# Variations of Hodge Structure, part 2

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Let $S$ be a complex manifold, let $f\colon X \to S$ be a family of compact Kähler manifolds, and let $s \in S$ be a base point. Fix an integer $k$, and let $V = H^k(X_s, \mathbb{R})$, considered as a real vector space together with the intersection form $\Phi\colon V \times V \to \mathbb{R}$ (alternating or symmetric as $k$ is odd or even). Then (locally around $s$) each point $s' \in \mathbb{R}$ determines a Hodge structure on $V$. If $D$ is the set of possible Hodge structures on $V$, then we get a period map $S \to D$ locally around $s$ (or, if you like, a map from the universal cover of $S$ into $D$). What we'd like to do now is give $D$ the structure of a complex manifold in such a way that $\pi\colon S \to D$ is holomorphic. The complex manifold $D$ is then a *period domain* for $X$.

## 1 Grassmannian varieties

Let $V$ be a complex vector space of dimension $n$. Let $\mathrm{Grass}(k, V)$ denote the set of complex subspaces $W \subset V$ of dimension $k$. The first order of business is to show that $\mathrm{Grass}(k, V)$ has the structure of a complex manifold. This is usually done using the Plücker embedding

$$\begin{aligned} \alpha\colon \mathrm{Grass}(k,V) &\to \mathbb{P}\left(\bigwedge^k V\right) \\ W &\mapsto \wedge^k W. \end{aligned}$$

That is, if $e_1, \dots, e_k$ is a basis for $W$, then $\alpha(W)$ is defined as the line in $\wedge^k V$ containing $e_1 \wedge \cdots \wedge e_k$. Of course, a change of basis only changes $e_1 \wedge \cdots \wedge e_k$ by a scalar, so this map is well-defined. One observes that $\alpha(W)$ determines

$W$, because

$$W = \left\{ v \in V \;\middle|\; w \wedge \alpha(W) = 0 \right\}.$$

It isn't terribly hard to characterize the image of $\alpha$: it is the set of lines $[v]$ with $v \in \wedge^k V$ such that the wedging-with-$v$ map $\beta\colon V \to \wedge^{k+1} V$ has a $k$-dimensional kernel, in which case that kernel is $W = \alpha^{-1}([v])$. (A priori, $\ker \beta$ has dimension at most $k$.) The condition that $\beta$ has a $k$-dimensional kernel is algebraic–in fact, the image of $\alpha$ is the intersection of a set of quadrics in $\mathbb{P}(\bigwedge^k V)$.

One can also give a complex structure on $\mathrm{Grass}(k, V)$ directly by means of charts. Let $W \subset V$ be a point of $\mathrm{Grass}(k, V)$, and let $C \subset V$ be a complimentary subspace, so that $V = W \oplus C$. A generic $k$-dimensional subspace of $V$ doesn't meet $C$. Let $\mathrm{Grass}(k, V)_C$ be the set of $k$-dimensional subspaces $W'$ such that $W' \cap C = 0$, so that $\mathrm{Grass}(k, V)_C$ contains $W$. Let $\pi_W$ and $\pi_C$ be the projections onto $W$ and $C$, respectively. The map $\pi_W|_{W'}\colon W' \to W$ is an isomorphism. We get a linear map $h_W\colon W \to C$, by $h_W = \pi_C \circ (\pi_W|_{W'})^{-1}$. Conversely, if $h\colon W \to C$ is any linear map, let

$$W' = \left\{ (w, h(w)) \;\middle|\; w \in W \right\} \subset W \oplus C = V;$$

then $h = h_{W'}$. We have shown that

$$\mathrm{Grass}(k, V)_C \cong \mathrm{Hom}(W, C) = \mathrm{Hom}(W, V/W),$$

so that $\mathrm{Grass}(k, V)_C$ is in bijection with a complex vector space. Note that $W$ on the left corresponds to 0 on the right. (To use this to show that $\mathrm{Grass}(k, V)$ really is a complex manifold, one has to show that the transition maps between charts are holomorphic...) As a consequence, the tangent space to the Grassmannian at $W$ is

$$T_W \,\mathrm{Grass}(k, V) = \mathrm{Hom}(W, V/W).$$

# 2 Variations of Hodge Structure

Let $S$ be a complex manifold. A *variation of Hodge structure* on $S$ is a local system $V$ on $S$, together with a Hodge structure on $V_s$ for each point $s \in S$. This is subject to two conditions:

1. The Hodge structure varies holomorphically with $s$.

2. Griffiths transversality is satisfied.

Let us explain each condition. Both are local conditions, so we might as well assume that $V$ is a fixed vector space, and for all $s$ we get a Hodge filtration on $V$. Switching to the point of view of filtrations, we get for each $p \in \mathbb{Z}$ a decreasing filtration $F_s^p V_{\mathbb{C}}$ of $V_{\mathbb{C}}$ which varies with $s$. We demand that $b_p = \dim F_s^p V_{\mathbb{C}}$ not depend on $s$. Let $\pi^p \colon S \to \operatorname{Grass}(b_p, V_{\mathbb{C}})$ be the map which sends $s$ to $F_s^p V_{\mathbb{C}}$. The holomorphicity condition is that $\pi^p$ be holomorphic for each $p$.

Griffiths transversality is a bit more subtle. In brief it is the statement that the derivative of a section of $F^p V_{\mathbb{C}}$ has to lie in $F^{p-1} V_{\mathbb{C}}$. More precisely, the period map $\pi^p \colon S \to \operatorname{Grass}(b_p, V_{\mathbb{C}})$ is differentiable, so we can take its derivative at $s \in S$:

$$T_s S \to T_{\pi^{p,k}(s)} \operatorname{Grass}(b_p, V_{\mathbb{C}}) = \operatorname{Hom}(F^p V_{\mathbb{C}}, V_{\mathbb{C}}/F^p V_{\mathbb{C}}).$$

Griffiths transversality is the condition that the image of this map lies in $\operatorname{Hom}(F^p V_{\mathbb{C}}, V_{\mathbb{C}}/F^p V_{\mathbb{C}})$.

The motivation behind this definition comes from differential geometry. Let $f \colon X \to S$ be a family of compact Kähler manifolds. Then $V = R^k f_*(\mathbb{R})$ is a local system of $\mathbb{R}$-vector spaces on $S$, whose fiber at $s \in S$ is $H^k(X_s, \mathbb{R})$. For each $s$ we get a Hodge filtration on $V_s$ coming from $X_s$.

**Theorem 2.0.1.** *$V$ is a variation of Hodge structures on $S$.*

For a proof, see Voisin's book.

If one restricts to primitive cohomology, one even gets a variation of *polarized* Hodge structures. Suppose $V$ is a variation of Hodge structures of weight $k$ on $S$. A polarization on $V$ is a nondegenerate bilinear pairing

$$\Psi \colon V \times V \to \mathbb{R},$$

which is alternating or symmetric as $n$ is odd or even. It is required that when we let

$$\begin{aligned} H \colon V_{\mathbb{C}} \times V_{\mathbb{C}} &\to \mathbb{C} \\ (v, w) &\mapsto i^k \Psi(x, \overline{y}), \end{aligned}$$

then the Hodge decompositions (for each $s$!) are orthogonal for $H$, and that $i^{p-q-k}(-1)^{k(k-1)/2} H(v, v) > 0$ for $\alpha \in V_s^{pq}$.

**Example 2.0.2.** Let $V$ be a vector space of dimension $n$, equipped with a nondegenerate symmetric form $\Psi$ of signature $(2, n-2)$. We have already seen that

$$\mathcal{D} = \left\{ u \in \mathbb{P}(V) \;\middle|\; \Psi(u,u) = 0,\ \Psi(u,\overline{u}) > 0 \right\}$$

classifies polarized Hodge structures of weight 2 on $V$ of type $h^{2,0} = 1$, $h^{1,1} = n-2$ and $h^{0,2} = 1$. (Note that the conditions make sense even though the quantities $\Psi(u,u)$ and $\Psi(u,\overline{u})$ aren't well-defined.) Being an open subset of the complex projective variety $\Psi(u,u) = 0$, $\mathcal{D}$ is itself a projective variety.

The Hodge decomposition corresponding to $u \in \mathcal{D}$ is

$$V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$

where

$$\begin{aligned} V^{2,0} &= u \\ V^{1,1} &= (u \oplus \overline{u})^{\perp} \\ V^{0,2} &= \overline{u}, \end{aligned}$$

so that

$$\begin{aligned} F^0 V_{\mathbb{C}} &= V_{\mathbb{C}} \\ F^1 V_{\mathbb{C}} &= u^{\perp} \\ F^2 V_{\mathbb{C}} &= u. \end{aligned}$$

We can check Griffiths transversality in this situation. I won't spell out every detail, but remember that the slogan is "the derivative of a section of $F^p$ lies in $F^{p-1}$". The only interesting case is $p = 2$. A section of $F^2$ is an assignment, to each $u \in \mathcal{D}$, a point on the line in $V_{\mathbb{C}}$ corresponding to $u$. Let $u(t) = (u_1(t), \ldots, u_n(t))$ be a curve in $V_{\mathbb{C}}$, which satisfies $\Psi(u(t), u(t)) = 0$ for each $t$. Thus

$$u_1(t)^2 + u_2(t)^2 - u_3(t)^2 - \cdots - u_n(t)^2 = 0.$$

To check transversality is to check that the derivative $u'(t)$ always lies in $u(t)^{\perp}$, but this is clear by applying $d/dt$ to the above equation.

In case that $n = 21$, the variation of Hodge structures on $\mathcal{D}$ comes from algebraic K3 surfaces. For our purposes, an *algebraic K3 surface* is a smooth 2-dimensional variety isomorphic to a hypersurface in $\mathbb{P}^3$ of degree 4. It can be shown that if $X$ is an algebraic K3 surface, then

- $X$ is simply connected, so that $H^1(X, \mathbb{Z}) = 0$.

- The canonical bundle of $X$ is trivial. This means that $\Omega^2_{X/\mathbb{C}}$ contains a nowhere vanishing holomorphic 2-form. This has to be unique up to scaling, because otherwise the ratio of two such forms would be a nonconstant function on $X$. This shows that $\dim H^{2,0}(X) = 1$.

- $\dim H^2(X, \mathbb{Z})_{\text{prim}} = \mathbb{Z}^{21}$. Because $H^2(X, \mathbb{R})_{\text{prim}}$ is a polarized Hodge structure of even weight with $\dim H^{2,0}(X) = 1$, it follows that the intersection form on $H^2(X, \mathbb{Z})$ has signature $(2, 19)$.

The Hodge structure on $H^2(X, \mathbb{R})$ therefore determines a point of the complex manifold $\mathcal{D}$ (where we have assumed $n = 21$. We can consider the set of all such $X$ up to isomorphism: this is the set of homogeneous polynomials of degree 4 in 4 variables, up to a linear change of variables (minus the locus where the resulting hypersurface is singular).

The number of homogenous monomials of degree 4 in 4 variables is $\binom{4+4-1}{4-1} = 35$, so the space of nonsingular projective quartic hypersurfaces in $\mathbb{P}^3$ is an open subset $U$ of $\mathbb{P}^{34}$. The group $\mathrm{PGL}_4(\mathbb{C})$ acts on $U$ by linear changes of variables. We have $\dim \mathrm{PGL}_4(\mathbb{C}) = 15$. Thus $\dim U/\mathrm{PGL}_4(\mathbb{C}) = 34 - 15 = 19$. Meanwhile, the period domain $\mathcal{D}$ is an open subset of a quadric surface is $\mathbb{P}^2 0$, so it is also 19-dimensional. The global Torelli theorem for K3 surfaces, proved by Piatetskii-Shapiro and Shafarevich, states that the period map $U/\mathrm{PGL}_4(\mathbb{C}) \to \mathcal{D}$ is an open immersion. The idea here is that the periods of 2-forms determine the isomorphism class of an algebraic K3 surface. Compare this to the case of abelian varieties, where the periods of 1-forms determine the isomorphism class.