

L-packets of S-unramified regular supercuspidal representations

Today: Explain a geom. real. of certain L-packets of supercuspidal reps.

s.c.

All reps over \mathbb{C} (\mathbb{Q}_ℓ)

I. Big Picture

central input in NT: p-adic gyps & reps.

$k =$ f. extn of \mathbb{Q}_p , or $\mathbb{F}_p((t))$, residue field \mathbb{F}_q .

$G =$ conn red gp / k .

Fact: Every irred repn. of $G(k)$ appears in the parabolic induction of a supercuspidal

History: Yu (exhaustion for $p \gg 0$; Kim, Fintzen)

Simplest recipe for a s.c. $G = SL_2$

- \mathbb{F}_q 1. Take an irred cuspidal repn of $SL_2 \mathbb{F}_q$
- \mathbb{O}_k 2. $SL_2(\mathbb{O}_k) \twoheadrightarrow SL_2 \mathbb{F}_q$. π can be viewed as a repn of $SL_2(\mathbb{O}_k)$.
- k 3. $c\text{-Ind}_{SL_2(\mathbb{O}_k)}^{SL_2(k)}(\pi)$ irred (and therefore s.c.)

• In gen: must use geometry to obtain ①

↳ cohom of DL var char sheaves

• "depth 0" reps here

to get "depth r "
 $r > 0$

replace ② by an algebraic recipe.

	$G(\mathbb{F}_q)$	$G_{\text{nr},0}(\mathbb{O}_k)$	$G(k)$
alg	sp. cases	Yu	Yu
geom	DL Lusztig	• Lusztig • Straszinski Z. Chen • C-Ivanov	• Sh. var., ADLV $\subset G/A_0, G/I, \dots$

sp. cases of X_r are closely rel to affinoids in LT tower @ mf. level $\cong \coprod X_r$

"higher level" ADLV $\subset G/A_{r,0}$ (Ivanov, C-Ivanov)

(gen param₀) Lusztig's conj. = "loop DL sets" $\subset G/U$

X_r

f. type
sum
sep
var. / \mathbb{F}_q

known to be var. for G/U & its inner forms.

(Lusztig, Boyarchenko, C., C-Ivanov)

↳ rel. obj $\subset G/B$

studied by A. Ivanov using

P. Scholze's perfectoid techniques.

TODAY:

"vertical compatibility"
alg vs. geom

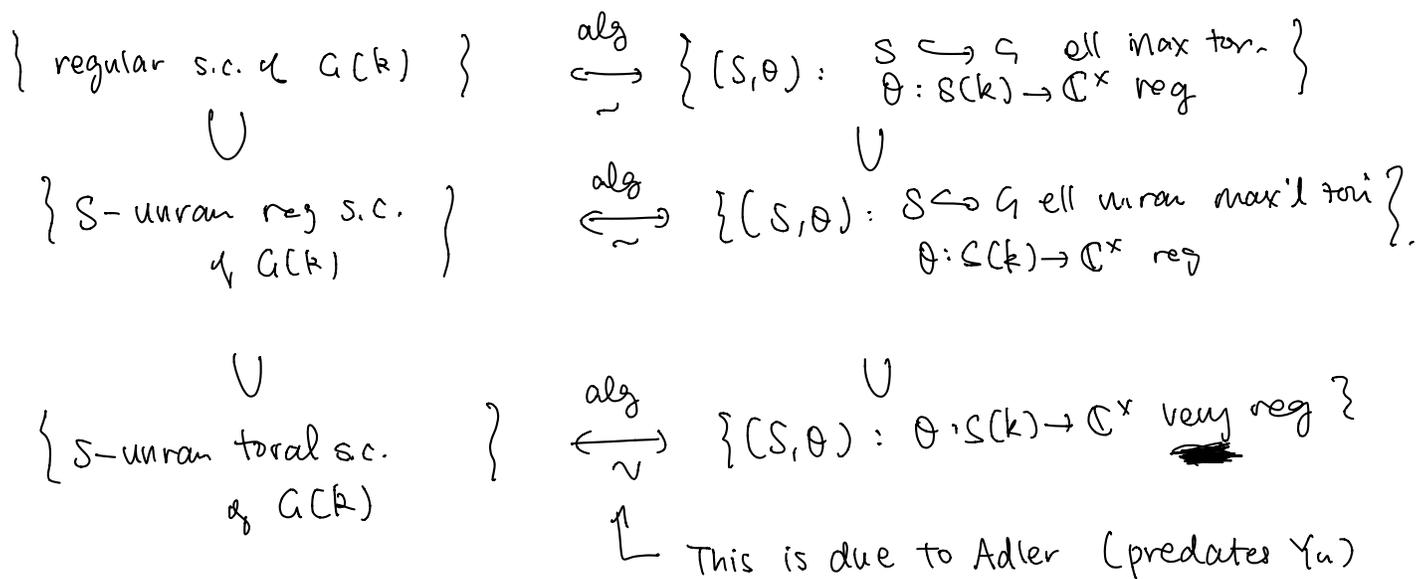
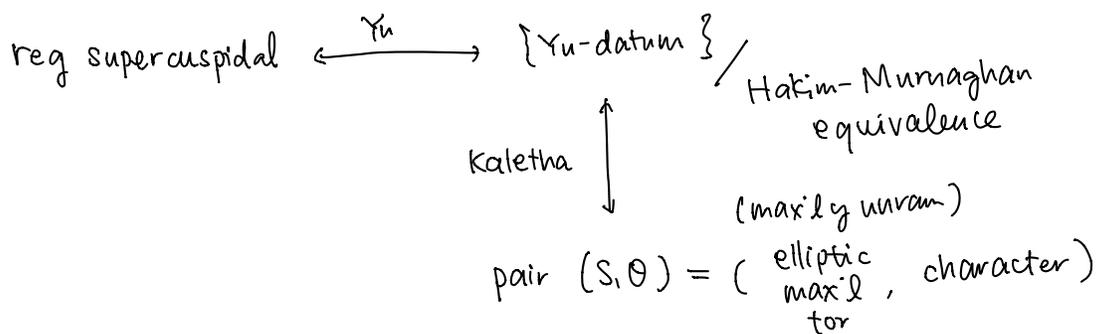
Main Thm. (C-Oi) These X_r 's (their coh) induces a geom. rell. of L-packets of S-unram toral supercuspidals.

II. Maximal Tori

By recent work of Kaletha, we know:

"regular locus" of supercuspidals \longleftrightarrow "regular" char. of elliptic max'l tori

This is due to a composition:



L-packets of the LHS should satisfy some props.

- stability: average over an L-packet gives a fn. constant on stable conj. classes

$$g \in G(k) \quad \{ hgh^{-1} \in G(k) : h \in G(\bar{k}) \}$$

\hookrightarrow there's a notion of a stable conj. class of (S, θ) .

Rmks. • In the depth 0 setting: have stability by considering

$$\{ \gamma_u(S, \theta) : (S, \theta) \text{ ranges over a stable conj. cl.} \}$$

- DeBacker-Reeder (geom = γ_u)
- Kazhdan-Varslavsky.

- Warning: geom \neq Yu in gen
- for S -unram toral s.c. (these are pos. depth).
 - Reeder (conj. defn of L-packet)
 - DeBacker-Spice: \curvearrowright not quite stable !!
 - \hookrightarrow can twist Θ by ξ ; once we do this, then stable. quadratic character

Strategy: prove geom = Yu \otimes DeBacker-Spice's ξ .

In progress: alg vs. geom for S -unram reg. s.c.
(w/ \mathcal{O}_i)

this is the most general setting that a geom answer for s.c is known.

III. Cohomological reps of parahoric subgroups (What is X_r ?)

$S \hookrightarrow G$ unram ell. max'l torus. \rightsquigarrow uniquely determines a $G_{X,0}$

\cap
 $B \supset U$ Frobenius σ on G .

for $r \in \mathbb{Z}_{\geq 0}$:

$$S_0 = S \cap G_{X,0} \quad X_r := \left\{ g \in G_{X,0} / G_{X,r+} : g^{-1} \sigma(g) \in U_0 / U_r \right\}$$

$$U_0 = U \cap G_{X,0} \quad \begin{matrix} \hookrightarrow \\ \uparrow \end{matrix} \quad \begin{matrix} G_{X,0}(\mathcal{O}_K) \\ S_0(\mathcal{O}_K) \end{matrix}$$

Ex: • If $r=0$: this is a classical DL var.

• If $G = SL_2$: $X_0 =$ Dinfeld curve \cong

$$\det \begin{pmatrix} x & \sigma(y) \\ y & \sigma(x) \end{pmatrix}$$

$$W(xy^q - x^q y = 1) \cong_{\mathbb{F}_q} \{ (x,y) \in \mathbb{A}^2 : x^{q+1} - y^{q+1} = 1 \}$$

$$X_r = \left\{ (x,y) \in W_r^{\oplus 2} : \det \begin{pmatrix} x & \sigma(y) \\ y & \sigma(x) \end{pmatrix} = 1 \right\}$$

If I start w/ $\Theta: S_0(\mathcal{O}_K) \rightarrow G^*$, depth r , then virtual

$$H_c^*(X_r)[\Theta] = a^v \text{ repn of } G_{X,0}(\mathcal{O}_K).$$

Idea of "char on vreg elts determin the repr"

↳ Henniart '90s. Galn, inner forms of Galn.

$G = \text{red gp} / \mathbb{F}_q$, $\pi \leftrightarrow \mathbb{G}$ max'l torus. (elliptic or not).
also OK

Thm: (C-0i) Assume $q \gg 0$.

If π is an irred repr of $G(\mathbb{F}_q)$ s.t. $\text{Tr}(g, \pi) = c \cdot \sum_{w \in W} \theta(wg w^{-1})$

then $\pi \cong c \cdot H_c^*(X_0)[\theta]$.

$\forall g$ "v. neg wrt T "

± 1 , indep of g

↓

for θ regular.

Galn: k_n/k deg n curve

$$\mathbb{G}_m: k_n^\times \rightarrow \mathbb{C}^\times$$

$$\pi \mapsto (-)^{n-1}$$

$$\mathbb{G}_m|_{\mathcal{O}_{k_n}} = 1$$

$$S \hookrightarrow G \rightsquigarrow {}^L S \hookrightarrow {}^L G$$

$$\left(\frac{\mathcal{O}_{k_n}}{\pi^{\mathbb{Z}}} \right)^\times$$

Galn settis: $W_k \rightarrow {}^L S \hookrightarrow {}^L G$

$$\cong \text{Ind}_{W_{k_n}}^{W_k} (\theta \cdot \frac{\mathbb{G}_m}{\mathbb{G}_m})$$

duo to Tam

In quat alg settis: X_2

$$X_r \quad (n-1)(n-1)$$

$$\cong \frac{\mathbb{G}_m}{\mathbb{Z}}$$

