Cohomology of the Drinfeld tower, a family affair

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Global motivation

(I) Fix a prime p and $L/\mathbb{Q}_p$ finite, "large enough" coefficient field, with residue field k_L.

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(II) For $K \leq \mathbb{GL}_2(\mathbb{A}_f)$ compact open, $Y(K) :=$ open modular curve $/\mathbb{Q}$ of level K.

(III) Let $K^p = \mathbb{GL}_2(\hat{\mathbb{Z}}^p)$ and

$$\hat{H}^{1,\mathrm{gl}} = \varprojlim_n (\varinjlim_{K_p} H^1_{\mathrm{et}}(Y(K_pK^p)_{\bar{\mathbb{Q}}}, \mathscr{O}_L/p^n))$$

$$= p - \text{adic completion of } \varinjlim_{K_p} H^1_{\mathrm{et}}(Y(K_pK^p)_{\bar{\mathbb{Q}}}, \mathscr{O}_L),$$

a $\mathrm{Gal}_{\mathbb{Q}} \times \mathbb{GL}_2(\mathbb{Q}_p)$-module.

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(I) Key extra symmetry: big spherical Hecke algebra $\mathbb{T}$

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$$\hat{H}^{1,\mathrm{gl}} = \bigoplus_{m \in \mathrm{Max}(\mathbb{T})} \hat{H}^{1,\mathrm{gl}}_m.$$

(II) Each m comes with a $\bar{\rho}_m : \mathrm{Gal}_{\mathbb{Q},\{p\}} \to \mathbb{GL}_2(k_L)$. For $\bar{\rho}_m$ absolutely irreducible, it lifts to

$$\rho_m : \mathrm{Gal}_{\mathbb{Q},\{p\}} \to \mathbb{GL}_2(\mathbb{T}_m)$$

with specified traces of Frobenius at primes $\neq p$.

Global motivation

(I) The key global result:

Theorem (Emerton, idealised)
As $\mathbb{T}_m[\mathbb{GL}_2(\mathbb{Q}_p) \times \mathrm{Gal}_{\mathbb{Q}}]$-modules

$$\hat{H}^{1,\mathrm{gl}}_m \simeq \Pi(\rho_m|_{\mathrm{Gal}_{\mathbb{Q}_p}})\widehat{\otimes}_{\mathbb{T}_m}\mathbb{T}^*_m \otimes_{\mathbb{T}_m} \rho_m,$$

where

- $\mathbb{T}^*_m = \mathscr{O}_L$-dual of $\mathbb{T}_m$.
- $\rho \to \Pi(\rho)$ is the p-adic local Langlands correspondence.

Global motivation

(I) This has many deep consequences, e.g.

- for $\rho : \mathrm{Gal}_{\mathbb{Q},\{p\}} \to \mathbb{GL}_2(L)$ odd and absolutely irreducible (plus technical hypotheses)

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbb{Q}}}(\rho, \hat{H}^{1,\mathrm{gl}}) \simeq \Pi(\rho|_{\mathrm{Gal}_{\mathbb{Q}_p}}).$$

- (under suitable assumptions on $\bar{\rho}$)

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbb{Q}}}(\bar{\rho}, \varinjlim_{K_p} H^1_{\mathrm{et}}(Y(K^pK_p)_{\bar{\mathbb{Q}}}, k_L)) = \pi(\bar{\rho}|_{\mathrm{Gal}_{\mathbb{Q}_p}}),$$

so LHS has finite length, highly nontrivial!

Global motivation

(I) Key facts used (all fail very badly in the local context):

• finiteness of $H^1_{\mathrm{et}}(Y(K^pK_p)_{\bar{\mathbb{Q}}}, ?) \rightsquigarrow$ admissibility of $\hat{H}^1$, nice topological behaviour.

• $H^2 = 0$ for $Y(K^pK_p)_{\bar{\mathbb{Q}}} \rightsquigarrow$ easy passage char $0-$ char p, good representation theoretic properties of $\hat{H}^1_m$.

• link between $\mathbb{GL}_2(\mathbb{Z}_p)$-algebraic vectors and crystalline Galois representations.

The Drinfeld tower

(I) Work over $\mathbb{C}_p$. Let

$$\mathscr{M}_0 = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$$

with the natural action of $G := \mathbb{GL}_2(\mathbb{Q}_p)$.

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(III) The "limit" $\mathscr{M}_\infty$ is perfectoid (Scholze and Weinstein) and

$$\mathrm{Gal}(\mathscr{M}_\infty / \mathscr{M}_0) = D^\times,$$

where $D =$ quaternion division algebra $/\mathbb{Q}_p$.

And its subtleties...

(I) What happens if we replace the $Y(K^p K_p)$'s by the $\mathscr{M}_n$'s? Issues:

- $\mathscr{M}_n$ not qc, no "reasonable" compactification known.

- $H^1_{\mathrm{et}}(\mathscr{M}_n, L)$ is huge, except for $n = 0$.

- no reasonable (co-)admissibility property.

- not clear/known if they are invariant under complete alg. closed extensions of $\mathbb{C}_p$.

- topology is a nightmare!

And its subtleties...

(I) Passage char $0-$ char p hard: no control on $H^2_{\mathrm{et}}(\mathscr{M}_n, k_L)$. If nonzero ($\Leftrightarrow \mathrm{Pic}(\mathscr{M}_n)$ not p-divisible), this space is an awful mess!

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(II) Contrast to:

- $H^2_{\text{et}}(X, k_L) = 0$ for X perfectoid quasi-Stein, e.g. $\mathscr{M}_\infty$ (Scholze$+$ Artin-Schreier$+$ Kedlaya-Liu).

- $H^2_{\text{et}}(X, \mathscr{O}^+/p)$ is almost 0 for a Stein curve X (Hansen).

- $\exists$ Stein curves X for which $H^2_{\text{et}}(X, \mathbb{F}_p) \neq 0$, e.g. open unit disc $/\mathbb{C}_p$ (but not over its spherical completion!).

Previous work

(I) Still:

Theorem (CDN)
For absolutely irreducible $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(L)$

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\rho, \varinjlim_n H^1_{\mathrm{et}}(\mathscr{M}_n, L(1))) = \begin{cases} JL(\rho) \otimes \Pi(\rho)^*, \text{ if } \rho \text{ is nice} \\ 0 \text{ if not} \end{cases}$$

nice: de Rham with weights $0, 1$, $WD(\rho)$ irreducible. Also

$$JL(\rho) := JL(LL(WD(\rho))) \in \mathrm{Irr}^{\mathrm{sm}}(D^\times).$$

Pending questions

(I) A few natural questions:

- integral or mod p analogue?
- description of $H^1_{\mathrm{et}}(\mathscr{M}_n, L)$ "à la Emerton"?
- where are the other Galois representations???

Limitations of the previous method

(I) The proof of the previous th. gives no clue:

• replace étale by pro-étale coh.

• describe pro-étale coh. via coherent and Hyodo-Kato coh.

$$0 \to \frac{\mathscr{O}(\mathscr{M}_n)}{\mathrm{cst}} \to H^1_{\mathrm{proet}}(\mathscr{M}_n, L(1)) \to (B^+_{\mathrm{st}}\widehat{\otimes}_{\breve{\mathbb{Q}}_p} H^1_{\mathrm{HK}}(\mathscr{M}_n))^{\varphi=p,N=0} -$$

• $\mathscr{O}(\mathscr{M}_n) \longleftrightarrow \Pi(\rho)$ via the Breuil-Strauch conjecture (Le Bras-D).

• HK coh. computed by p-adic uniformisation and l-adic ($l \neq p$, sic!) non-abelian Lubin-Tate theory.

Limitations of the previous method

(I) Harder for étale coh:

$$H^1_{\mathrm{et}}(\mathscr{M}_n, L(1)) \simeq (B^+_{\mathrm{st}} \widehat{\otimes}_{\breve{\mathbb{Q}}_p} H^{1,G-\mathrm{bd}}_{\mathrm{HK}}(\mathscr{M}_n))^{\varphi=p,N=0} \cap \Omega^{1,G-\mathrm{bd}}(\mathscr{M}_n),$$

but

- $H^{1,G-\mathrm{bd}}_{\mathrm{HK}}(\mathscr{M}_n) \longleftrightarrow \widehat{\Pi}$ with Π discrete series rep. of G, and $\widehat{\Pi}$ is huge.

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(II) Contrary to $\Omega^1(\mathscr{M}_n)$ (described by DL), $\Omega^{1,G-\mathrm{bd}}(\mathscr{M}_n)$ is quite mysterious (not coadmissible).

The key new result

(I) Saw: $\mathrm{Hom}(\rho, H^1(\mathscr{M}_n, L)) =$ dual of a finite length Banach G-representation. More delicate:

Theorem (CDN) $\mathrm{Hom}_{\mathrm{Gal}_{\mathbb{Q}_p}}(\bar{\rho}, H^1(\mathscr{M}_n, k_L))$ is the dual of a finite length smooth G-module $\forall \bar{\rho} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(k_L)$.

The method of proof is completely different and quite indirect.

Enters Paskūnas' theory

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(II) Gabriel's theory $\rightsquigarrow$

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(III) Each B is finite and

$$\mathscr{C}_B \simeq \text{compact } E_B - \text{modules},$$

$$\pi \to \mathrm{Hom}_G(P_B, \pi^\vee), \ M \to (M \otimes_{E_B} P_B)^\vee.$$

where

$$E_B = \mathrm{End}_G(P_B), \ P_B = (\text{inj. envelope of } \oplus_{\pi \in B} \pi)^\vee.$$

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(II) "Simplest" example:

$$\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(k_L) \text{ abs. irr} \rightsquigarrow B = B_{\bar{\rho}} = \{\pi(\bar{\rho})\}.$$

$$\rho^{\mathrm{un}} : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(R_{\bar{\rho}}) = \text{universal deformation of } \bar{\rho}$$

$$E_B = R_{\bar{\rho}}, \ P_B = R_{\bar{\rho}} - \text{dual of } \Pi(\rho^{\mathrm{un}}).$$

Completed cohomology of the Drinfeld tower

(I) Define

$$H^1_{k_L} = \varinjlim_{j} H^1(\mathscr{M}_j, k_L)^{\mathrm{Gal}_{\mathbb{Q}_p}-\mathrm{sm}}$$

The finiteness theorem+previous discussion $\rightsquigarrow$

$$H^1_{k_L} = \bigoplus_{\substack{\bar{\rho}:\mathrm{Gal}_{\mathbb{Q}_p}\to \mathbb{GL}_2(k_L) \\ \mathrm{ss}}} \overline{JL}_{\bar{\rho}} \otimes_{E_{\bar{\rho}}} P_{\bar{\rho}},$$

with $\overline{JL}_{\bar{\rho}}$ a smooth $D^{\times}$-module with action of $\mathrm{Gal}_{\mathbb{Q}_p}$.

Completed cohomology of the Drinfeld tower

(I) Can define similarly $\hat{H}^1$ for the Drinfeld tower and

$$\hat{H}^1 = \widehat{\bigoplus_{\bar{\rho}\ \mathrm{ss}}} JL_{\bar{\rho}} \mathbin{\widehat{\otimes}}_{E_{\bar{\rho}}} P_{\bar{\rho}}.$$

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$$\hat{H}^1 = \widehat{\bigoplus_{\bar{\rho}\ \mathrm{ss}}} JL_{\bar{\rho}} \widehat{\otimes}_{E_{\bar{\rho}}} P_{\bar{\rho}}.$$

(II) What is $JL_{\bar{\rho}}$? Need to "compute"

$$\mathrm{Hom}_G(\pi^\vee, H^1_{\mathrm{et}}(\mathscr{M}_n, \mathscr{O}_L/p^k)^{\mathrm{Gal}_{\mathbb{Q}_p}-\mathrm{sm}})$$

for π a smooth finite length G-module.

Completed cohomology of the Drinfeld tower

(I) Rationally: "easy". Proof: p-adic comparison theorems+Breuil-Strauch conjecture, as before.

Theorem (CDN) For $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(L)$ absolutely irreducible

$$\mathrm{Hom}_G(\Pi(\rho)^*, \varinjlim_n H^1_{\mathrm{et}}(\mathscr{M}_n, L(1))) \simeq \begin{cases} JL(\rho) \otimes \rho, \text{ if } \rho \text{ is nice} \\ 0 \text{ if not} \end{cases}$$

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(II) For π smooth modulo p: p-adic uniformisation+LGC $\rightsquigarrow$ $\mathrm{Hom}_G(\pi^*, H^1_{\mathrm{et}}(\mathscr{M}_n, k_L(1)))$ is linked to Scholze's functor.

Enters Scholze's functor

(I) Duality isomorphism (Faltings, Fargues, Scholze, Weinstein)

$$\mathscr{M}_\infty \simeq \mathrm{LT}_\infty$$

with the infinite level Lubin-Tate space, a pro-étale G-torsor of $\mathbb{P}^1$:

$$f : \mathrm{LT}_\infty \to \mathbb{P}^1.$$

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$$f : \mathrm{LT}_\infty \to \mathbb{P}^1.$$

(II) Scholze: π smooth mod p G-rep. $\rightsquigarrow$ smooth $D^\times$-modules with continuous $\mathrm{Gal}_{\mathbb{Q}_p}$-action

$$S^i(\pi) = H^i(\mathbb{P}^1, \mathscr{F}_\pi), \;\; \mathscr{F}_\pi = (f_* \underline{\pi})^G.$$

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(I) The construction extends to smooth $\mathscr{O}_L$-torsion modules, to p-adic Banach reps, and works in families.

Theorem (Hansen, Ludwig, Scholze)

a) $\pi \to S^i(\pi)$ preserves admissibility.

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b) $S^2(\pi) = 0$ if $\pi \pmod{p}$ is principal series or supersingular.

(II) Paskūnas used this to study $S^1(\Pi)$ when Π is a Banach representation.

Paskūnas, Schraen, D. (in progress): $S^1(\Pi)$ has finite length if Π is irreducible and corresponds to a Galois representation whose difference of Hodge-Tate weights $\notin \mathbb{Z}$.

Enters Scholze's functor

(I) The link to Scholze's functor:

Theorem (CDN) If π is a locally finite length smooth representation of G, killed by p^k and belonging to a generic bloc, then

$$\mathrm{Hom}_G(\pi^\vee, H^1(\mathscr{M}_\infty, \mathscr{O}_L/p^k)) \simeq S^1(\pi).$$

Simple idea: analyse the Cech spectral sequence for the covering $f : \mathrm{LT}_\infty \to \mathbb{P}^1$. Problem: describe $H^i(\mathrm{LT}_\infty \times G^k, \underline{\pi})$, which comes down to controlling certain $R^1\mathrm{lim}$.

Enters Scholze's functor

(I) One gets a spectral sequence

$$E_2^{p,q} = H^p(G, \mathrm{Hom}^{\mathrm{cont}}(\pi^\vee, H^q(\mathscr{M}_\infty, \mathscr{O}_L/p^k))) \Longrightarrow S^{p+q}(\pi).$$

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(II) $E_2^{p,0}$-terms controlled by:

- Strauch's description of $\pi_0(\mathscr{M}_\infty)$
- Fust's comparison theorem between continuous cohomology and Ext groups
- results of Paskūnas to kill these Ext groups.

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(III) Example (Paulina Fust): $H^2(\mathrm{SL}_2(\mathbb{Q}_p), \pi) = 0$ for π supersingular.

Completed cohomology of the Drinfeld tower

(I) The "simplest" case:

Theorem (CDN) If $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(\mathscr{O}_L)$ has absolutely irreducible reduction mod p, then

$$\mathrm{Hom}(\bar{\rho}, \varinjlim_n H^1(\mathscr{M}_n, k_L)) = \pi(\bar{\rho})^* \otimes_{k_L} \mathrm{Hom}(\bar{\rho}, S^1(\pi(\bar{\rho}))$$

and

$$\mathrm{Hom}(\rho, \hat{H}^1) = \Pi(\rho)^* \widehat{\otimes}_{\mathscr{O}_L} \mathrm{Hom}(\rho, S^1(\Pi(\rho)))).$$

The proof is quite tricky (in particular uses the compatibility of Scholze's functor and patching to avoid the problem of $H^2(\mathscr{M}_n, k_L)$ being unmanageable).

Link with potentially crystalline deformation rings

(I) Fix n and an irreducible L-representation σ of $\mathrm{Gal}(\mathscr{M}_n/\mathscr{M}_0)$, of dimension > 1. If X is a $D^\times$-module write

$$X[\sigma] = \mathrm{Hom}_{D^\times}(\sigma, X).$$

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(II) Define

$$\tilde{H}^1(\mathscr{M}_n, L(1)) = (\varprojlim_k H^1(\mathscr{M}_n, \mathscr{O}_L/p^k(1))^{\mathrm{Gal}_{\mathbb{Q}_p}-\mathrm{sm}})[1/p],$$

a sort of completed cohomology of the tower $\mathscr{M}_n/F$ for $F/\mathbb{Q}_p$ finite.

Link with potentially crystalline deformation rings

(I) One gets:

Theorem (CDN) For each semi-simple $\bar{\rho}: \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{GL}_2(k_L)$ there is a quotient $R^{\mathrm{ps}}_{\mathrm{tr}(\bar{\rho})}[1/p] \to R^{\sigma}_{\bar{\rho}}$ and a rank 2 Galois representation $V^{\sigma}_{\bar{\rho}}$ over $R^{\sigma}_{\bar{\rho}}$ such that

$$\tilde{H}^1(\mathscr{M}_n, L(1))[\sigma] = \widehat{\bigoplus_{\bar{\rho}}} \Pi(V^{\sigma}_{\bar{\rho}})^* \widehat{\otimes}_{R_{\bar{\rho}}} R^*_{\bar{\rho}} \otimes_{R_{\bar{\rho}}} V^{\sigma}_{\bar{\rho}}.$$

For $\bar{\rho}$ absolutely irreducible $V^{\sigma}_{\bar{\rho}}$ is the universal potentially crystalline deformation of $\bar{\rho}$ with Hodge-Tate weights $0, 1$ and Weil-Deligne type determined by σ.