The Baily-Borel compactification

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Let G be a simple algebraic group defined over $\mathbb{Q}$, and suppose that $X = G(\mathbb{R})/K$ is a hermitian symmetric domain. (Note that this already places restrictions on $G_{\mathbb{R}}$–the simple factors of its Lie algebra have to be of type A_n, B_n, C_n, D_n, E_6 or E_7.) For instance, G could be SL_2 and X could be $\mathcal{H}$, the upper half-plane. Or, G could be Sp_{2g} and X could be $\mathcal{H}_g$, the Siegel upper half-space. Or, G could be $SO(2, n-2)$ and X could be $G(\mathbb{R})/S(O(2) \times O(n-2))$. In those cases, $G_{\mathbb{R}}$ was simple, but we do not want to exclude situations where $G_{\mathbb{Q}}$ is simple but $G_{\mathbb{R}}$ is not.

We can try to give meaning to the expression $G(\mathbb{Z})$. For instance, one can always embed G into GL_n (as algebraic groups over $\mathbb{Q}$), and then set $G(\mathbb{Z}) = \mathrm{GL}_n(\mathbb{Z}) \cap G(\mathbb{Q})$. Call a subgroup $\Gamma \subset G(\mathbb{Q})$ *arithmetic* if the intersection $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$. The theorem of Baily-Borel states that for an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$, the quotient $Y = \Gamma \backslash X$ is a quasi-projective variety. This is proved by constructing a compactification $\overline{Y}$ (as a complex analytic space) and an isomorphism of $\overline{Y}$ with a closed subset of some projective space. By GAGA, $\overline{Y}$ is a projective variety, and therefore Y is quasi-projective.

1 Basic examples: Modular curves and Hilbert modular surfaces

1.1 Modular curves

Suppose $G = \mathrm{SL}_2 / \mathbb{Q}$. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ be an arithmetic subgroup. Then $Y = \Gamma \backslash \mathcal{H}$ is a can be compactified by adding finitely many cusps. Recall that a *cusp* for Γ is a coset in $\Gamma \backslash \mathbf{P}^1$. Let $\overline{Y}$ be the topological space obtained by adding the cusps of Γ to Y. This is given the following structure of a complex manifold by declaring that a basis of neighborhoods of the cusp ∞ is $\left\{ z \in \mathcal{H} \;\middle|\; \mathrm{Im} z > B \right\} \cup \{\infty\}$ (for $B > 0$); for large enough B this set is in bijection with an open subset of $\mathbb{C}$ via $z \mapsto e^{2\pi i/N}$, where N is the index of $\Gamma \cap \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}$ in $\begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}$. Then $\overline{Y}$ is a compact Riemann surface, and therefore a projective curve.

Already we can see a connection between cusps and parabolic subgroups of G. A *parabolic subgroup* P of an algebraic group G is one for which G/P is a projective variety; in the case of $G = \mathrm{SL}_2 / \mathbb{Q}$, the parabolic subgroups are all conjugate to the subgroup P of upper-triangular matrices, and $G(\mathbb{Q})/P(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q})$ is the set of all parabolic subgroups of $G(\mathbb{Q})$.

Consider the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (and brush aside issues arising from the elliptic fixed points). We know that $\overline{Y} = \mathbb{P}^1$ via the j-invariant, but we can also see this using modular forms. The ring of modular forms of Γ is a graded ring isomorphic to $\mathbb{C}[x, y]$ (where x has weight 4 and y has weight 6). Then we can form $\mathrm{Proj}\mathbb{C}[x, y]$, which is $\mathbb{P}^1$. In other words: for each even k, we get a map $Y \to \mathbb{C}^N$ by $z \mapsto [f_0(z) : \cdots : f_N(z)]$, where $f_0, \ldots, f_N$ is a basis for the space of weight k forms. For $k = 12$ this morphism separates points and tangent directions: $z \mapsto [E_4(z) : E_4(z)^3 - E_6(z)^2] = j(z)$ (modulo some constants). This is going to be the pattern in general: if we can find enough automorphic forms, we can find an embedding in projective space.

1.2 Hilbert modular surfaces

Now suppose $F = \mathbb{Q}(\sqrt{d})$ is a real quadratic field, with real embeddings i_1 and i_2. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}_2)$, so that $G(\mathbb{R}) = \mathrm{SL}_2 \times \mathrm{SL}_2$; the corresponding hermitian symmetric space is $X = \mathcal{H} \times \mathcal{H}$. This has an action of $G(\mathbb{Q}) = \mathrm{SL}_2(\mathcal{O}_F)$, by $\gamma(z_1, z_2) = (i_1(\gamma)z, o_2(\gamma)z_2)$. For now let's consider the case of the arithmetic subgroup $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$. Let $Y = \Gamma \backslash X$. It turns out that $\Gamma \backslash Y$ can be given a moduli interpretation in terms of polarized abelian varieties endowed with an action of an order in F.

Indeed, for $z = (z_1, z_2) \in X$, put

$$\Lambda_z = \left\{ i_1(\alpha)z_1 + i_1(\beta), i_2(\alpha)z_2 + i_2(\beta) \;\middle|\; (\alpha, \beta) \in \mathcal{O}_F \times \mathcal{O}_F \right\},$$

so that $\Lambda_z \approx \mathcal{O}_F \times \mathcal{O}_F$ is a subgroup of $\mathbb{C}^2$ of rank 4. In fact it is a lattice in $\mathbb{C}^2$, because $\Lambda_z \otimes \mathbb{R} = (\mathbb{R}z_1 + \mathbb{R}) \times (\mathbb{R}z_2 + \mathbb{R}) = \mathbb{C}^2$ (since $z_1, z_2 \in \mathcal{H}$). Let $A_z = \mathbb{C}^2/\Lambda_z$. For $r \in F$, define

$$E_r \colon \mathcal{O}_F^2 \times \mathcal{O}_F^2 \to \mathbb{Q}$$

by

$$E_r((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \operatorname{tr}_{F/\mathbb{Q}}(r(\alpha_1\beta_2 - \alpha_2\beta_1))$$

Then E_r is $\mathbb{Q}$-bilinear alternating, and has integer values if r belongs to $D_{F/\mathbb{Q}}^{-1}$ (the inverse to the different ideal). We have $E_r(\gamma \cdot \alpha, \beta) = E_r(\alpha, \gamma \cdot \beta)$ for all $\gamma \in \mathcal{O}_L$, where $\gamma \cdot (\alpha_1, \alpha_2) = (\gamma\alpha_1, \gamma^\sigma\alpha_2)$. Consider E_r as a form on Γ_z, and extend $\mathbb{R}$-linearly to get an alternating form $E_{r,z}$ on the real vector space $\mathbb{C}^2$.

If we let $H_{r,z}$ be the hermitian form on $\mathbb{C}^2$ defined by

$$H_{r,z}((x_1, x_2), (y_1, y_2)) = \frac{x_1\overline{y}_1 i_1(r)}{\operatorname{im} z_1} + \frac{x_2\overline{y}_2 i_2(r)}{\operatorname{im} z_2},$$

then $H_{r,z}$ is positive definite when r is totally positive. We claim $\operatorname{im} H_{r,z} = E_{r,z}$. It is enough to check this on Λ_z, and since E_z is alternating, it suffices to check this on pairs of vectors where the first one is in $\mathcal{O}_F \cdot (z_1, z_2)$ and the second is in $\mathcal{O}_F \cdot (1, 1)$:

$$\begin{aligned}
\operatorname{im} H((i_1(\alpha)z_1, i_2(\alpha)z_2), (i_1(\beta), i_2(\beta))) &= \operatorname{im} \frac{i_1(\alpha)z_1\overline{i_1(\beta)}i_1(r)}{\operatorname{im}(z_1)} + \dots \\
&= i_1(\alpha)i_1(\beta)i_1(r) + \dots \\
&= \operatorname{tr}_{F/\mathbb{Q}} \alpha\beta r \\
&= E((\alpha, 0), (0, \beta)) \\
&= E_z(\alpha \cdot (z_1, z_2), \beta \cdot (1, 1))
\end{aligned}$$

We conclude that A_z is an abelian variety, since (for r totally positive in $D_{F/\mathbb{Q}}^{-1}$) it is polarized by $H_{r,z}$. The form $H_{r,z}$ corresponds to a polarization $\lambda_{r,z} \colon A_z \to A_z^\vee$ of degree $N(rD_{F/\mathbb{Q}})^2$. If we choose a particular r with $(r) = D_{F/\mathbb{Q}}^{-1}$, then $\lambda_{r,z}$ is prinicipal. The abelian variety A_z comes equipped with an action of $\mathcal{O}_F$, and the polarization is $\lambda_{r,z}$ is $\mathcal{O}_F$-linear.

Theorem 1.2.1. *$\Gamma\backslash X$ is in bijection with the set of triples (A, ι, λ), where $A/\mathbb{C}$ is an abelian variety, $\iota \colon \mathcal{O}_F \to \operatorname{End} A$ is a homomorphism, and $\lambda \colon A \to A^\vee$ is an$\mathcal{O}_F$-linear principal polarization.*

To compactify $Y = \Gamma\backslash X$, we need to add a cusp for each Γ-orbit of rational parabolic subgroups of $G/\mathbb{Q}$. As with the modular curve case, the set of such subgroups is a projective space, $\mathbb{P}^1(F)$. Thus the cusps of Y are in bijection with Γ-orbits on $\mathbb{P}^1(F)$. In fact the quotient set is the class group of F. Proof: given a ratio $[\alpha : \beta]$ in $\mathbb{P}^1(F)$, associate the fractional ideal $\alpha\mathcal{O}_F + \beta\mathcal{O}_F$. This is well-defined because if you scale α and β by the same $\gamma \in F^\times$, then the ideal changes by a principal ideal, and clearly translating by $\mathrm{SL}_2(\mathcal{O}_F)$ doesn't change anything. It is surjective because every ideal is generated by two elements.

For injectivity: say

$$\alpha\mathcal{O}_F + \beta\mathcal{O}_F = \alpha'\mathcal{O}_F + \beta'\mathcal{O}_F$$

in the class group of F. After replacing α, β by a scalar multiple, we may assume that this is an equality of ideals. Let I^{-1} be the common ideal, so that

$$\mathcal{O}_F = \alpha I + \beta I = \alpha' I + \beta' I.$$

Let a, b, a', b' be such that

$$1 = \alpha a + \beta b = \alpha' a' + \beta' b'.$$

Let

$$M = \begin{pmatrix} \alpha & -b \\ \beta & a \end{pmatrix}, \; N = \begin{pmatrix} \alpha' & -b' \\ \beta' & a' \end{pmatrix},$$

so that $\det M = \det N = 1$. We have $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $N\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$, so that $NM^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha'\beta'$.

Let $\overline{Y}$ be the disjoint union of Y and $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{P}^1(F)$. Turn $\overline{Y}$ into a topological space by declaring a system of neighborhoods of $(i\infty, i\infty)$ to be

$$\left\{ (z_1, z_2) \in \mathcal{H} \times \mathcal{H} \;\middle|\; \operatorname{im} z_i > r \right\}, \; r \in \mathbb{R}_{>0}$$

Acting on these by elements of $\mathrm{SL}_2(F)$, we get neighborhoods for all the cusps. At this point one gives $\overline{Y}$ the structure of a complex analytic space (but it is quite complicated). It turns out that unlike the case of modular curves, the cusps are actually quite singular.

Hirzebruch constructed a desingularization of $\overline{Y}$, whose geometry involves the continued fraction expansion of $\sqrt{d}$. Using this he was able to prove:

Theorem 1.2.2. *Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$. Assume that $h(\mathbb{Q}(\sqrt{p})) = 1$. Write $\sqrt{p} = [a_0, \overline{a_1, \ldots, a_s}]$ with s minimal. Then*

$$h(\mathbb{Q}(\sqrt{-p}) = \frac{1}{3}\sum_{j=1}^{s}(-1)^j a_j.$$