A comparison thm for ordinary p-adic modular forms

· jt work in progress w G. Mantovan & J. Newton

§1. Ordinary completed cohomology (Hida, Emerton)

§2. Higher Hida theory (Hida, Pilloni, Boxer-Pilloni)

§3. Preliminaries

§4. The canonical part

§5. The anti-canonical part

Motivation: Eichler-Shimura isomorphism in families

Ohta, Cais : ordinary case

Andreatta - Iovita - Stevens

Chojecki - Hansen - Johansson, Andreatta - Iovita

→ finite slope case

§1. Notation: $G = GL_2/\mathbb{Q}$

$$\cup$$

$$B = T \ltimes N$$

$B_0 = B(\mathbb{Z}_p)$, $T_0 = T(\mathbb{Z}_p)$, $N_0 = N(\mathbb{Z}_p)$

$$N_0 \cong \mathbb{Z}_p$$

$\Lambda = \mathbb{Z}_p[\![T_0]\!]$

$\kappa : T_0 \longrightarrow \Lambda^\times$ univ. character

$K = K^p K_p \subset G(\mathbb{A}_f)$ $\leadsto$
neat

(fix & ignore K^p)

$X_K \hookrightarrow X_K^* / \mathbb{Q}$

$\mathbb{Q} \hookrightarrow \mathbb{C}$

$\hookrightarrow C = \widehat{\overline{\mathbb{Q}}}_p$

completed cohomology: general notion of mod p^n modular form

$$\widetilde{R\Gamma}_c(\mathbb{Z}/p^n\mathbb{Z}) := \varinjlim_{K_p \subset G(\mathbb{Q}_p)} R\Gamma_c(X_K(\mathbb{C}), \mathbb{Z}/p^n\mathbb{Z})$$

$\circlearrowleft$
$G(\mathbb{Q}_p)$

↑

$\uparrow$ object in derived category of smooth $G(\mathbb{Q}_p)$-representations w admissible cohomology

$$\widetilde{R\Gamma}_c(N_0, \mathbb{Z}/p^n\mathbb{Z}) = \varinjlim_{N_0 \subset K_p \subset G(\mathbb{Q}_p)} R\Gamma_c(X_K(\mathbb{C}), \mathbb{Z}/p^n\mathbb{Z})$$

$$= R\Gamma_{cont}(N_0, \widetilde{R\Gamma}_c(\mathbb{Z}/p^n\mathbb{Z}))$$

$u = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ $\quad u N_0 u^{-1} \subset N_0$

$$\begin{array}{ccc} \widetilde{R\Gamma}_c(N_0, \mathbb{Z}/p^n\mathbb{Z}) & \xrightarrow[u^*]{} & \widetilde{R\Gamma}_c(u N_0 u^{-1}, \mathbb{Z}/p^n\mathbb{Z}) \\ & \searrow^{U_p^B} & \downarrow \mathrm{tr} \\ & & \widetilde{R\Gamma}_c(N_0, \mathbb{Z}/p^n\mathbb{Z}) \end{array}$$

take ordinary parts w.r.t. U_p^B

$$\widetilde{R\Gamma_c}(N_0, \mathbb{Z}/p^n\mathbb{Z})^{ord} :=$$

$$\widetilde{R\Gamma_c}(N_0, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}[\mathcal{U}_p^B]} \mathbb{Z}/p^n\mathbb{Z}[\mathcal{U}_p^B, (\mathcal{U}_p^B)^{-1}]$$

Remarks: 1). Because $\widetilde{R\Gamma_c}(\mathbb{Z}/p^n\mathbb{Z})$ has admissible cohomology, this def'n of ordinary parts agrees w:

- Hida's definition via an idempotent
- Emerton's definition via locally $\mathcal{U}_p^B$-finite vectors

2). Can take $(R)\varprojlim_n$ & form

$$\widetilde{R\Gamma_c}(\mathbb{Z}_p), \quad \widetilde{R\Gamma_c}(N_0, \mathbb{Z}_p)^{ord}$$

[Mittag-Leffler OK by admissibility]

3). for modular curve, this complex is concentrated in degree 1:

$$\tilde{H}^1_c := \widetilde{R\Gamma_c}(N_0, \mathbb{Z}_p)^{ord}$$

$\updownarrow$

p-adically admissible rep'n of T_0

injective object because:

$\widetilde{R\Gamma_c}(B_0, \mathbb{Z}_p)^{ord}$ is also concentrated in degree 1.

$$\mathcal{M}_n : \left\{ \begin{array}{c} \text{smooth} \\ \text{admissible} \\ \text{reps of } T_0 \\ \text{w } \mathbb{Z}/p^n\mathbb{Z}\text{-coeffs} \end{array} \right\} \to \left\{ \begin{array}{c} \text{fin. gen.} \\ \Lambda_n\text{-modules} \end{array} \right\}$$

Pontryagin duality $\quad \Lambda_n = \mathbb{Z}/p^n\mathbb{Z}[\![T_0]\!]$

$$\mathcal{M}_n(H_n) := \mathrm{Hom}_{\Lambda_n}(H_n^\vee, \Lambda_n)$$

$$\mathcal{M}(H) = \varprojlim_n \mathcal{M}_n(H/p^n)$$

$$\widetilde{\mathcal{M}}^1_c := \mathcal{M}(\widetilde{H}^1_c)$$

§2. Consider

$\mathfrak{X}^{*,\mathrm{ord}} / \mathrm{Spf}\, \mathbb{Z}_p$ ordinary locus

affine

$\mathcal{E}[p^\infty]\big|_{X^{*,\mathrm{ord}}}$ p-div gp w slope filtration

$$0 \to \mathcal{G}^\circ \to \mathcal{G} \to \mathcal{G}^{et} \to 0$$

Igusa tower: trivializes graded pieces of slope filtration

$$\mathrm{Ig}^* \xrightarrow{\pi} \mathfrak{X}^{*,\mathrm{ord}}$$

T_0-torsor

pro-finite étale

Hecke correspondence:

· mod p

$$\begin{array}{ccc} & \overline{\mathcal{J}_g^*} \ (p^{-1}) & \\ {\scriptstyle F}\ \swarrow {\scriptstyle p_1} & & \searrow {\scriptstyle \wr\ p_2} \\ \overline{\mathcal{J}_g^*} & & \overline{\mathcal{J}_g^*} \end{array}$$

relative Frobenius

$$\rightsquigarrow \quad p_2^* \mathcal{O}_{\overline{\mathcal{J}_g^*}} \longrightarrow p_1^! \mathcal{O}_{\overline{\mathcal{J}_g^*}}$$

tr_F:

F has a canonical lift $\tilde{F}$ to char 0

(moduli-theoretically: quotient by canonical subgp)

$$\rightsquigarrow \quad U_p^{\text{naive}} := \mathrm{tr}_{\tilde{F}} : p_2^* \mathcal{O}_{\mathcal{J}_g^*} \longrightarrow p_1^! \mathcal{O}_{\mathcal{J}_g^*}$$

$$H^0 := H^0\left(\mathcal{I}_g^*, \Omega^1_{\mathcal{I}_g^*, \log} \otimes \mathcal{J}\right)^{U_p^{\text{naive}}\text{-ord}} \quad (\&)$$

$\circlearrowleft T_0$

$\mathcal{J} \subset \mathcal{O}_{\mathcal{I}_g^*}$ ideal of cusps

$$\cong$$

$$H^0(\mathcal{I}_g^*, \mathcal{J})^{U_p\text{-ord}}$$

$$U_p = \gamma_p U_p^{\text{naive}}$$

$$\pi_*\left(\mathcal{J} \hat{\otimes}_{\mathbb{Z}_p} \Lambda\right)^{T_0} =: \omega^\kappa$$

$$M^0 := H^0\left(\mathfrak{X}^{*,\text{ord}}, \omega^\kappa\right)^{U_p\text{-ord}}$$

$$\uparrow \qquad = M(H^0)$$

fin proj Λ-module (Hida)

(by affineness of $\mathfrak{X}^{*,\text{ord}}$)

Can also consider

$$H^1_c := H^1_c(\mathcal{I}_g^*, \mathcal{I})^{\tilde{F}\text{-ord}}$$

$$M^1_c = H^1_c(\mathcal{X}^{*,\text{ord}}, \omega^k)^{\tilde{F}\text{-ord}}$$

$$= M(H^1_c)$$

↑ fin. proj Λ-module (by Serre duality)

Expected thm (C-Mantovan-Newton)

$C = \widehat{\overline{\mathbb{Q}}_p} \rightsquigarrow \Lambda_{\mathcal{O}_C} = \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$

We have a Hecke-equivariant short exact sequence of $\Lambda_{\mathcal{O}_C}$-modules

$$0 \longrightarrow M^1_c \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{(2)} \tilde{M}^1_c \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$$

$$\downarrow (1)$$

$$M^0 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \longrightarrow 0$$

$M^1_c \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$, $M^0 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$: ord. coherent coh. of $\mathcal{J}$ on Ig^*

$\tilde{M}^1_c \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$: ordinary completed cohomology w. compact support

where:

(2) is equivariant w.r.t. automorphism of $\Delta_{\mathcal{O}_C}$ induced by

$$t \mapsto t_1^2 \, \kappa(t)$$

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

(2) —11— induced by $t \mapsto t^{w_0} = \begin{pmatrix} t_2 & 0 \\ 0 & t_1 \end{pmatrix}$

Remarks:

1). We have a Hecke-equivariant identification of

$$\tilde{M}^1_c = \varprojlim_{N_0 \subset K_p} H^1_c(X_{K^pK_p}(\mathbb{C}), \mathbb{Z}_p)^{*\text{-ord}}$$

ordinary parts w.r.t. $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

via Λ-adic Poincaré duality given on homology by

$(R)\mathrm{Hom}_\Lambda(\ \ , \Lambda)$

$\leadsto$ this recovers thm of Ohta (works over $\mathbb{Z}_p^{cycl}$, Galois action, splitting)

2). Idea of proof: use geometry of Hodge-Tate period morphism:

$$\begin{array}{ccccc} \mathcal{X}^* & \supset & \mathcal{X}^{*,ord} & = & \left(\mathcal{I}g_C^{*,perf} \times G(\mathbb{Q}_p)\right)/B(\mathbb{Z}_p) \\ \downarrow \pi_{HT} & & \downarrow & \pi_{HT} & \downarrow \\ \mathbb{P}^{1,ad} & \supset & \mathbb{P}^1(\mathbb{Q}_p) & = & G(\mathbb{Q}_p)/B(\mathbb{Q}_p) \end{array}$$

compute ordinary parts of induced rep'n using Bruhat decomposition [geometric version of computation of derived ordinary parts due to Hauseux, ten author paper]

3). Related work: AIS, CHJ, AI but we treat two Bruhat cells equally, Juan Esteban Rodriguez.

§ 3. Preliminaries :

1). We apply primitive comparison isom (Scholze, Faltings)

$$\mathcal{H}^1_c := H^1_{proet}(\mathcal{X}^*_{N_0}, \hat{\mathcal{I}}^+)^{ord}$$

$$\uparrow \cong^a$$

$$\tilde{H}^1_c \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$$

where

$$\mathcal{X}^*_{N_0} = \varprojlim \mathcal{X}^*_{K^p K_p}$$

$N_0 \subset K_p$ as diamonds

$\hat{\mathcal{I}}^+ \subseteq \hat{\mathcal{O}}^+$ sections that vanish along cusps

2). Bruhat decomposition on $\mathbb{P}^{1,ad}$:

$$G(\mathbb{Q}_p) = B(\mathbb{Q}_p) w_0 B(\mathbb{Q}_p) \sqcup B(\mathbb{Q}_p)$$

open: $\mathbb{P}^{1,ad} = \mathbb{A}^{1,ad} \sqcup \{id\}$ — $\{id\}$ closed (from generic fibre) — for map (1)

$= \overline{\mathbb{B}^{\circ}(w_0, 1)} \sqcup \mathbb{B}(id, 1)$

$\overline{\mathbb{B}^{\circ}(w_0, 1)}$: gc closed — for map (2)

$\mathbb{B}(id, 1)$: Stein open (by taking tubes from special fibre)

corresponding to these,

have 2 distinguished Δ's at level N_0,

$\mathcal{U}_p$-equivariant:

Notation: if $\mathcal{U} \subset \mathbb{P}^{1,ad}$ is N_0-stable locally closed

$\mathcal{X}^{*}_{N_0}(\mathcal{U})$ = preimage under

π_{HT}/N_0 of $\mathcal{U}/N_0 \subset |\mathbb{P}^1/N_0|$ (top space)

these 2 distinguished Δ's are canonically identified

Coh w support in $\overline{\mathbb{B}^\circ(w_0, \varepsilon)}$ is indep of ε after applying ordinary parts.

4). <u>the canonical part:</u>

$$R\Gamma_{proet}(X^*_{N_0}, \hat{\mathcal{J}}^+)^{ord} \to R\Gamma_{proet}(X^*_{N_0}(id), \mathcal{J}^+)^{ord}$$

<u>key points:</u>

have commutative diagram:

$$\begin{array}{ccc} (\mathcal{I}g^{*,perf}_{\mathcal{O}_C})^{ad}_\eta & \longrightarrow & X^*(id) \\ \downarrow & & \downarrow \;\; N_0\text{-torsor away from cusps} \\ (\mathcal{I}g^{*}_{\mathcal{O}_C})^{ad}_\eta & \longrightarrow & X^*_{N_0}(id) \end{array}$$

$(T_p\mu_{p^\infty})_\eta$-torsor away from cusps

$(T_p\mu_{p^\infty})_\eta \cong \mathbb{Z}_p(1)$

where have an action of

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$$(V_p \mu_{p^\infty}) \rtimes T(\mathbb{Q}_p)$$

$$\curvearrowright$$

$$\widetilde{\mu}_{p^\infty} \rtimes T(\mathbb{Q}_p) = \mathrm{Aut}(\widetilde{\mathfrak{X}})$$

$$\circlearrowleft$$

$$\mathcal{I}_g^{*,\mathrm{perf}} \otimes C$$

where $\mathfrak{X} = \mu_{p^\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p$

(C-Scholze, Howe)

- Bhatt-Morrow-Scholze (§8 of 1st paper)

If $\mathfrak{X}$ / Spf $\mathcal{O}_C$ smooth formal scheme w adic generic fibre X

$\nu : X_{\text{proét}} \longrightarrow \mathfrak{X}_{\text{Zar}}$

$$\mathcal{H}^i\left(L\eta_{(\zeta_p - 1)} R\nu_* \hat{\mathcal{O}}^+\right) \simeq \Omega^i_{\mathfrak{X}/\mathcal{O}_C}\{-i\}$$

$\hat{\mathcal{J}}^+$

$L\eta_{(\zeta_p-1)}$ kills $(\zeta_p - 1)$-torsion in cohomology

$\{-i\}$: Breuil-Kisin-Fargues twist

forthcoming work of Hansheng Diao & Zijian Yao

applies to case $\mathfrak{Z} \subset \mathfrak{X}$ simple normal crossings divisors & $\hat{\mathcal{J}}^+ \subset \hat{\mathcal{O}}^+$ defined by $\mathfrak{Z}^{\text{ad}}_{\eta}$.

$$\mathcal{H}^i(L\eta_{(\zeta_p - 1)} R\nu_* \hat{\mathbb{J}}^+) \simeq \Omega^i_{\mathfrak{X}, \log} \otimes \mathbb{J}\{-i\}$$

Take $\mathfrak{X} = \mathfrak{I}g^{*/tor}_{\mathcal{O}_C}$

Simplifications when you apply ordinary parts:

$$R\Gamma_{proet}(\mathcal{X}^*_{N_0}(id), \hat{\mathbb{J}}^+)^{ord}$$

$$\wr\!\!\mid$$

$$H^0(\mathfrak{I}g^*_{\mathcal{O}_C}, \Omega^1_{\mathfrak{I}g^*, \log} \otimes \mathbb{J})^{U_p\text{-naive-ord}}[-1]$$

$$\wr\!\!\mid$$

$$(H^0 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C)[-1]$$

<u>Because:</u>

1). $U_p^B = U_p^{naive}$

acts top nilpotently

on $\Theta_{J_g}^{*} \mathcal{O}_C$

2). $(\gamma_p - 1)$-torsion in

ordinary parts is

almost

($u = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ acts as $\tilde{F}^{-1}$)

on $J_g^{*, perf} \mathcal{O}_C$

3). ordinary is overconvergent

(mod m_Λ : weights $k \geq 3$

ordinary is classical)

4). anti-canonical part:

1). near $X^*_{N_0}(w_0)$,

$\mathcal{U}_p^B$ acts as $\widetilde{\Gamma}$
$\leftrightarrow$ $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ $\swarrow$

2). control them:
from $X^*_{N_0}(w_0)$
to $X^*_{\Gamma_0(p)}(w_0)$

"higher Coleman theory"

$\widetilde{M}^d_c \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \twoheadrightarrow H^0 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$

ord comp. coh. Hida theory

$G = GL_2/F$
p splits completely