RAMpAGe

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Mod p Hecke algebras and perverse $\mathbb{F}_p$-sheaves

Local Langlands

E - non. arch. local field, residue char $p>0$
$G = GL_n$

Frobenius semisimple n'dim'l complex Weil-Deligne reps $\leftrightarrow$ Smooth irr reps of $G(E)$

↳ studied with local Hecke algebras

Two modifications:

1) p-adic Langlands program: use $\mathbb{Q}_p$ coefficients mod p coefficients

2) geometric Langlands: replace automorphic functions w/ automorphic sheaves on Bun_G.

Main Results

① Mod p version of geometric Satake

② Geometric construction of central elt's in Iwahori mod p Hecke algebra

Future: - (w. C Pépin) restriction to Levi for ①

- (w. C Pépin, T. Schmidt) relate to mod p Kazhdan-Lusztig theory + mod p Galois rep's.

Geometric Satake

$\mathcal{O}_E$ = ring of integers $\qquad K = G(\mathcal{O}_E)$

Unramified local Langlands reduces to Satake iso, which describes

$$\mathcal{H}_{K,\mathbb{C}} = \{ f : G(E) \to \mathbb{C} \ , \ f \text{ has compact supp., } K \text{ bi-invariant} \}.$$

Suppose E has char $p > 0$, residue field $\mathbb{F}_q$.

Affine Grassmannian

$Gr = LG / L^+G$

$LG : R \to G(R((t)))$
$L^+G : R \to G(R[[t]])$

Gr is ind-projective / $\mathbb{F}_q$. $Gr(\mathbb{F}_q) = G(E)/k$

Certain L^+G-equivariant perverse sheaves on Gr $\xrightarrow{\sim}$ $Rep_{\mathbb{Q}_\ell}(GL_n)$
$\mathbb{Q}_\ell$

Mod p Geometric Satake

Étale $\mathbb{F}_p$ sheaves on $X / \mathbb{F}_q$

1) $Rf_!$, Rf^*, $\overset{L}{\otimes}$ preserve constructibility
Rf_*, $Rf^!$, Rhom do not.

2) No duality

3) Have function-sheaf dictionary

$P_{L^+G}(Gr, \mathbb{F}_p)$ - monoidal category of L^+G-equivariant perverse $\mathbb{F}_p$-sheaves on Gr.

Thm (C): There exists an affine, pro-solvable monoid scheme / $\mathbb{F}_p$, M_G, such that

$$P_{L^+G}(Gr, \mathbb{F}_p) \xrightarrow{\sim} Rep_{\mathbb{F}_p}(M_G).$$

Proof uses BD-Grassmannians, but exploits more facts about affine Schubert varieties

$$M_G \longrightarrow \operatorname{Spec} \mathbb{F}_p[X_*(T)^+]$$

$T \subset B \subset G$ $\quad L^+G$ orbits in $Gr \longleftrightarrow X_*(T)^+$

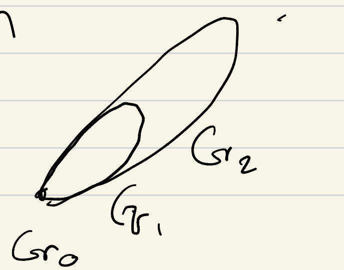
$$Gr_\mu \leftarrow \mu$$

Ex: $G = SL_2$ $\quad X_*(T)^+ \cong \mathbb{Z}_{\geq 0}$

$$\left(t \to \begin{pmatrix} t^n & \\ & t^{-n} \end{pmatrix}\right) \mapsto n$$

Gr has one connected component

$\dim Gr_n = 2n$



$$\overline{Gr_\mu} = Gr_{\leq\mu} = \bigcup_{\lambda\leq\mu} Gr_\lambda$$

Thm (C):

1) Simple objects in $P_{L^+G}(Gr, \mathbb{F}_p)$ are the constant sheaves
$$IC_\mu = \mathbb{F}_p[\dim Gr_{\leq\mu}] \text{ on } Gr_{\leq\mu}.$$

2) $IC_{\mu_1} * IC_{\mu_2} = IC_{\mu_1+\mu_2}$.

3) $$\mathbb{F}_p[X_*(T)^+] \xrightarrow[(\cancel{\sim})]{} K_0(P_{L^+G}(Gr, \mathbb{F}_p)) \otimes \mathbb{F}_p$$
$$\mu \longmapsto [IC_\mu]$$

Proof of 1) uses F-singularities

2) Says that constant sheaf preserved under convolution
$$\widetilde{Gr}_{\mu_1,\mu_2} \xrightarrow{m} Gr_{\leq\mu_1+\mu_2}$$
$$Rm_!(\mathbb{F}_p) \simeq \mathbb{F}_p$$

Artin Schreier sequence $0 \to \mathbb{F}_p \to \mathcal{O}_X \xrightarrow{F-1} \mathcal{O}_X \to 0$

+ $Gr_{\leq\mu}$ has rational singularities (Pappas-Rapoport, Faltings)

Kovács

Relation to mod p Satake

Herzig, Henniart – Vignéras

$$S : \mathcal{H}_{K,\mathbb{F}_p} \xrightarrow{\sim} \mathbb{F}_p[X_*(T)^+]$$

$$\begin{tikzcd}
\mathbb{F}_p[X_*(T)^+] \arrow[r, "(*)"'] \arrow[dr, bend right, "S^{-1}"'] & K_0(P_{L^+G}(\mathrm{Gr}, \mathbb{F}_p)) \otimes \mathbb{F}_p \arrow[d, "\text{function-sheaf}"] \\
& \mathcal{H}_{K,\mathbb{F}_p}
\end{tikzcd}$$

Iwahori mod p Hecke algebra

$I \subset K$ Iwahori

Vignéras, Ollivier

$$T : Z(\mathcal{H}_{I,\mathbb{F}_p}) \xrightarrow{\sim} \mathcal{H}_{K,\mathbb{F}_p}$$
$$f \mapsto f * \underbrace{\mathbb{1}_K}_{\text{char. fcn. of } K}$$

Affine flag variety $\quad \mathcal{F}\ell = LG/L^+I$

$$\mathcal{F}\ell(\mathbb{F}_q) = G(E)/I$$

$\pi : Fl \to Gr$ $\quad R\pi_!$ lifts $*1_K$ to the level of sheaves

$P_{L^+G}(Gr, \mathbb{F}_p)^{ss}$ semisimple objects

$p > 2$, G_{der} is absolutely almost simple

Thm (c): There exists a functor

$$Z : P_{L^+G}(Gr, \mathbb{F}_p)^{ss} \longrightarrow P_I(Fl, \mathbb{F}_p)$$

such that

1) $R\pi_!(Z(F)) \cong F$
2) $Z(F_1 * F_2) \cong Z(F_1) * Z(F_2)$
3) $Z(F_1) * F_3 \cong F_3 * Z(F_1)$.

Z lifts T^{-1} to sheaves.

Can describe $Z(IC_\mu)$ in terms of the μ-admissible set. (it is a constant sheaf)

Proof uses F-singularities + equal char. analogues of local models of Shimura varieties (Pappas-Zhu)

Perversity of constant sheaf

(A, m) strictly henselian local ring
char p, dim d.

$i : \operatorname{Spec} k(m) \to \operatorname{Spec}(A)$.

Perversity of $\mathbb{F}_p[d]$ requires
$$R^n i^! \mathbb{F}_p = 0 \text{ for } n < d.$$

$R^n i^! \mathbb{F}_p$ can be computed in terms of local cohomology modules
$$H^*_m(A) = H^{*-1}(U, \mathcal{O}_U)$$
$$U = \operatorname{Spec} A - \operatorname{Spec} k(m).$$

Ex: $B = k[x]$, $m = (x)$
$$H^1_m(B) = k[x, x^{-1}] / k[x]$$

Thm (C): If A is Cohen-Macaulay, then $\mathbb{F}_p[d]$ is perverse.

Uses that $H^i_m(A) = 0$ for $i < d$

$$j : U \to \operatorname{Spec} A$$

$$Ri^! \mathbb{F}_p \longrightarrow \mathbb{F}_p \longrightarrow Rj_* \mathbb{F}_p$$

$\hookrightarrow$ has coh. in degree 0

To compute $Rj_* \mathbb{F}_p$, use Artin-Schreier sequence.

$A[F]$, polynomial ring $Fa = a^p F$.

$H^i_m(A)$ is an $A[F]$-module.

Simplicity of $\mathbb{F}_p[d]$ requires $R^d i^! \mathbb{F}_p[d] = 0$.

If $d \geq 2$, need $(H^d_m(A))^F = 0$.

Lemma: $(H^d_m(A))^F = 0$ if $H^d_m(A)$ is a simple $A[F]$-module.

Defn (Smith, Hochester-Huneke):

A is F-rational if $H^d_m(A)$ is a simple $A[F]$-module.

Thm (C): If $p \nmid |\pi_1(G_{der})|$ then the local rings of $Gr_{\leq \mu}$ are F-rational. In fact, $Gr_{\leq\mu}$ is globally F-regular.

The local models are F-regular, as are certain schubert varieties in BD Grassmannians.

F-regularity

An F-splitting of $X/\mathbb{F}_q$ is an $\mathcal{O}_X$-linear splitting of

$$\mathcal{O}_X \longrightarrow F_* \mathcal{O}_X$$
$$a \longmapsto a^p$$

Global F-regularity is a condition which guarantees many F-splittings compatible with effective Cartier divisors. (Smith)

Globally F-regular $\Rightarrow$ F-rational $\Rightarrow$ normal, Cohen-Macaulay, rational singularities

(Smith, Hochester - Huneke)

Proof of global F-regularity uses that affine Demazure resolutions of $Gr_{\leq \mu}$ are split along an ample effective Cartier divisor.

Faltings, Pappas-Rapoport

Zhu constructed splittings of local models.

special fiber lives in Fl $\longrightarrow$ $M_\mu \longleftarrow Gr_{\leq \mu}$ $\rightsquigarrow$ generic fiber

Richarz-Haines - local models are normal, Cohen-Macaulay.

Mehta-Ramanathan - classical Schubert varieties in G/B.

X Globally F-regular $\Rightarrow H^i(X, \mathcal{O}_X) = 0$, $i > 0$.