L-packets of S-unramified regular supercuspidal representations

Today: Explain a geom. real. of certain L-packets of supercuspidal reps.
s.c.

All reps over $\mathbb{C}$ $(\overline{\mathbb{Q}}_\ell)$

I. Big Picture

central input in NT: p-adic gps & repns.

k = f. extn of $\mathbb{Q}_p$, or $\mathbb{F}_p((t))$, residue field $\mathbb{F}_q$.

G = conn red gp / k.

Fact: Every irred repn. of $G(k)$ appears in the parabolic inductn of a supercuspidal

History: Yu (exhaustion for $p \gg 0$; Kim, Fintzen)

Simplest recipe for a s.c. $G = SL_2$

$\mathbb{F}_q$ 1. Take an irred cuspidal repn π of $SL_2 \mathbb{F}_q$

$\mathcal{O}_k$ 2. $SL_2(\mathcal{O}_k) \twoheadrightarrow SL_2 \mathbb{F}_q$. π can be viewed as a repn of $SL_2(\mathcal{O}_k)$.

k 3. $\text{c-Ind}_{SL_2(\mathcal{O}_k)}^{SL_2(k)}(\pi)$ irred (and therefore s.c.)

- In gen: must use geometry to obtain 1
  ↳ coh of DL var
  char sheaves

- "depth 0" reps here
  to get "depth r" $r>0$ replace 2 by an algebraic recipe.

	$G(\mathbb{F}_q)$	$G_{x,0}(\mathcal{O}_k)$	$G(k)$
alg	sp. cases	→ Yu	→ Yu
geom	DL Lusztig	• Lusztig • Stasinski Z. Chen • C-Ivanov	• Sh. var., ADLV $\subset G/G_0$, G/I, ...

"higher level" ADLV $\subset G/G_{x,r}$ (Ivanov, C-Ivanov)

(gen parahorics) Lusztig's conj. : "loop DL sets" $\subset G/U$
known to be var. for GL_n & its inner forms.
(Lusztig, Boyarchenko, C., C-Ivanov)

↳ rel. obj $\subset G/B$ studied by A. Ivanov using P. Scholze's perfectoid techniques.

X_r
f. type
sum
sep
var. / $\overline{\mathbb{F}}_q$.

sp. cases of X_r are closely rel to affinoids in LT tower @ inf. level $\cong \bigsqcup_{G/G_{x,0}} X_r$

$\cong$

TODAY:
"vertical compatibility"
alg vs. geom

Main Thm. (C-Oi) These X_r's (their coh) induces a geom. real. of L-packets of S-unram toral supercuspidals.

II. Maximal Tori

By recent work of Kaletha, we know:

"regular locus" of supercuspidals $\longleftrightarrow$ "regular" char. of elliptic max'l tori

This is due to a composition:

$$\text{reg supercuspidal} \xleftrightarrow{\text{Yu}} \{\text{Yu-datum}\}/\text{Hakim-Murnaghan equivalence}$$

$$\text{Kaletha} \updownarrow$$

(maxily unram)

$$\text{pair } (S,\theta) = (\text{elliptic max'l tor}, \text{ character})$$

$$\left\{ \text{regular s.c. of } G(k) \right\} \underset{\sim}{\overset{\text{alg}}{\longleftrightarrow}} \left\{ (S,\theta) : \begin{array}{l} S \hookrightarrow G \text{ ell max tor.} \\ \theta : S(k) \to \mathbb{C}^\times \text{ reg} \end{array} \right\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\left\{ \begin{array}{c} \text{S-unram reg s.c.} \\ \text{of } G(k) \end{array} \right\} \underset{\sim}{\overset{\text{alg}}{\longleftrightarrow}} \left\{ (S,\theta) : \begin{array}{l} S \hookrightarrow G \text{ ell unram max'l tori} \\ \theta : S(k) \to \mathbb{C}^\times \text{ reg} \end{array} \right\}.$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\left\{ \begin{array}{c} \text{S-unram toral s.c.} \\ \text{of } G(k) \end{array} \right\} \underset{\sim}{\overset{\text{alg}}{\longleftrightarrow}} \left\{ (S,\theta) : \theta : S(k) \to \mathbb{C}^\times \text{ very reg} \right\}$$

$\hookrightarrow$ This is due to Adler (predates Yu)

L-packets of the LHS should satisfy some props.

- stability: average over an L-packets gives a fn. constant on stable conj. classes
  $g \in G(k)$ $\{hgh^{-1} \in G(k) : h \in G(\bar{k})\}$.
  $\hookrightarrow$ there's a notion of a stable conj. class of (S,θ).

Rmks: • In the depth 0 setting: have stability by considering

$$\{ \text{Yu}(S,\theta) : (S,\theta) \text{ ranges over a stable conj. cl.} \}.$$

- DeBacker-Reeder (geom = Yu)
- Kazhdan-Varshavsky.

- Warning: geom $\neq$ Yu in gen
- for S-unram toral s.c. (these are pos. depth):
  - Reeder (conj. defn of L-packet)
  - DeBacker–Spice: not quite stable !!
    - ↳ can twist θ by $\underset{\sim}{\xi}$; once we do this, then stable.
      quadratic character

<u>Strategy</u>: prove geom $=$ Yu $\otimes$ DeBacker-Spice's ξ.

<u>In progress</u>: alg vs. geom for $\underset{\sim}{\text{S-unram reg. s.c.}}$
(w/ Oi)

this is the most general setting that a geom constr for s.c. is known.

III. Cohomological reps of parahoric subgps (What is X_r?)

$S \hookrightarrow G$ <u>unram</u> ell. max'l torus. $\rightsquigarrow$ uniquely determines a $G_{x,0}$

$\cap$ floating Frobenius σ on G.

$B \supset U$

For $r \in \mathbb{Z}_{\geq 0}$:

$S_0 = S \cap G_{x,0}$

$U_0 = U \cap G_{x,0}$

$$X_r := \{ g \in G_{x,0}/G_{x,r+} : g^{-1}\sigma(g) \in U_0/U_r \}.$$

$G_{x,0}(\mathcal{O}_k) \circlearrowright X_r \circlearrowleft S_0(\mathcal{O}_k)$

<u>Ex</u>: • If $r=0$: this is a classical DL var.

• If $G = SL_2$, $X_0 =$ Drinfeld curve

$$\det\begin{pmatrix} x & \sigma(y) \\ y & \sigma(x) \end{pmatrix}$$

$$W(xy^q - x^q y = 1) \underset{\mathbb{F}_{q^2}}{\cong} = \{ (x,y) \in \mathbb{A}^2 : x^{q+1} - y^{q+1} = 1 \}$$

$$X_r = \{ (x,y) \in \mathbb{W}_r^{\oplus 2} : \det\begin{pmatrix} x & \sigma(y) \\ y & \sigma(x) \end{pmatrix} = 1 \}.$$

If I start w/ $\theta : S_0(\mathcal{O}_k) \to \mathbb{C}^\times$, depth r, then

$H_c^*(X_r)[\theta] =$ a virtual repn of $G_{x,0}(\mathcal{O}_k)$.

Thm. (C–Ivanov) ① If $\theta: S_0(\mathcal{O}_k) \to \mathbb{C}^\times$ is v. reg., then $H_c^*(X_r)[\theta]$ is irred.

② $\theta: S_0(\mathcal{O}_k) \to \mathbb{C}^\times$ arb.

For $g \in S_0(\mathcal{O}_k)$ s.t. $\bar{g} \in S_0(\mathbb{F}_q)$ is regular in $G_{x,0:0+}(\mathbb{F}_q)$

then $\mathrm{Tr}(g; H_c^*(X_r)[\theta])$

$$= \sum_{w \in W} \theta(wgw^{-1}).$$

[ex. of v. reg: $GL_2(\mathbb{F}_q[[t]]) \supseteq \mathbb{F}_{q^2}[[t]]^\times$

v. reg. char of $\mathbb{F}_{q^2}[[t]]^\times$ means:

- $\theta: \mathbb{F}_{q^2}[[t]]^\times / 1+t^n\mathbb{F}_{q^2}[[t]] \to \mathbb{C}^\times$
- $\theta|_{1+t^{n-1}\mathbb{F}_{q^2}[[t]]}$ = nontriv & has triv $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$-stab.)

IV. Alg. vs. geom.

For $\theta: S_0(\mathcal{O}_k) \to \mathbb{C}^\times$ very reg., Yu's rep $= \mathrm{Ind}_{S_0(\mathcal{O}_k) G_{x,s}(\mathcal{O}_k)}^{G_{x,0}(\mathcal{O}_k)} (\sigma_\theta)$.

Pf. of Main Thm.

Step 1. $\mathrm{Hom}_{S_{0+} G_{x,s}}(H_c^*(X_r)[\theta], \sigma_\theta) \neq 0$.

↳ • uses Thm from III.

• knowledge of the character of σ_θ on $S_0^{reg} G_{x,s}$.

trick! simplest setting: $S_0 = S_{0+} \sqcup S_0^{res}$.

$$\Rightarrow \langle\ ,\ \rangle_{S_0 G_{x,s}} = \langle\ ,\ \rangle_{S_{0+} G_{x,s}} + \underbrace{\langle\ ,\ \rangle_{S_0^{reg} G_{x,s}}}_{\neq 0}$$

Step 2: $\exists!$ quad. char ξ of $S_0(\mathcal{O}_k)$ s.t.

chars of $H_c^*(X_r)[\theta]$ and $\mathrm{Ind}_{S_0 G_{x,s}}^{G_{x,0}}(\sigma_{\theta\cdot\xi})$

agree (up to a constant) on $S_0(\mathcal{O}_k)^{reg}$.

& ξ = the ξ of DeBacker–Spice.

Step 3: For these irreps, the char. on the v. reg elts determine the rep

II

Idea of "char on v.reg elts determine the repn"

↳ Henniart '90s. GL_n, inner forms of GL_n.

G = red gp / $\mathbb{F}_q$, $T \hookrightarrow G$ max'l torus. (elliptic or $\underbrace{\text{not}}_{\text{also OK}}$).

Thm: (C-Oi) Assume $\underline{q \gg 0}$.

If π is an irred repn of $G(\mathbb{F}_q)$ s.t. $\mathrm{Tr}(g;\pi) = \overset{\pm 1,\ \text{indep of } g}{c} \cdot \sum_{w \in W} \theta(wgw^{-1})$

then $\pi \cong c \cdot H_c^*(X_0)[\theta]$. $\forall g$ "v.reg wrt T"

for θ regular.

GL_n: k_n/k deg n unr.

$\xi : k_n^\times \to \mathbb{C}^\times$

$\varpi \mapsto (-1)^{n-1}$

$\xi|_{\mathcal{O}_{k_n}} = 1$

$S \hookrightarrow G \rightsquigarrow {}^LS \hookrightarrow {}^LG$ $\left(\mathcal{O}_{k_n} / \pi^N\right)^\times$

GL_n setting: $\underbrace{W_k \to {}^LS \hookrightarrow {}^LG}$

$\cong \mathrm{Ind}_{W_{k_n}}^{W_k}(\theta \cdot \xi)$.

due to Tam

In quot alg setting: X_2

X_r $(n-1)(r-1)$