $(x,y)\mapsto(2x,2y)$. Then X is diffeomorphic to $S^1\times S^3$, which implies that its Betti numbers are $1,1,0,1,1$. Since β_1 is odd, X cannot be Kähler.

1.2 Lefshetz decomposition

Our manifold X comes equipped with a Kähler class $[\varpi]\in H^2(X,\mathbb{R})$. Wedging with $[\varpi]$ defines a map

$$L\colon H^k(X,\mathbb{R})\to H^{k+2}(X,\mathbb{R}),$$

called the Lefschetz operator. By Poincaré duality, $H^k(X,\mathbb{R})$ is dual to $H^{2n-k}(\mathbb{R})$. The hard Lefshetz theorem says that for $k\le n$,

$$L^{n-k}\colon H^k(X,\mathbb{R})\to H^{2n-k}(X,\mathbb{R})$$

is an isomorphism.

Now let

$$H^k(X,\mathbb{R})_{\mathrm{prim}}=\ker L^{n-k+1},$$

the set of "primitive classes" in degree k. There is a decomposition of $H^k(X,\mathbb{R})$ into primitive subspaces coming via iterates of L from degrees $k,k-2,\ldots$:

$$H^k(X,\mathbb{R})=\bigoplus_r L^r H^{k-2r}(X,\mathbb{R})_{\mathrm{prim}}.$$

Each $H^k(X,\mathbb{R})_{\mathrm{prim}}$ has its own Hodge structure, so we can talk about $H^k(X,\mathbb{C})_{\mathrm{prim}}$ splitting into subspaces $H^{p,q}(X)_{\mathrm{prim}}$.

1.3 Polarization

The map L^{n-k} is an isomorphism between $H^k(X,\mathbb{R})$ and its dual $H^{2n-k}(X,\mathbb{R})$, so we have just set up a bilinear pairing. For $i\le n$, let Ψ be the nondegenerate bilinear form Ψ (the *intersection form*) defined on each $H^k(X,\mathbb{R})$ by

$$\Psi(\alpha,\beta)=\int_X \omega^{n-k}\wedge\alpha\wedge\beta.$$

It is alternating if k is odd and symmetric if k is even. Furthermore, it respects the Hodge structure in the sense that $H^{p,q}(X)$ and $H^{r,s}(X)$ are orthogonal unless $(p,q)=(s,r)$. We refer to this state of affairs by saying that $V = H^k(X, \mathbb{R})$ is a *polarized Hodge structure of weight k*. In Deligne's fancy language, the polarization is a morphism of Hodge structures $\Psi\colon V \times V \to \mathbb{R}(-k)$ (where $\mathbb{R}(-k)$ refers to the Tate twist). It means that if we define a form H on $V_{\mathbb{C}}$ by

$$H(\alpha, \beta) = i^k \Psi(\alpha, \overline{\beta}),$$

then H is Hermitian (not necessarily positive definite). The *Hodge index theorem* computes the signature of the Hermitian form Ψ on $H^k(X, \mathbb{C})$; in fact it becomes definite when you restrict it to the primitive part $H^{p,q}(X)_{\text{prim}}$, with sign $i^{p-q-k}(-1)^{k(k-1)/2}$ (see 7.2.1 in Voisin).

1.4 Integral polarized Hodge structures

Recall that a complex manifold X is projective if and only if it admits a Kähler form ω whose class lies in $H^2(X, \mathbb{Z})$ (the Kodaira embedding theorem). In that case, wedging with ω carries $H^k(X, \mathbb{Z})$ onto $H^{k+2}(X, \mathbb{Z})$, and the whole discussion (Lefshetz decompsition, polarization) carries over onto $H^k(X, \mathbb{Z})$, so that we can talk about $H^k(X, \mathbb{Z})_{\text{prim}}$. This inspires the following definition, with V playing the role of $H^k(X, \mathbb{Z})_{\text{prim}}$:

Definition 1.4.1. An *integral polarized Hodge structure of weight k* is a $\mathbb{Z}$-module $V_{\mathbb{Z}}$ of finite rank, equipped with a pairing

$$\Psi\colon V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z},$$

alternating if k is odd and symmetric otherwise, and a Hodge decomposition

$$V_{\mathbb{C}} = \bigoplus V^{pq},$$

where $\overline{V}^{pq} = V^{qp}$, such that if

$$H(\alpha, \beta) = i^k \Psi(\alpha, \overline{\beta}),$$

then the V^{pq} are orthogonal with respect to H, and is definite on V^{pq} with sign $i^{p-q-k}(-1)^{k(k-1)/2}$.

2 Variations of Hodge Structure

Let $f: X \to S$ be a morphism of complex manifolds, such that every fiber is a Kähler manifold. For every $s \in S$ we have an abelian groups $H^i(X_s, \mathbb{Z})$, which form a local system $\mathcal{L}$ on S. In sheaf-theoretic language, $\mathcal{L} = R^i f_*(\mathbb{Z})$.

And what is a local system? It's a locally constant sheaf of abelian groups $\mathcal{L}$ on (the underlying topological space) S. This means that you get a stalk $\mathcal{L}_s$ for each $s \in S$, together with some data about how to move sections of $\mathcal{L}_s$ around. For each homotopy class of path $r : s \to s'$ in S, there's a corresponding isomorphism $\rho(r): \mathcal{L}_s \to \mathcal{L}_s$, and these have to satisfy the obvious transitivity property. Of course if $r: s \to s$ is a loop in $\pi_1(S, s)$, then we get an automorphism of $\rho(r)$ of $\mathcal{L}_s$; in brief we get a representation $\rho: \pi_1(S, s) \to \operatorname{Aut} \mathcal{L}_s$, and this representation carries exactly the information of the local system $\mathcal{L}$ (for S connected anyway).

Now let's mix in the Hodge structures. Let $V = H^k(X_s, \mathbb{R})$. For each path $r : s \to s'$ in S, we get a Hodge structure on $H^k(X_{s'}, \mathbb{R})_{\mathrm{prim}}$ coming from $X_{s'}$, which gives a Hodge structure V_r^{pq} on V via $\rho(r)$. Note that the subspaces V_r^{pq} vary with r, but the original integral and polarization structures on V do not. Let D be a space whose points represent Hodge structures of weight k on V, with the same combinatorial data as the Hodge structure coming from X_s itself (that is, we hold the dimension of V^{pq} fixed). We get a map $\pi: \tilde{S} \to D$, where $\tilde{S}$ is the universal cover. This is a *period map*. A point of $\tilde{S}$ is a homotopy class of paths $p: s \to s'$, and π sends this to the Hodge structure V_r^{pq} on V.

Essentially the period map $\pi: \tilde{S} \to D$ measures how the integrals $\int_{\gamma_s} \omega_s$ behave, where the cycle γ_s (resp., the k-form ω_s) vary continuously with s, and ω_s is constrained to be of type (p, q).

Example 2.0.2. Let $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and let $X \to S$ be the Legendre family of elliptic curves, so that X_λ as equation

$$y^2 = x(x-1)(x-\lambda),$$

for all $\lambda \in S$. The differential form $\omega = dx/2y$ forms a basis for the $\mathbb{C}$-vector space $H^{1,0}(X_\lambda)$ of holomorphic 1-forms on X_λ. Let $\lambda \in S$ be a base point, and let $V = H^1(X_\lambda, \mathbb{Z})$. In this case the relevant space D of Hodge structures is the set of positive complex structures on a 2-dimensional symplectic space; we know this to be $\mathcal{H}$, the upper half-plane. Thus our period map is $\pi: \tilde{S} \to \mathcal{H}$.

Explicitly, the period map works like this. E_λ is a double cover of $\mathbb{P}^1$, branched at $0, 1, \lambda, \infty$. The loop in $\mathbf{P}^1$ which encircles 0 and 1 lifts to a cycle $\alpha \in H_1(X_\lambda, \mathbb{Z})$, and the loop encircling λ and ∞ lifts to a cycle β. Let us use the basis α, β to identify $H_1(X, \mathbb{Z})$ with $\mathbb{Z}^2$; we can use the dual basis to identify $V_\mathbb{Z} = H^1(X, \mathbb{Z})$ with $\mathbb{Z}^2$ as well.

The Hodge decomposition is $V_\mathbb{C} = H^{1,0}(X_\lambda) \oplus H^{0,1}(X_\lambda)$. We have just identified $V_\mathbb{C}$ with $\mathbb{C}^2$. With respect to this, $H^{1,0}$ is the line spanned by

$$\left(\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \int_\lambda^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \right) \in \mathbb{C}^2$$

The ratio of these integrals is $\pi(\lambda) \in \mathcal{H}$. As λ varies, so does $\pi(\lambda)$, but when you move λ through a nontrivial element of $\pi_1(S)$, the cycles α and β are going to change (monodromy!). So really π is only well-defined on the universal cover of S.

On the other hand, we already know the universal cover of S: it is $\mathcal{H}$, and the quotient map $\mathcal{H} \to S$ is the modular λ-function, with symmetry group $\Gamma(2) \cong F_2 = \pi_1(S)$. In this case, the period map $\pi\colon \mathcal{H} \to \mathcal{H}$ is the identity map; this reflects the fact that the periods of X_λ determine X_λ up to isomorphism (the Torelli theorem).

In the next lecture, we'll take on the question of what sort of period spaces D can arise when considering a general family of smooth projective varieties $X \to S$.