Hermitian Symmetric Domains

November 11, 2013

1 The Deligne torus, and Hodge structures

Let S be the real algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. Thus $S(\mathbb{R}) = \mathbb{C}^\times$.

If V is a finite-dimensional real vector space, the data of a Hodge structure on V is equivalent to the data of a morphism $h\colon S \to \mathrm{GL}(V)$ of real algebraic groups. Let's recall how this works: if such an h is given, then for all pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ we can define a subspace $V^{pq} \subset V_{\mathbb{C}}$ by

$$V^{pq} = \left\{ v \in V_{\mathbb{C}} \;\middle|\; h(z) = z^{-p}\overline{z}^{-q} v \; z \in \mathbb{C}^\times \right\}.$$

Then the V^{pq} are obviously disjoint, and $\overline{V}^{pq} = V^{qp}$. Further, the complex characters of S are exactly the $z \mapsto z^{-p}\overline{z}^{-q}v$, and any representation of S on a complex vector space has to break up into such characters, so V has to be the direct sum of the V^{pq}.

Conversely, if the V^{pq} are given, then it is easy to define $h\colon S \to \mathrm{GL}(V)$ in terms of the above formula.

We remark that V is homogeneous of weight k if and only if, for all $t \in \mathbb{R}^\times$, we have $h(t) = t^k I \in \mathrm{GL}(V)$.

Lemma 1.0.1. *Two Hodge structures on V have the same combinatorial data (ie the dimensions of the V^{pq}) if and only if the corresponding morphisms $h\colon S \to \mathrm{GL}(V)$ are conjugate.*

Proof. If the two morphisms are conjugate by some $g \in \mathrm{GL}(V)$, then g induces an automorphism of $V_{\mathbb{C}}$ carries the V^{pq} for one Hodge decomposition onto the V^{pq} for the other, so that these have the same dimension. Conversely, if $\{V_1^{pq}\}$ and $\{V_2^{pq}\}$ are two Hodge structures on V with the same combinatorial data, we can find $g\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ which carries V_1^{pq} onto V_2^{pq} and which satisfies $g(\overline{v}) = \overline{g(v)}$. Since g commutes with complex conjugation, it descends to an automorphism of the real vector space V; this automorphism carries one h onto the other. □

In the situation of a family of Hodge structures which comes from family of smooth Kähler manifolds $X \to B$, the combinatorial data of those Hodge structures (i.e., the dimensions of the V^{pq}) is constant. Therefore, the period map out of B can be said to have values in a single conjugacy class of morphisms $h\colon S \to \mathrm{GL}(V)$.

Deligne's formalism of period domains generalizes this picture by replacing $\mathrm{GL}(V)$ by a general real algebraic group G. (In the following, I am going to confuse the algebraic group G with the Lie group $G(\mathbb{R})$ a little bit.) Let X be a conjugacy class of maps $h\colon S \to G$. We impose the following assumption:

$$h(\mathbb{R}^\times) \text{ lies in the center of } G, \text{ for all (equivalently, one) } h \in X.$$

If $h \in X$, then $X = G/K$, where $K \subset G$ is the stabilizer of h, a Lie subgroup of G. Thus X gets the structure of a smooth manifold. Our first order of business is to give X the structure of a complex manifold. To do this, it is necessary to give T_hX the structure of a complex vector space. We have $T_hX = \operatorname{Lie} G / \operatorname{Lie} K$.

2 The complex structure on $X = G/K$

The real vector space $L = \operatorname{Lie} G$ is a G-module through the adjoint action $\mathrm{ad}\colon G \to \mathrm{GL}(L)$–this is the derivative of the action of $G \to \operatorname{Aut} G$ on itself through conjugation. Composing $h\colon S \to G$ with ad, we get a Hodge structure on L. Note that since $h(\mathbb{R}^\times)$ lies in the center of G, $\mathrm{ad}\, h(\mathbb{R}^\times)$ is the identity on L, which is to say that L is homogeneous of weight 0. Let $L^{00} \subset L$ be the $(0,0)$-part of this Hodge structure; that is, L^{00} is the space of vectors fixed by every $h(z)$, $z \in S$.

Lemma 2.0.2. $L^{00} = \operatorname{Lie} K$.

Proof. Since K is by definition the stabilizer of h in G, we have $(\mathrm{ad}\, h(z))(k) = k$ for all $k \in K$. Differentiating, we find that $(\mathrm{ad}\, h(z))(v) = v$ for all $v \in \operatorname{Lie} K$, so that $\operatorname{Lie} K \subset L^{00}$. Conversely, if $v \in L^{00}$, then $(\mathrm{ad}\, h(z))(v) = v$;

exponentiating gives $(\operatorname{ad} h(z))(\exp v) = \exp v$, so that $\exp v \in K$, which means that $v \in \operatorname{Lie} K$. □

Lemma 2.0.3. *The natural inclusion $L \to L_{\mathbb{C}}$ induces an isomorphism $L/L^{00} \to L_{\mathbb{C}}/F^0 L_{\mathbb{C}}$.*

Proof. The image of L^{00} lands in the $(0,0)$-part of $L_{\mathbb{C}}$, which is certainly contained in $F^0 L_{\mathbb{C}}$. Thus $L/L^{00} \to L_{\mathbb{C}}/F^0 L_{\mathbb{C}}$ is well-defined.

If $v \in L$ lies in $F^0 L_{\mathbb{C}} = \bigoplus_{p \geq 0} L_{\mathbb{C}}^{pq}$, then $\overline{v} = v$ lies in $\overline{F^0 L_{\mathbb{C}}} = \bigoplus_{p \leq 0} L_{\mathbb{C}}^{pq}$. Thus v has to lie in the intersection of these two spaces, namely in the part where $p = 0$. But since L has weight 0, the index q has to be 0 as well. Thus $v \in L_{\mathbb{C}}^{00} \cap L = L^{00}$.

For surjectivity, let $v \in L_{\mathbb{C}}^{pq}$. At least one of v and $\overline{v}$ has to lie in $F^0 L_{\mathbb{C}}$ (again using the fact that L has weight 0), so that in the quotient $L_{\mathbb{C}}/F^0 L_{\mathbb{C}}$, v has the same class as $v + \overline{v}$, a real vector. □

The previous two lemmas show that $T_h X = L/L^{00} \cong L_{\mathbb{C}}/F^0 L_{\mathbb{C}}$ has a natural complex structure. This gives X the structure of an almost complex manifold. To show that X really is a complex manifold, we are going to embed X in a flag variety $\mathcal{F}$, which we already know is a complex manifold, and show that this embedding respects the almost complex structures on either side.

We choose a faithful representation $G \to \mathrm{GL}(V)$. Then for each $h \in X$ we get a Hodge structure on V. The combinatorial data of those Hodge structures doesn't depend on $h \in X$, though, since these are all conjugate. Let $\mathcal{F}$ be the variety of filtrations of $V_{\mathbb{C}}$ by subspaces of the appropriate dimension. Thus we get a map $\phi \colon X \to \mathcal{F}$. Since the Hodge filtration determines the Hodge structure, this map is injective. Now let's check it preserves complex structures at every $h \in X$. First, we'll give a convenient description of the tangent space to $\mathcal{F}$ at $\phi(h)$. The Hodge structure on V induces a Hodge structure on $\operatorname{End} V$, with respect to which

$$F^p \operatorname{End} V_{\mathbb{C}} = \left\{ f \in \operatorname{End} V_{\mathbb{C}} \;\middle|\; f(F^q V_{\mathbb{C}}) \subset F^{p+q} V_{\mathbb{C}},\ q \in \mathbb{Z} \right\}.$$

Lemma 2.0.4. $T_{\phi}(h)\mathcal{F} = \operatorname{End}(V_{\mathbb{C}})/F^0 \operatorname{End}(V_{\mathbb{C}})$.

Proof. The flag variety $\mathcal{F}$ is the closed subspace of the product $\prod_p \operatorname{Grass}(b^p, V_{\mathbb{C}})$ consisting of those sequences $(F^p V_{\mathbb{C}})_p$ which are decreasing: $F^p V_{\mathbb{C}} \supset F^{p+1} V_{\mathbb{C}}$. We have identified the tangent space to $\operatorname{Grass}(b^p, V_{\mathbb{C}})$ at $F^p V_{\mathbb{C}}$ with $\operatorname{Hom}(F^p V_{\mathbb{C}}, V_{\mathbb{C}}/F^p V_{\mathbb{C}})$.

It is not hard to show that the tangent space to $\mathcal{F}$ is the subset of the direct sum $\bigoplus_p \operatorname{Hom}(F^pV_{\mathbb{C}}, V_{\mathbb{C}}/F^pV_{\mathbb{C}})$ consisting of sequences (f_p) satisfying the compatibility condition

$$f_p|_{F^{p+1}V_{\mathbb{C}}} \equiv f_{p+1} \pmod{F^pV_{\mathbb{C}}}.$$

Now if $f \in \operatorname{End} V_{\mathbb{C}}$ is arbitrary, it gives rise to a sequence (f_p) by $f_p = f|_{F^pV_{\mathbb{C}}} \pmod{F^pV_{\mathbb{C}}}$; this is easily seen to satisfy the compatibility condition. The condition that $f_p = 0$ for all p is equivalent to the condition that $f(F^pV_{\mathbb{C}}) \subset F^pV_{\mathbb{C}}$ for all p, which is the same as the condition that $f \in F^0 \operatorname{End} V_{\mathbb{C}}$. Thus $f \mapsto (f_p)$ is injective. A dimension count shows it is surjective (exercise). □

The derivative of the action $G \to \mathrm{GL}(V)$ is a map of Lie algebras $\alpha\colon L \to \operatorname{End} V$. We have an adjoint action of G on $\operatorname{End} V$, and then $h\colon S \to G$ induces a Hodge structure on $\operatorname{End} V$. Then the map α preserves Hodge structures, and in particular it takes F^0 to F^0. Thus α induces a map

$$L_C/F^0L_{\mathbb{C}} \to \operatorname{End}(V_{\mathbb{C}})/F^0\operatorname{End}(V_{\mathbb{C}}).$$

This is actually the derivative of $\phi\colon X \to \mathcal{F}$ at h. It is now manifest that ϕ preserves almost complex structure. This shows that X really is a complex manifold. Note that the complex structure on X doesn't depend on the choice of V.

A nice consequence of this calculation is that whenever V is a representation of G, we get a family of Hodge structures on V parametrized by X which varies holomorphically.

3 Griffiths transversality

We can now state what is probably the most important theorem of this lecture:

Theorem 3.0.5. *The variation of Hodge structures on V satisfies Griffiths transversality if and only if the Hodge structure on L is of type* $\{(-1,1),(0,0),(1,-1)\}$.

The condition on L doesn't depend on V. Thus if one V satisfies Griffiths transversality, then they all must.

Proof. Recall the map $\alpha\colon L \to \operatorname{End} V$ (the derivative of $G \to \mathrm{GL}(V)$). We have seen that the derivative of $\phi\colon X \to \mathcal{F}$ at h is the map

$$\alpha\colon L_{\mathbb{C}}/F^0 L_{\mathbb{C}} \to \operatorname{End}(V_{\mathbb{C}})/F^0 \operatorname{End}(V_{\mathbb{C}})$$

induced from α. Griffiths transversality is the condition that, for indices p, a vector f in the image of this map should send $F^p V_{\mathbb{C}}$ into $F^{p-1}V_{\mathbb{C}}/F^p V_{\mathbb{C}} \subset V_{\mathbb{C}}/F^p V_{\mathbb{C}}$. But this is precisely the condition that $f \in F^{-1} \operatorname{End} V_{\mathbb{C}}$. Since α is injective, this means that $L_{\mathbb{C}} = F^{-1} L_{\mathbb{C}}$. Since L has weight 0, the only possible Hodge summands appearing in $L_{\mathbb{C}}$ are $(1, -1)$, $(0, 0)$ and $(-1, 1)$. □

4 Cartan involutions

The following is a digression on Cartan involutions which we'll need when we talk about polarizable variations of Hodge structure.

Let G be a real algebraic group, and let σ be an involution of G; that is, an automorphism of order 2. The real form G^σ is a real algebraic group whose $\mathbb{R}$-points are

$$G^\sigma(\mathbb{R}) = \left\{ g \in G(\mathbb{C}) \;\middle|\; \sigma(g) = \overline{g} \right\}.$$

Generally, if A is an $\mathbb{R}$-algebra, the A-points of G^σ are

$$G^\sigma(A) = \left\{ g \in G(A \otimes \mathbb{C}) \;\middle|\; \sigma(g) = \overline{g} \right\},$$

where $g \mapsto \overline{g}$ is interpreted as coming from the automorphism $1 \otimes z \mapsto 1 \otimes \overline{z}$ of $A \otimes \mathbb{C}$. Note that G and G^σ have the same base change to $\mathbb{C}$.

Example 4.0.6. For instance, let σ be the automorphism of $G = GL_n$ which sends g to $(g^t)^{-1}$. Then $G^\sigma = U(n)$, a compact group.

Example 4.0.7. For another example, let $G = O(n)$ and let $\sigma(g) = JgJ^{-1}$, where

$$J = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$$

has signature (r, s) $(r + s = n)$. I claim that $G^\sigma = O(r, s)$. Indeed, let

$$\Psi = \operatorname{diag}(1, \ldots, 1, i, \ldots, i),$$

so that $\Psi^2 = J$. We have

$$G^\sigma(\mathbb{R}) = \left\{ g \in G(\mathbb{C}) \;\middle|\; gg^t = I,\ \overline{g} = JgJ^{-1} \right\}.$$

The map $g \mapsto \Psi g \Phi^{-1}$ maps $G^\sigma(\mathbb{R})$ isomorphically onto the set of h which satisfy the conditions $hJh^t = J$ and $\overline{h} = h$, which is exactly $O(r, s)$. Note that $O(r, s)$ is only compact if $rs = 0$.

Note that a non-compact real group G can have a compact form G^σ. (Pedantic note: *compact* means that $G^\sigma(\mathbb{R})$ is compact, and that $G^\sigma(\mathbb{R})$ meets every connected component of $G^\sigma(\mathbb{C}) = G^\sigma(\mathbb{C})$.) We say that σ is a *Cartan involution* if G^σ is compact.

Theorem 4.0.8. *A connected group G is reductive if and only if it admits a Cartan involution. In that case, any two Cartan involutions are conjugate.*

(A *reductive* algebraic group is one whose unipotent radical is trivial.)

An obvious source of involutions on G arises when you consider elements $C \in G(\mathbb{R})$ for which C^2 lies in the center of G. Then $\sigma(g) := CgC^{-1}$ is an involution of G. Call a representation V of G *C-polarizable* if there exists a G-invariant bilinear form $V \times V \to \mathbb{R}$ such that $\Psi(x, Cy)$ is symmetric and positive definite. The following theorem gives a condition for σ to be a Cartan involution.

Theorem 4.0.9. *Let G be a real algebraic group, let $C \in G(\mathbb{R})$ be such that C^2 is central, and let $\sigma \in \operatorname{Aut} G$ be conjugation by C. The following are equivalent.*

1. *σ is a Cartan involution.*

2. *Every representation of G is C-polarizable.*

3. *G admits a faithful C-polarizable representation.*

Proof. We'll only sketch the proof that (1) and (3) are equivalent. First we observe that if K is a compact Lie group, and V is a complex representation of K, then V admits a K-invariant positive definite hermitian form H. Indeed, let H_0 be any positive definite hermitian form on V, and let

$$H(v, w) = \int_{k \in K} H_0(kv, kw)\,dk,$$

where dk is the Haar measure on K. Conversely, if K is a Lie group, and V is a faithful representation of K admitting a K-invariant symmetric positive definite form H, then K is compact, because it is a closed subset of $U(H)$, which is compact.

There is a version of this observation for real representations: if K is a compact real Lie group, and V is a real representation of K, then V admits a K-invariant positive definite symmetric form ψ. Conversely, if K is a Lie group, and V is a faithful representation of K admitting a K-invariant symmetric positive definite form, then K is compact.

Assume (3). Let V be a faithful representation of G equipped with a C-polarization Ψ. Then $\Psi_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is a symmetric bilinear form which is $G(\mathbb{C})$-invariant. Let

$$H(u, v) = \Psi_{\mathbb{C}}(u, \overline{v}),$$

so that H is a positive definite hermitian form. Then the hermitian form H is G-invariant in the sense that

$$H(gu, \overline{g}v) = H(u, v)$$

for $u, v \in V_{\mathbb{C}}$. We claim that

$$H^{\sigma}(u, v) = H(u, Cv)$$

is a $G^{\sigma}(\mathbb{R})$-invariant positive definite hermitian form on $V_{\mathbb{C}}$. First we check that H^{σ} is hermitian:

$$\overline{H^{\sigma}(u, v)} = \Psi_{\mathbb{C}}(\overline{u}, C\overline{v}) = \Psi_{\mathbb{C}}(\overline{v}, C\overline{u}) = H^{\sigma}(v, u).$$

Note that we used that $\Psi(u, Cv)$ was symmetric. Now we check $G^{\sigma}(\mathbb{R})$-invariance: for $g \in G^{\sigma}(\mathbb{R})$,

$$H^{\sigma}(gu, gv) = H(gu, Cgv) = H(gu, CgC^{-1}Cv) = H(gu, \overline{g}Cv) = H(u, Cv) = H^{\sigma}(u, v).$$

Finall, H^{σ} is positive definite, since for $u \in V$ nonzero we have

$$H^{\sigma}(u, u) = H(u, Cu) = \Psi(u, Cu) > 0.$$

Thus $G^{\sigma}(\mathbb{R})$ admits a faithful representation admitting an invariant positive definite hermitian form, and so must be compact, so that we get (1).

Now assume (1), so that $G^\sigma(\mathbb{R})$ is compact. Let V be any faithful (and real) representation of $G^\sigma(\mathbb{R})$; then there exists a $G^\sigma(\mathbb{R})$-invariant positive definite symmetric form ψ on V. Let $\Psi\colon V \times V \to \mathbb{R}$ be

$$\Psi(u, v) = \psi(u, C^{-1}v).$$

Then $\Psi(u, Cv) = \psi(u, v)$ is positive definite and $G(\mathbb{R})$-invariant (since it is real-valued and invariant by $G(\mathbb{C})$). □

5 Variations of polarizable Hodge structures

Let $h\colon S \to \mathrm{GL}(V)$ be a Hodge structure of weight k on a real vector space V, so that we get a decomposition $V_C = \bigoplus V^{pq}$. A *polarization* on V is a bilinear form $\Psi\colon V \times V \to \mathbb{R}$ which turns V into a polarized Hodge structure. Recall that this means that Ψ is alternating or symmetric as k is odd or even, and if we let $H\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{R}$ be the associated Hermitian form

$$H(\alpha, \beta) = i^k \Psi(\alpha, \overline{\beta}),$$

then the V^{pq} are orthogonal with respect to H, and H has sign i^{p-q-k} when restricted to V^{pq}. (In earlier lectures there was an additional factor of $(-1)^{k(k-1)/2}$, in accordance with Voisin's book. Since k is fixed throughout the discussion, we can ignore this difference in sign.)

In terms of h, this means the following.

Lemma 5.0.10. *A bilinear form $\Psi\colon V \times V \to \mathbb{R}$ is a polarization if and only if it satisfies the following two conditions:*

$$\Psi(h(z)v, h(z)w) = |z|^{-k}\, \Psi(v, w),$$

for all $v, w \in V$, $z \in S(\mathbb{R}) = \mathbb{C}^\times$, and

$$\Psi(v, h(i)w) \text{ is symmetric and positive definite}$$

This elegant characterization is a small advertisement for the "h" point of view. If $\mathbb{R}(n)$ denotes the vector space $\mathbb{R}$ together with the Hodge structure $z \mapsto |z|^n$, then in the context of the lemma we have a morphism of Hodge structures $\Psi\colon V \otimes V \to \mathbb{R}(-k)$ (it commutes with the hs).

Proof. We'll do the "if" direction and leave the converse as an exercise. Since V has weight k, $h(-1) = (-1)^k I_V$. The symmetry of $\Psi(v, h(i)w)$ means that

$$\Psi(w,v) = (-1)^k \Psi(w, h(i)^2 v) = (-1)^k \Psi(h(i)v, h(i)w) = (-1)^k \Psi(v,w),$$

so that Ψ is alternating or symmetric as k is odd or even. Now let $H(v,w) = i^k \Psi(v, \overline{w})$ as usual. We claim that the V^{pq} are orthogonal with respect to H. Let $v \in V^{pq}$ and $w \in V^{rs}$. We have

$$\begin{aligned} |z|^{-k} H(v,w) &= H(h(z)v, h(z)w) \\ &= H(z^{-p}\overline{z}^{-q}, z^{-r}\overline{z}^{-s}w) \\ &= \Psi(z^{-p}\overline{z}^{-q}, z^{-s}\overline{z}^{-r}\overline{w}) \\ &= z^{-p-s}\overline{z}^{-q-r} H(v,w) \end{aligned}$$

This means that $H(v,w) = 0$ unless $r = p$ and $s = q$. Thus the V^{pq} are orthogonal with respect to H.

The condition that $\Psi(v, h(i)w)$ be positive definite means that $\Psi(v, h(i)\overline{w}) = i^{-k} H(v, h(i)w)$ is a positive definite hermitian form. For $v \in V^{pq}$ nonzero we have

$$0 < i^{-k} H(v, h(i)v) = i^{p-q-k} H(v,v),$$

and therefore $H(v,v)$ has the correct sign. □

We return to Deligne's setting. Let G be a real algebraic group, and let X be the conjugacy class of maps $S \to G$. Keep the assumption that each $h(\mathbb{R}^\times)$ be central.

Let V be a faithful representation of G, so that each point of X gives a Hodge structure on V. Call V *polarizable* if there exists a bilinear form $\Psi \colon V \times V \to \mathbb{R}$, such that each $h \in X$ determines a Hodge structure on V which is polarized with respect to Ψ.

A little explanation is in order. V might not be homogeneous of a particular weight, but (because of our assumption on $h(R^\times)$) it does break up into G-invariant summands V_k, which are homogenous of weight k. Polarizability now means that for each k, V_k admits a bilinear form (symmetric or alternating, as k is even or odd), such that each $h \in X$ determines a polarized Hodge structure on V_k.

Remarkably, the condition that V be polarizable is intrinsic to X. That is, if one V is polarizable, then they all are. This follows from the following theorem.

Theorem 5.0.11. *Let G_1 be the smallest subgroup of G through which all the $h \in X$ factor. V is polarizable if and only if the following conditions hold: G_1 is reductive, and for some $h \in X$ (equivalently, all of them), conjugation by $h(i)$ is a Cartan involution on the adjoint group G_1^{ad}.*

We remark that if G is connected then its adjoint group is simply what you get when you mod out by the center.

Let's examine the criterion that V be polarizable. In what follows, we assume that V is homogeneous of weight k. Let $\Psi\colon V \times V \to \mathbb{R}$ be a polarization. It will be useful to introduce the subgroup $G_2 \subset G_1$, defined as the smallest subgroup containing $h(U^1)$ for each $h \in X$ (where $U^1 \subset \mathbb{C}^\times$ is the subgroup of norm 1 elements). For all $z \in U^1$, and all $v \in V^{pq}$, we have (by the last lemma) $\Psi(h(z)v, h(z)w) = \Psi(v, w)$, so that Ψ is G_2-invariant. We also have that $\Psi(v, h(i)w)$ is symmetric and positive definite for all $h \in X$. This means that V is a representation of G_2 admitting an $h(i)$-polarization. From this we get that $h(i)$ is a Cartan involution on G_2.

For its part, the group G_1 is generated by G_2 together with the elements $h(t)$, where $h \in X$ and $t \in \mathbb{R}^\times$. The latter elements are central in G. From here one can show that G_1 and G_2 have the same adjoint group. This gives one direction of the theorem. For the other, suppose that conjugation by $h(i)$ is a Cartan involution on $G_1^{\mathrm{ad}} = G_2^{\mathrm{ad}}$. Since G_2 is generated by compact subgroups, its center is compact (?), so that an involution on G_2 is Cartan if and only if it is Cartan on G_2^{ad}. Thus conjugation by $h(i)$ is Cartan on G_2. Let V be a faithful representation of G, and let Ψ be an $h(i)$-polarization on V with respect to G_2. By definition, Ψ is G_2-invariant. This means that V is a polarizable.

6 Hermitian symmetric domains

We are thus led to study pairs (G, X), where G is a real algebraic group and X is a conjugacy class of nontrivial morphisms $h\colon S \to G$, which satisfy the following conditions. (If an $h \in X$ satisfies one of these conditions, then they all do.)

1. $h(\mathbb{R}^\times)$ is central.
2. The Hodge structure on $\operatorname{Lie} G$ induced by $\operatorname{ad} \circ h$ is of type $\{(1,-1),(0,0),(-1,1)\}$.
3. Conjugation by $h(i)$ is a Cartan involution on the adjoint group of G_1.

(Actually, Deligne wants to consider only connected X, in which case X is a conjugacy class under the neutral component of $G(\mathbb{R})$.)

Let's first examine the case that $G = G_1$ is a simple adjoint group. Let $h \in X$. Let $K \subset G(\mathbb{R})$ be the centralizer of X, so that $X = G(\mathbb{R})/K$ (or a connected component thereof). Let $\sigma = \operatorname{ad} h(i)$, so that G^σ is compact. We have $K \subset G^\sigma$ is a closed subgroup, so that K is compact as well. We can give X a $G(\mathbb{R})$-invariant Riemannian metric, by putting a K-invariant symmetric positive definite form on $T_h X$, and using G to translate this to all of X.

The tangent space to X at h is $\operatorname{Lie}(G)/\operatorname{Lie}(G)^{00}$, and $\operatorname{ad} h(i)$ acts on it by -1. This means that for all $h \in X$ there exists an involution σ_h of X (namely, conjugation by $h(i)$) which has p as an isolated fixed point, and whose derivative at p is the scalar -1. It can be shown that σ_h is an isometry of X.

On the other hand, X is a complex manifold. The complex and Riemannian structures combine to give X the structure of a hermitian manifold, which happens to have an involutive symmetry at each point.

Definition 6.0.12. A *hermitian symmetric space* is a hermitian manifold X with these properties:

1. The automorphism group of X acts transitively.
2. At each $p \in X$, there exists an involution s_p of X which has p as an isolated fixed point.

X is a *hermitian symmetric domain* if in addition it is noncompact.

Theorem 6.0.13. *The manifolds X constructed from pairs (G, X) as above are exactly the hermitian symmetric domains.*

7 Classification of hermitian symmetric domains in terms of simple complex algebraic groups

We would like to classify the pairs (G, X), where G is a real algebraic group and X is a conjugacy class of maps $h\colon S \to G$ satisfying the conditions (1)–(3) above. In our classification, we're going to make two simplifying assumptions on G:

- G is simple (no connected normal subgroups other than 1 and G). Eg, $\mathrm{SL}(n)$.
- G is adjoint (the adjoint map $\mathrm{ad}\colon G \to \mathrm{GL}(\mathrm{Lie}\, G)$ is a faithful representation of G.

Examples of simple adjoint groups: $\mathrm{PSL}(n) = \mathrm{SL}(n)/\{\pm I\}$, $\mathrm{PSp}(n) = \mathrm{Sp}(n)/\{\pm I\}$, $\mathrm{PSU}(p,q)$ (the special unitary group $\mathrm{SU}(p,q)$ modulo its center), $SO(p,q)/\{pmI\}$.

Now the conditions on $h\colon S \to X$ can be restated this way:

1. With respect to h, the adjoint representation $\mathrm{Lie}\, G$ is of type $\{(1,-1),(0,0),(-1,1)\}$.
2. Conjugation by $h(i)$ is a Cartan involution of G.
3. h is nontrivial.

Condition (1) above means that $\mathrm{ad} \circ h(\mathbb{R}^\times)$ acts trivially on $\mathrm{Lie}\, G$. Since ad is a faithful representation, this means that $h(\mathbb{R}^\times) = 1$. Also, given (1) and (2), the condition (3) is equivalent to saying that G is noncompact. Indeed, if h is trivial, then by (2) we have that the identity is a Cartan involution, which means that G is compact. Conversely, if G is compact, then the uniqueness of the Cartan involution shows that $\mathrm{ad} \circ h(i)$ is the identity. But by (1), $\mathrm{ad} \circ h(i)$ acts on $\mathrm{Lie}\, G/(\mathrm{Lie}\, G)^{00}$ by -1. This shows that $\mathrm{Lie}\, G = (\mathrm{Lie}\, G)^{00}$, and therefore that (for all z) $\mathrm{ad}\, h(z)$ is the identity. Since ad is faithful, h is trivial.

We will now classify pairs (G,X), where G is a simple adjoint group over $\mathbb{R}$ and X is a conjugacy class of h satisfying (1)–(3) above. Of course, it's easier to classify objects over $\mathbb{C}$ than over $\mathbb{R}$. So let's examine what happens when we base change everything to $\mathbb{C}$.

Let $\mathbb{G}_{\mathrm{m}}$ be the multiplicative group, considered as an algebraic group over $\mathbb{C}$. We have an isomorphism of complex algebraic groups $S_{\mathbb{C}} \to \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$ which sends (a,b) to $(a+bi, a-bi)$. Let $\mathbb{G}_{\mathrm{m}} \to S_{\mathbb{C}}$ be the inclusion onto the first coordinate. If $h\colon S \to G$ is a morphism, let $\mu_h\colon \mathbb{G}_{\mathrm{m}} \to G_{\mathbb{C}}$ be the composite

$$\mu_h\colon \mathbb{G}_{\mathrm{m}} \to S_{\mathbb{C}} \to^h G_{\mathbb{C}}.$$

Then the $G(\mathbb{C})$-conjugacy class of μ_h only depends on the $G(\mathbb{R})$-conjugacy class of h. If V is a (real) representation of G, so that h determines a Hodge structure on V, then μ_h is characterized by

$$\mu_h(z)v = z^p v,\ z \in \mathbb{C}^\times,\ v \in V^{pq} \subset V_{\mathbb{C}}$$

The characters of the algebraic group $\mathbb{G}_{\mathrm{m}}$ are the power maps $z \mapsto z^n$, for $n \in \mathbb{Z}$. Any representation $\mathbb{G}_{\mathrm{m}} \to \mathrm{GL}(V)$ (V a finite-dimensional $\mathbb{C}$-vector space) breaks up to characters of this type.

Lemma 7.0.14. *Assume that $h\colon \mathbb{G}_m \to G$ satisfies property (1) (that $h(\mathbb{R})$ is central in G). The morphism h satisfies property (2) above (that $\operatorname{Lie} G$ is of type $\{(1,-1),(0,0),(-1,1)\}$ with respect to the Hodge structure induced by $\operatorname{ad}\circ h$) if and only if representation $\operatorname{ad}\circ\mu_h$ of $\mathbb{G}_m$ on $\operatorname{Lie} G_{\mathbb{C}}$ contains only characters of type $z, 1, z^{-1}$.*

Proof. $\operatorname{Lie} G_{\mathbb{C}}$ is a direct sum of spaces $(\operatorname{Lie} G_{\mathbb{C}})^{pq}$, where $p+q=0$ (condition (1) guarantees that the Hodge structure on $\operatorname{Lie} G$ has weight 0). On $(\operatorname{Lie} G_{\mathbb{C}})^{pq}$, $\mu_h(z)$ acts by z^p. Thus p takes only values $1, 0, -1$ if and only if (p,q) takes the values $(1,-1),(0,0),(-1,1)$. □

The following theorem tells how to recover the pair (G, X) from its "complexification".

Theorem 7.0.15. *Let H be a simple complex adjoint algebraic group. There is a bijection between the following two sets:*

- *Pairs (G, X), where G is a real form of H, and X is a conjugacy class of maps $h\colon S \to G$ satisfying (1)–(3),*
- *$H(\mathbb{C})$-conjugacy classes of nontrivial maps $\mu\colon \mathbb{G}_m \to H$ such that only $z, 1, z^{-1}$ appear in $\operatorname{Lie} H$.*

Proof. By the lemma, the complexification of a (G, X) satisfying (1)–(3) gives such a μ. Now suppose μ is given. Since H is simple, it is reductive and therefore admits a compact real form G^*. It is not hard to see that $h \mapsto \mu_h$ is a bijection between $G^*(\mathbb{R})$-conjugacy classes of $h\colon S \to G^*$ (which are trivial on $\mathbb{R}^\times$ and $H(\mathbb{C})$-conjugacy classes of $\mu\colon \mathbb{G}_{\mathrm{m}} \to G^*_{\mathbb{C}} = H$. Thus we have a map $h\colon S \to G^*$, defined over $\mathbb{R}$.

Let G be the real form of G^* corresponding to the involution $g \mapsto h(i)gh(i)^{-1}$. Then conjugation by $h(i)$ is a Cartan involution on G. Via the lemma, the condition on μ translates into the required condition on h. □

If G is a simple adjoint algebraic group over $\mathbb{C}$, what are the $\mu\colon \mathbb{G}_{\mathrm{m}} \to G$ which satisfy the condition in Thm. 7.0.15?

Let's recall some basics from the theory of simple complex Lie groups. Let $\mathfrak{g} = \operatorname{Lie} G$, a simple complex Lie algebra. There is a nondegenerate bilinear form (the Killing form) $\mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$, defined by $(x, y) = \operatorname{tr}((\operatorname{ad} x)(\operatorname{ad} y))$, where $\operatorname{ad} x\colon \mathfrak{g} \to \mathfrak{g}$ is $y \mapsto [x, y]$. (In fact, B being nondegenerate is equivalent to $\mathfrak{g}$ being semisimple–this is Cartan's criterion.)

Let T be a maximal torus of G. Let

$$\begin{aligned} X(T) &= \operatorname{Hom}(T, \mathbb{G}_{\mathrm{m}}) \\ Y(T) &= \operatorname{Hom}(\mathbb{G}_{\mathrm{m}}, T) \end{aligned}$$

be the character and cocharacter groups of T. These are written additively. Then $X(T)$ and $Y(T)$ are dual under the pairing $X(T) \times Y(T) \to \operatorname{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) = \mathbb{Z}$. The Lie algebra $\mathfrak{g}$ admits an action of T via ad, and must therefore break up into characters of T. The nontrivial characters $\alpha \in X(T)$ which appear in $\mathfrak{g}$ are the *roots* of G. Let Φ be the set of roots. We have the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the α-eigenspace:

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; (\operatorname{ad} t)(x) = \alpha(t)x,\ t \in T \right\}.$$

The restriction of the Killing form to $\mathfrak{t}$ is nondegenerate, so that it supplies an isomorphism between $\mathfrak{t}$ and $\mathfrak{t}^\vee$, its $\mathbb{C}$-linear dual.

We have a map $X(T) = \operatorname{Hom}(T, \mathbb{G}_{\mathrm{m}}) \to \operatorname{Hom}_{\mathbb{C}}(\mathfrak{t}, \operatorname{Lie}\mathbb{G}_{\mathrm{m}}) = \mathfrak{t}^\vee$, so we can think of Φ as a subset of $\mathfrak{t}^\vee$. From this point of view,

$$\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [H, x] = \alpha(H)x,\ H \in T \right\}.$$

We claim that Φ spans $\mathfrak{t}^*$ as a complex vector space. Otherwise there would be a nonzero $H \in \mathfrak{t}$ for which $\alpha(H) = 0$ for all roots α. But then $\operatorname{ad} H$ kills $\mathfrak{g}_\alpha$ for all α, so that $\operatorname{ad} H$ kills all of $\mathfrak{g}$, and then we would have $H = 0$.

Let $V = \mathfrak{t}^\vee \otimes \mathbb{R}$. It so happens that the Killing form restricts to a positive definite form on V. The roots Φ span V, and have the following properties:

1. The only scalar multiples of $\alpha \in \Phi$ within Φ are $\pm\Phi$.

2. Φ is invariant under the reflections s_α, defined by

$$s_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

3. If $\alpha, \beta \in \Phi$, then $2(\alpha, \beta)/(\alpha, \alpha)$ is an integer. (The projection of α onto the line through β is a half-integer multiple of β.)

This means that Φ is a *root system*.

It is possible to choose a set of positive roots $\Phi^+ \subset \Phi$ which contains exactly one of $\alpha, -\alpha$ for each $\alpha \in \Phi$, and which has the property that for all $\alpha, \beta \in \Phi^+$ for which $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Phi^+$. (The choice of Φ^+ is equivalent to the choice of Borel subgroup of G containing T. The Weyl group of T acts transitively on the set of such Borels.)

Within Φ^+ we have the set of simple roots B: these are the roots which are not the sum of two other positive roots. Every positive root can be written uniquely as $\sum_{\alpha \in B} m_\alpha \alpha$, where $m_\alpha \geq 0$. V is spanned by B.

The *Dynkin diagram* of the root system has one vertex for each simple root α. Two roots are joined by $\{0, 1, 2, 3\}$ edges, as the angle between α and β is $\{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$. In the latter two cases, an arrow is drawn which points to the shorter vector.

The *Weyl chamber* corresponding to B is

$$\left\{ v \in V \;\middle|\; (\alpha, v) \geq 0 \;\alpha \in B \right\};$$

this is an intersection of half-planes in V. The Weyl group permutes the Weyl chambers simply transitively.

The last fact we need concerns the *highest root*:

Theorem 7.0.16. *There exists a unique root $\alpha_+ = \sum_{\alpha \in B} n_\alpha \alpha$, such that if $\sum_\alpha m_\alpha \alpha$ is any other root, then $n_\alpha \geq m_\alpha$ for all α. (In particular $n_\alpha \geq 1$ for all α.*

Theorem 7.0.17. *Conjugacy classes of nontrivial minuscule $\mu \colon \mathbb{G}_m \to G$ are in correspondence with simple roots α with $n_\alpha = 1$.*

If $\mu \colon \mathbb{G}_\mathrm{m} \to G$ is a cocharacter, then the image of μ is contained in some maximal torus; after replacing μ with a conjugate we assume that $\mu \colon \mathbb{G}_\mathrm{m} \to T$ is actually a cocharacter: $\mu \in Y(T)$. Finally, we can translate μ by an element of the Weyl group to assume that

$$(\alpha, \mu) \geq 0, \; \alpha \in B$$

The condition that only the characters $z, 1, z^{-1}$ appear in $\operatorname{Lie} G$ is equivalent to the condition that $(\alpha, \mu) \in \{1, 0, -1\}$ for all roots $\alpha \in \Phi$. Combining this with the above, we find that $(\alpha, \mu) \in \{0, 1\}$ for all $\alpha \in B$. Assuming that μ is nontrivial, we have that $(\alpha_0, \mu) = 1$ for at least one $\alpha_0 \in B$. We have $(\alpha_+, \mu) = \sum_\alpha n_\alpha(\alpha, \mu) = 1$, so that $n_{\alpha_0} = 1$. Note that this forces $(\alpha, \mu) = 0$ for all other simple roots $\alpha \neq \alpha_0$.

Conversely, let $\alpha_0 \in B$ be a simple root with $n_{\alpha_0} = 1$. Let $\mu \in Y(T)$ be defined by the conditions $(\alpha_0, \mu) = 1$ and $(\alpha, \mu) = 0$ for all other simple roots α. We claim that $(\beta, \mu) \in \{1, 0, -1\}$ for all roots β. It is enough to assume that $\beta \in \Phi^+$, and then β is a sum of simple roots: $\beta = \sum_\alpha m_\alpha \alpha$, with $m_\alpha \geq 0$. Then $(\beta, \mu) = m_\alpha \leq n_\alpha = 1$, so that $m_\alpha \in \{0, 1\}$.

The nodes of the Dynkin diagram corresponding to simple roots α with $n_\alpha = 1$ are called *special nodes*.

8 Examples

8.1 The A_n root system

Let $V \subset \mathbb{R}^{n+1}$ be the subspace of vectors whose coordinates sum to 0. The A_n root system is $\Phi = \left\{ v \in V \cap \mathbb{Z}^{n+1} \;\middle|\; |v| = \sqrt{2} \right\}$. For the simple roots we can take $\alpha_i = e_i - e_{i+1}$, $i = 1, \ldots, n$. The positive roots take the form

$$e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1},$$

for $j > i$. Thus the highest root is

$$\alpha_+ = \alpha_1 + \cdots + \alpha_n = e_1 - e_{n+1}.$$

We see that every node of A_n is special. Let $1 \leq p \leq n$.

A_n is the root system corresponding to the simple group PGL_n. If μ is the cocharacter corresponding to the (special) node α_p, so that

$$\mu(z) = \mathrm{diag}(1, 1, \ldots, 1, z, z, \ldots, z) \text{ with } p \text{ 1s},$$

then the real form corresponding to μ is $G = \mathrm{PU}(p, q)$, and the corresponding hermitian symmetric domain is $U(p, q)/U(p) \times U(q)$.

8.2 The B_n root system

The B_n root system is $\Phi = \left\{ v \in \mathbb{Z}^n \;\middle|\; |v| = 1 \text{ or } \sqrt{2} \right\}$. For the simple roots we can take $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_n = e_n$. Then the highest root is

$$\alpha_+ = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n.$$

Thus there is only one special node, namely the node corresponding to α_1.

B_n is the root system corresponding to the simple group $SO(2n+1)$. The cocharacter corresponding to α_1 is

$$\mu(z) = \begin{pmatrix} (z+z^{-1})/2 & (z-z^{-1})/2i \\ -(z-z^{-1})/2i & (z+z^{-1})/2 \end{pmatrix} \bigoplus I_{2n-1}.$$

The real form corresponding to μ is $G = SO(2, 2n-1)$, and the corresponding hermitian symmetric domain is $O(2, 2n-1)/O(2) \times O(2n-1)$.

In fact, the only root systems whose Dynkin diagrams admit special nodes are those of type A_n, B_n, C_n, D_n, E_6 and E_7. The remaining root systems (those of type F_4, E_8 and G_2) do not have associated hermitian symmetric domains (and therefore no Shimura varieties).